THE GREAT OÏD

(DDD '24/25 X, XI & XII [RRS])

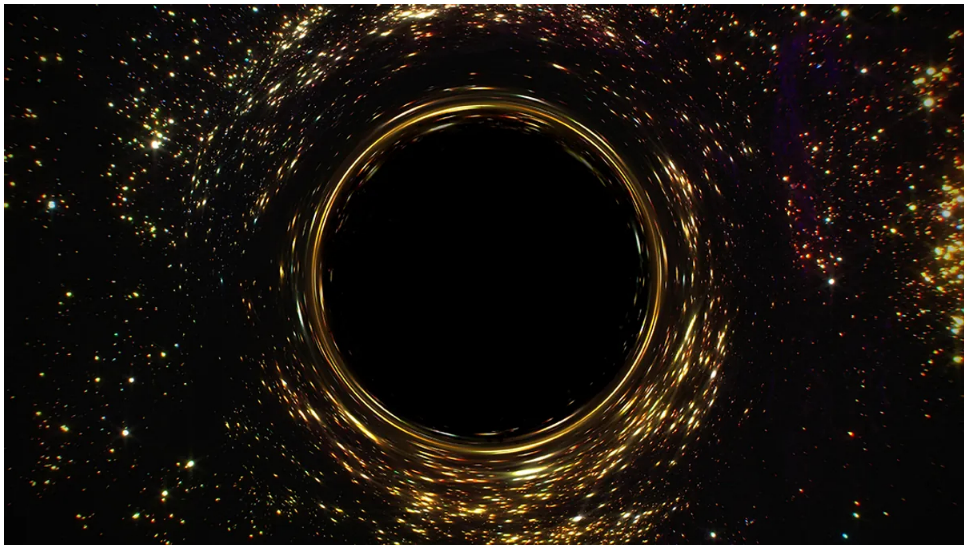

FIGURE 1. An artist's impression of a cosmic Void. One of the largest such spaces in the Universe is The Boötes Void, poetically described as The Great Nothing. Unlike the cosmic Nothings large and small, those in our understanding and modelling of symmetries in field (string, M-, *etc.*) theory can and ought to be filled with Meaningful Stuff. One such procedure encountered along the Path to Great Categorification consists in ingeniously dropping the "V"...

CONTENTS

We have, by now, an *almost* complete geometric framework in which to formulate the Gauge Principle—as intuited by theoretical physicists in the $2^{\rm nd}$ half of the last century of the previous millenium, on the wave of The Grand Unification—with the symmetry model given by a group action $\lambda : {\rm G} \longrightarrow {\rm Diff}(M)$ on the space of internal degrees of freedom M of a field theory. In order to spell out that Principle in full, we still need to go to the tangent of the many constructions discussed and lift to that tangent the local splitting of the total space of the bundle into its base B and the typical fibre M in a consistent manner, a construction which goes under the name of connection. Prior to rigorously formulating and—immediately thereafter—attaining that goal, though, we first generalise the symmetry model itself, with a view to encompassing rather natural geometric situations such as the one described in Weinstein's motivating example of Sec. IV.2. in Lecture I. Thus, in what follows, we consider—after [SS25]—a symmetry model given by a Lie groupoid, which—whenever it comes with object manifold M and arrow manifold $\mathscr{G}$—we denote as $\mathscr{G} \rightrightarrows M$, or—by a mild and common abuse of notation—simply as $\mathscr{G}$. Its group of bisections, about to play a pivotal rôle in our field-theoretic constructions, shall be denoted as $\mathbb{B}$.

1. The mother principaloid bundle, its shadow that matters, & the duck sitting in between

Let us recapitulate our assumptions: We want a *fibre bundle* with base given by our preferred spacetime B and typical fibre (with the interpretation of the 'space of internal degrees of freedom') given by the object manifold M of a Lie groupoid $\mathscr{G} \rightrightarrows M$. The transition 1-cocycle invariably takes values in a subgroup of $\mathrm{Diff}(M)$, and it precisely there where we encode the novel symmetry model—we restrict to those diffeomorphisms which are engendered by the Lie groupoid[1], *i.e.*, the group of bisections in the shadow realisation $t_*(\mathbb{B}) \subset \mathrm{Diff}(M)$. At this stage, we might look for a 'parent' fibre-bundle structure from which the thus constrained local transition data of the 'matter bundle' (and subsequently also a connection) would be induced, in analogy with the classic setting. The first thing that springs to one's mind is a principal bundle for the structure group chosen, but that is quite problematic as we are dealing with an infinite-dimensional (generically) Lie group, and so the corresponding principal bundle would have to be infinite dimensional, too. In order to avoid technical (and conceptual) problems naturally associated with differential geometry in infinite dimension, we shall, instead, look for a finite dimensional construction that cleverly encodes the same topological information (over B) as a transition 1-cocycle of a principal $\mathbb{B}$-bundle. In so doing, we draw intuition from the nontrivial prototype inscribed in the Trident (VII-VIII-IX.9). There, the associated bundle $\mathsf{P} \times_\lambda M$ is the prototype of our 'matter bundle', and over it, we find the peculiar object $\mathsf{P} \times M$, which carries the information about the principal bundle, and that of the associating action (in the typical fibre). Taking into account a convenient model of the principal bundle:

$$\mathsf{P} = \bigsqcup_{i \in I} (O_i \times \mathrm{G}) / \sim_{g_{\cdot\cdot}} ,$$

written in terms of the transition maps $g_{ij} : O_{ij} \longrightarrow \mathrm{G}$ associated with local trivialisations over elements O_i of an open cover of B, we readily derive a groupoidal rendering of the identification of typical fibres of $\mathsf{P} \times M$ over the intersections O_{ij}:

$$\big(x, (g, m), j\big) \sim \big(x, \big(g_{ij}(x) \cdot g, m\big), i\big) \equiv \big(x, \big(g_{ij}(x), \lambda_g(m)\big).(g, m), i\big) .$$

Above, the binary operation represented by . is that in the action Lie groupoid. Thinking in terms of the corresponding bisections, we rewrite the above as

$$\big(x, (g, m), j\big) \sim \big(x, \big(g_{ij}(x) \cdot g, m\big), i\big) \equiv \big(x, \big(g_{ij}(x), \cdot\big)\big(t(g, m)\big).(g, m), i\big) \equiv \big(x, L_{(g_{ij}(x), \cdot)}(g, m), i\big) ,$$

where, clearly,

$$\big(g_{ij}(x), \cdot\big) \in \mathrm{Bisec}(\mathrm{G} \ltimes_\lambda M) .$$

We are thus led to

Definition 1. A **principaloid bundle** $\mathscr{P}$ over B is a fibre bundle $(\mathscr{P}, B, \mathscr{G}, \pi_{\mathscr{P}})$ with structure group $L(\mathbb{B}) \subset \mathrm{Diff}(\mathscr{G})$.

Corollary 1. For every principaloid bundle $\mathscr{P}$ with a transition 1-cocycle $\beta_{ij} \in C^\infty(O_{ij}, \mathbb{B})$, associated with an open cover $\{O_i\}_{i \in I}$ of the base B of $\mathscr{P}$, there exists a canonical bundle isomorphism

$$\mathscr{P} \cong \bigsqcup_{i \in I} (O_i \times \mathscr{G}) / \sim_{L_{\beta_{\cdot\cdot}}} , \tag{1}$$

which puts the bundle on the right-hand side in the rôle of a model of $\mathscr{P}$.

[1] Our choice of the paradigm in which to formulate the postulates—that of the theory of fibre bundles—plays a decisive rôle. Had we relinquished the idea of a global model of the 'space of internal degrees of freedom', we might now contemplate more general geometric structures.

Proof: A simple consequence of The Clutching Theorem (V.3) for fibre bundles. □

Remark 1. It is to be emphasised at this stage that principaloid bundles, while offering us the practical advantage of finite-dimensionality, actually carry—as anticipated in our general considerations—the same Čech-cohomological information as the underlying infinite-dimensional principal $\mathbb{B}$-bundles. Indeed, the latter are represented—in virtue of The Clutching Theorem just invoked—by (classes of) the corresponding transition 1-cocycles $\{\beta_{ij}\}_{(i,j)\in I_{\mathcal{O}}^{\times 2}}$, and these can readily be recovered from their realisations $L_{\beta_{ij}}$ on the typical fibre $\mathscr{G}$ through evaluation on the identity bisection $\mathrm{Id}(M)\subset\mathscr{G}$,

$$L_{\beta_{ij}(x)}(\mathrm{Id}_m)\equiv\big(\beta_{ij}(x)\big)\big(t(\mathrm{Id}_m)\big).\mathrm{Id}_m=\big(\beta_{ij}(x)\big)(m)\,.$$

This observation highlights the strength and naturality of Def. 1.

Having found the 'parent' bundle, we next recover the 'matter' bundle from

Theorem 1. Every principaloid bundle $\mathscr{P}$ canonically induces a fibre bundle $(\mathscr{F},B,M,\pi_{\mathscr{F}})$ with structure group $t_*(\mathbb{B})\subset\mathrm{Diff}(M)$ and model

$$\mathscr{F}\cong\bigsqcup_{i\in I}\big(O_i\times M\big)/\sim_{t_*\beta_{\cdot\cdot}}\,, \tag{2}$$

written in terms of the transition 1-cocycle $\{\beta_{ij}\}_{(i,j)\in I_{\mathcal{O}}^{\times 2}}$ of $\mathscr{P}$. It comes with a bundle map

$$\begin{tikzcd} \mathscr{P} \arrow[rr, "\mathscr{D}"] \arrow[dr, "\pi_{\mathscr{P}}"'] & & \mathscr{F} \arrow[dl, "\pi_{\mathscr{F}}"] \\ & \Sigma & \end{tikzcd}\,, \tag{3}$$

locally modelled on $t:\mathscr{G}\to M$. The triple $(\mathscr{P},\mathscr{F},\mathscr{D})$ carries a canonical structure of a principal-$\mathscr{G}$-bundle object in the category of fibre bundles over Σ,

$$\begin{tikzcd} \Sigma \arrow[d, "\mathrm{id}_\Sigma"'] & \mathscr{P} \arrow[l, "\pi_{\mathscr{P}}"'] \arrow[d, "\mathscr{D}"'] \arrow[dr, "\mu"] & \mathscr{G} \arrow[d, shift left] \arrow[d, shift right] \\ \Sigma & \mathscr{F} \arrow[l, "\pi_{\mathscr{F}}"] & M \end{tikzcd}\,, \tag{4}$$

with ($\mathbb{B}$-equivariant) moment map $\mu:\mathscr{P}\longrightarrow M$ locally modelled on s, and action $\varrho:\mathscr{P}\,{}_\mu\!\times_t\mathscr{G}\longrightarrow\mathscr{P}$ locally modelled on the right-fibred action of $\mathscr{G}$ on itself.

Proof: Formula (2) implies that the induced bundle admits local trivialisations

$$\mathscr{F}\tau_i\ :\ \pi_{\mathscr{F}}^{-1}(O_i)\xrightarrow{\ \cong\ }O_i\times M \tag{5}$$

such that the corresponding transition mappings are

$$\mathscr{F}\tau_i\circ\mathscr{F}\tau_j^{-1}\ :\ O_{ij}\times M\longrightarrow O_{ij}\times M\ :\ (x,m)\longmapsto\big(x,t_*\big(\beta_{ij}(x)\big)(m)\big)\,.$$

Now, the local mappings

$$\mathscr{D}_i\ :\ \pi_{\mathscr{P}}^{-1}(O_i)\longrightarrow\pi_{\mathscr{F}}^{-1}(O_i)\ :\ \mathscr{P}\tau_i^{-1}\big(x,g\big)\longmapsto\mathscr{F}\tau_i^{-1}\big(x,t(g)\big)$$

glue up smoothly at $x'\in O_{ij}$ as

$$\begin{aligned}\mathscr{D}_j\big(\mathscr{P}\tau_i^{-1}\big(x',g\big)\big)&=\mathscr{D}_j\big(\mathscr{P}\tau_j^{-1}\big(x',L_{\beta_{ji}(x')}\big(g\big)\big)\big)\equiv\mathscr{F}\tau_j^{-1}\big(x',t\big(\beta_{ji}(x')\big(t(g)\big).g\big)\big)\\ =\ \ &\mathscr{F}\tau_j^{-1}\big(x',t_*\beta_{ji}(x')\big(t(g)\big)\big)=\mathscr{F}\tau_i^{-1}\big(x',t_*\beta_{ij}(x')\circ t_*\beta_{ji}(x')\big(t(g)\big)\big)\end{aligned}$$

$$= \mathscr{F}\tau_i^{-1}\big(x', t_*\big(\beta_{ij}(x')\cdot\beta_{ji}(x')\big)\big(t(g)\big)\big) = \mathscr{F}\tau_i^{-1}\big(x', t(g)\big) \equiv \mathscr{D}_i\big(\mathscr{P}\tau_i^{-1}\big(x', g\big)\big)\,.$$

For the structure of a right $\mathscr{G}$-module on $\mathscr{P}$, define smooth maps

$$\mu_i \;:\; \pi_{\mathscr{P}}^{-1}(O_i) \longrightarrow M \;:\; \mathscr{P}\tau_i^{-1}(x,g) \longmapsto s(g)\,.$$

On overlaps $O_{ij}\times\mathscr{G} \ni (x,g)$ of trivialisation charts, we find

$$\mu_j\big(\mathscr{P}\tau_i^{-1}(x,g)\big) = \mu\big(\mathscr{P}\tau_j^{-1}\big(x,\beta_{ji}(x)\rhd g\big)\big) \equiv s\big(\beta_{ji}(x)\rhd g\big) = s(g) \equiv \mu_i\big(\mathscr{P}\tau_i^{-1}(x,g)\big)\,.$$

Hence, the μ_i are restrictions $\mu_i = \mu\restriction_{\pi_{\mathscr{P}}^{-1}(O_i)}$ of a globally smooth map

$$\mu \;:\; \mathscr{P} \longrightarrow M\,.$$

The map is readily seen to intertwine the defining action $\mathscr{R}$ of $\mathbb{B}$ on $\mathscr{P}$ with the shadow action $t_*\circ\mathrm{Inv}$ on M: For ever y $(x,g)\in O_i\times\mathscr{G}$ and $\beta\in\mathbb{B}$,

$$\big(\mu\circ\mathscr{R}_\beta\big)\big(\mathscr{P}\tau_i^{-1}(x,g)\big) \equiv \mu\big(\mathscr{P}\tau_i^{-1}(x,g\lhd\beta)\big) \equiv s(g\lhd\beta) = t_*\big(\beta^{-1}\big)\big(s(g)\big) \equiv \big(t_*\big(\beta^{-1}\big)\circ\mu\big)\big(\mathscr{P}\tau_i^{-1}(x,g)\big)\,,$$

where the third equality follows from identities (i) of Prop. 6. Thus, indeed, $\mu\circ\mathscr{R}_\beta = t_*\big(\beta^{-1}\big)\circ\mu$.

Next, consider smooth maps

$$\varrho_i \;:\; \pi_{\mathscr{P}}^{-1}(O_i)\,{}_{\mu_i}\!\times_t\mathscr{G} \longrightarrow \pi_{\mathscr{P}}^{-1}(O_i) \;:\; \big(\mathscr{P}\tau_i^{-1}(x,g),h\big) \longmapsto \mathscr{P}\tau_i^{-1}(x,g.h)\,.$$

Again, the above glue to a globally smooth map

$$\varrho \;:\; \mathscr{P}\,{}_\mu\!\times_t\mathscr{G} \longrightarrow \mathscr{P}$$

since – for arbitrary $(x,g)\in O_{ij}\times\mathscr{G}$ and $h\in t^{-1}(s(g))$ –

$$\begin{aligned}\varrho_j\big(\mathscr{P}\tau_i^{-1}(x,g),h\big) &= \varrho_j\big(\mathscr{P}\tau_j^{-1}\big(x,\beta_{ji}(x)\rhd g\big),h\big) \equiv \mathscr{P}\tau_j^{-1}\big(x,\big(\beta_{ji}(x)\rhd g\big).h\big) = \mathscr{P}\tau_j^{-1}\big(x,\beta_{ji}(x)\rhd(g.h)\big)\\ &= \mathscr{P}\tau_i^{-1}\big(x,g.h\big) \equiv \varrho_i\big(\mathscr{P}\tau_i^{-1}(x,g),h\big)\,,\end{aligned}$$

where in the third equality we used Eq. (28). The triple $(\mathscr{P},\mu,\varrho)$ satisfies axioms (GrM1)–(GrM3) from Def. IV-75. of a right $\mathscr{G}$-module. This follows immediately from the fact that it is locally modelled on the canonical right-$\mathscr{G}$-module structure of Ex. IV-77.

Finally, we demonstrate the existence, on the fibred square $\mathscr{P}\times_{\mathscr{F}}\mathscr{P} \equiv \mathscr{P}\,{}_{\mathscr{D}}\!\times_{\mathscr{D}}\mathscr{P}$, of a unique map

$$\phi_{\mathscr{P}} \;:\; \mathscr{P}\times_{\mathscr{F}}\mathscr{P} \longrightarrow \mathscr{G} \tag{6}$$

with the property $(\mathrm{pr}_1,\phi_{\mathscr{P}}) = (\mathrm{pr}_1,\varrho)^{-1}$, which can be expressed more explicitly as

$$t\circ\phi_{\mathscr{P}}(p_1,p_2) = \mu(p_1)\,, \qquad p_2 = \varrho\big(p_1,\phi_{\mathscr{P}}(p_1,p_2)\big)\,, \tag{7}$$

with arbitrary $(p_1,p_2)\in\mathscr{P}\times_{\mathscr{F}}\mathscr{P}$. Consider smooth maps

$$\phi_i \;:\; \pi_{\mathscr{P}}^{-1}(O_i)\times_{\mathscr{F}}\pi_{O_i}^{-1}(O_i) \longrightarrow \mathscr{G} \;:\; \big(\mathscr{P}\tau_i^{-1}(x,g_1),\mathscr{P}\tau_i^{-1}(x,g_2)\big) \longmapsto g_1^{-1}.g_2\,,$$

whose well-definedness hinges on the identity $s(g_1^{-1}) = t(g_1) = t(g_2)$, derived from

$$\mathscr{F}\tau_i^{-1}\big(x,t(g_1)\big) \equiv \mathscr{D}\circ\mathscr{P}\tau_i^{-1}(x,g_1) = \mathscr{D}\circ\mathscr{P}\tau_i^{-1}(x,g_2) \equiv \mathscr{F}\tau_i^{-1}\big(x,t(g_2)\big)\,.$$

These maps satisfy the equality (written for any $x\in O_{ij}$)

$$\begin{aligned}&\phi_j\big(\mathscr{P}\tau_i^{-1}(x,g_1),\mathscr{P}\tau_i^{-1}(x,g_2)\big) = \phi_j\big(\mathscr{P}\tau_j^{-1}\big(x,\beta_{ji}(x)\rhd g_1\big),\mathscr{P}\tau_j^{-1}\big(x,\beta_{ji}(x)\rhd g_2\big)\big)\\ \equiv\;& \big(\beta_{ji}(x)\rhd g_1\big)^{-1}.\big(\beta_{ji}(x)\rhd g_2\big) = \big(g_1^{-1}\lhd\beta_{ij}(x)\big).\big(\beta_{ji}(x)\rhd g_2\big) = g_1^{-1}.\big(\beta_{ij}(x)\rhd\big(\beta_{ji}(x)\rhd g_2\big)\big)\\ =\;& g_1^{-1}.\big(\big(\beta_{ij}(x)\cdot\beta_{ji}(x)\big)\rhd g_2\big) = g_1^{-1}.g_2 \equiv \phi_i\big(\mathscr{P}\tau_i^{-1}(x,g_1),\mathscr{P}\tau_i^{-1}(x,g_2)\big)\,,\end{aligned}$$

in whose derivation we have invoked Eqs. (27) and (29). Hence, they glue up to a globally smooth map

$$\phi_{\mathscr{P}} \;:\; \mathscr{P}\times_{\mathscr{F}}\mathscr{P} \longrightarrow \mathscr{G}\,, \qquad \phi_{\mathscr{P}}\restriction_{\pi_{\mathscr{P}}^{-1}(O_i)\times_{\mathscr{F}}\pi_{O_i}^{-1}(O_i)} \equiv \phi_i\,,$$

The desired properties (7) are readily verified in the local presentation. □

Definition 2. We call $\mathscr{F}$ the **shadow** (**bundle**) of $\mathscr{P}$ and $\mathscr{D} : \mathscr{P} \longrightarrow \mathscr{F}$ the **sitting-duck map**. We call the map (6) the **division map** of $\mathscr{P}$.

The shadow bundle is our model of the configuration bundle of a field theory with the global symmetry $\mathscr{G} \rightrightarrows M$ gauged. It is illuminating to draw a picture of its local sections (representing 'matter', or 'Higgs' fields), and relate it to the intuitions derived in Sec. VII-VIII-IX.2.

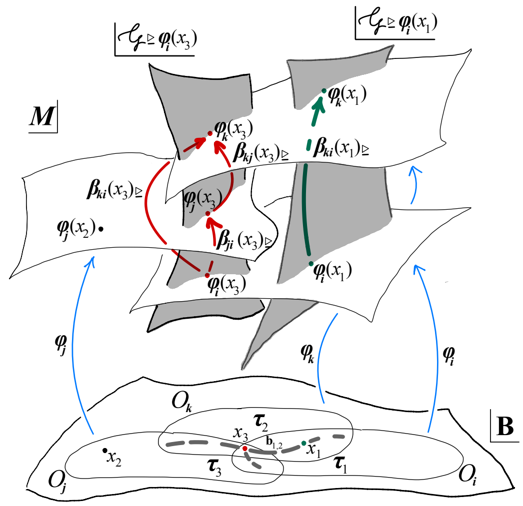

FIGURE 2. The shadow bundle $\mathscr{F} \longrightarrow B$ and its global section $\varphi \in \Gamma(\mathscr{F})$ represented by locally smooth maps $\varphi_i : O_i \longrightarrow M$ glued by the transition 1-cocycle $\{t_* \beta_{ij}\}_{(i,j)\in I_{\mathcal{O}}^{\times 2}}$ of $\mathscr{F}$ associated with a trivialising cover $\mathcal{O} = \{O_i\}_{i\in I}$ of the spacetime base B. The gluing models local-field discontinuities within $\mathscr{G}$-orbits.

Remark 2. In the light of The Godement Criterion of Thm. I.21., the shadow bundle $\mathscr{F}$ is diffeomorphic to $\mathscr{P}//\mathscr{G}$ and the sitting-duck map is the quotient map $\mathscr{P} \to \mathscr{P}//\mathscr{G}$. Note, however, that in contrast to ordinary Lie-group actions, freeness and properness of a $\mathscr{G}$-action does not, in general, give a principaloid bundle.

We immediately note the following very natural 'symmetry' property of the principaloid bundle:

Corollary 2. Every principaloid bundle $\mathscr{P}$ is endowed with a canonical right action of the group of bisections $\mathbb{B}$ of its structure groupoid $\mathscr{G}$, given by the formula

$$\mathscr{R} : \mathscr{P} \times \mathbb{B} \longrightarrow \mathscr{P} : (p, \beta) \longmapsto p \blacktriangleleft \beta^{-1}\big(\mu(x)\big)^{-1}.$$

<u>*Proof:*</u> The statement follows directly from Prop. 7 (itself a counterpart of Prop. IV-78. for right actions), with $\mathscr{R} \equiv \mathbb{B}\varrho$. $\square$

One is tempted to enquire if, perhaps, this is actually a defining property. The answer turns out to be much subtler than expected...

Theorem 2. For every Id-reducibile Lie groupoid $\mathscr{G}$, the commutant, within $\mathrm{Diff}(\mathscr{G})$,

$$C_{\mathrm{Diff}(\mathscr{G})}(R(\mathbb{B})) \equiv R(\mathbb{B})' := \{\ \Psi \in \mathrm{Diff}(\mathscr{G}) \mid \forall \beta \in \mathbb{B} : \Psi \circ R_\beta = R_\beta \circ \Psi\ \}$$

of the subgroup $R(\mathbb{B})$, *i.e.*—in other words—the group of $R(\mathbb{B})$-equivariant diffeomorphisms of $\mathscr{G}$, is $L(\mathbb{B})$, the group of left-multiplications by bisections:

$$\mathrm{Diff}_{\mathbb{B}}(\mathscr{G}) \equiv R(\mathbb{B})' = L(\mathbb{B}).$$

Proof: To be provided in the next lecture. □

The last result enables us to reformulate the definition of a principaloid bundle *for the distinguished class of structure Lie groupoids.*

Proposition 1. For every Id-reducibile Lie groupoid $\mathscr{G}$, the following statements are equivalent:

(PdB1) $(\mathscr{P}, B, \mathscr{G}, \pi_{\mathscr{P}})$ is a principaloid bundle;

(PdB2) $(\mathscr{P}, B, \mathscr{G}, \pi_{\mathscr{P}})$ is a right $\mathbb{B}$-space in the category of fibre bundles with typical fibre $\mathscr{G}$, such that, in a local trivialisation, $\mathbb{B}$ acts by right-multiplication on the fibre $\mathscr{G}$.

Proof: The implication (PdB1) $\Longrightarrow$ (PdB2) is a consequence of Cor. 2. For the converse implication, note that—by definition—the group of bisections $\mathbb{B}$ acts on the right $\mathbb{B}$-space object $\mathscr{P}$ by bundle automorphisms which cover the identity on B, *i.e.*, there exists a group homomorphism

$$\mathscr{R} \;:\; \mathbb{B} \longrightarrow \mathrm{Aut}_{\mathbf{Bun}(B)}(\mathscr{P})_{\mathrm{vert}}\,. \tag{8}$$

The object $\mathscr{P}$ admits, furthermore, an open cover $\{O_i\}_{i\in I}$ of its base B with local trivialisations

$$\mathscr{P}\tau_i \;:\; \pi_{\mathscr{P}}^{-1}(O_i) \xrightarrow{\cong} O_i \times \mathscr{G}\,, \qquad i \in I \tag{9}$$

which are $\mathbb{B}$-equivariant,

$$\mathscr{R}_\beta \circ \mathscr{P}\tau_i^{-1}(x,g) = \mathscr{P}\tau_i^{-1}\big(x, R_\beta(g)\big)\,, \qquad (x,g) \in O_i \times \mathscr{G}\,.$$

For every $x' \in O_{ij}$ and $g \in \mathscr{G}$, we obtain

$$\mathscr{P}\tau_i \circ \mathscr{P}\tau_j^{-1}(x',g) = \big(x', t_{ij}(x',g)\big)$$

for some smooth map

$$t_{ij} \;:\; O_{ij} \longrightarrow \mathrm{Diff}_{\mathbb{B}}(\mathscr{G}).$$

Here, for simplicity, $t_{ij}(x,g) \equiv (t_{ij}(x))\,(g)$. (PdB1) now follows directly from Thm. 2 : The transition maps t_{ij} take the special form:

$$t_{ij}(\sigma) = L_{\beta_{ij}(\sigma)}$$

expressed in terms of a Čech transition 1-cocycle

$$\beta_{ij} \;:\; O_{ij} \longrightarrow \mathbb{B}\,, \qquad (i,j) \in I_{\mathcal{O}}^{\times 2}\,.$$

□

Example 1. The relevance of the assumption of Id-reducibilty in Thm. 2, and so also in Prop. 1, can readily be demonstrated[2] on the previously introduced example of a non-Id-reducible pair groupoid $\mathrm{Pair}(M_{12})$ of the disjoint sum $M_{12} = M_1 \sqcup M_2$ of two non-diffeomorphic manifolds M_1 and M_2 of the same dimension. Assuming the M_α connected, we obtain the following decomposition of the arrow manifold of $\mathrm{Pair}(M_{12})$ into disjoint connected components

$$M_{12} \times M_{12} = M_1 \times M_1 \sqcup M_1 \times M_2 \sqcup M_2 \times M_1 \sqcup M_2 \times M_2\,.$$

Consider a diffeomorphism of $M_{12} \times M_{12}$ with the corresponding decomposition

$$\Phi = \mathrm{id}_{M_1\times M_1} \sqcup \mathrm{id}_{M_1\times M_2} \sqcup (f_2 \times \mathrm{id}_{M_1}) \sqcup \mathrm{id}_{M_2\times M_2}\,,$$

written for some $f_2 \in \mathrm{Diff}(M_2) \smallsetminus \{\mathrm{id}_{M_2}\}$. As the diffeomorphism has a trivial restriction to the second argument in its domain, it manifestly commutes with

$$R(\mathbb{B}) = \big\{\; \mathrm{id}_{M_{12}} \times (f_1 \sqcup f_2) \mid (f_1, f_2) \in \mathrm{Diff}(M_1) \times \mathrm{Diff}(M_2) \;\big\} \subset \mathrm{Diff}(M_{12})\,.$$

On the other hand, we have

$$L(\mathbb{B}) = \big\{\; (f_1 \sqcup f_2) \times \mathrm{id}_{M_{12}} \mid (f_1, f_2) \in \mathrm{Diff}(M_1) \times \mathrm{Diff}(M_2) \;\big\} \subset \mathrm{Diff}(M_{12})\,,$$

[2] The author thanks Damian Kayzer and Jakub Filipek for suggesting and working out (a special case of) this example.

and so, clearly,

$$\Phi \notin L(\mathbb{B})\,.$$

Thus, the observed feature of principaloid bundles is *not* a defining one in general, and so we shall keep the original sheaf-theoretic definition for the sake of keeping the class of admissible structure Lie groupoids as wide as possible.

2. Automorphisms of principaloid bundles

Now that the basic objects of our theory have been introduced, it is time to indicate the class of bundle automorphisms which are going to model groupoidal 'symmetry transformations between the local frames', along the lines (re)drawn in Lectures VII, VIII and IX. As in the case of transition maps, we fix the latter in local presentations, choosing from among diffeomorphisms of the typical fibre $\mathscr{G}$. And once again, we impose a groupoidal nature of the structure maps under considerations by restricting this choice to $L(\mathbb{B}) \subset \mathrm{Diff}(\mathscr{G})$ (for the principaloid bundle) resp. $t_*(\mathbb{B}) \subset \mathrm{Diff}(M)$ (for the shadow bundle). We start with

Definition 3. Let $(\mathscr{P}, B, \mathscr{G}, \pi_{\mathscr{P}})$ be a principaloid bundle and let $\mathcal{O} \equiv \{O_i\}_{i \in I}$ be a trivialising cover of its base B, with the associated transition 1-cocycle $\{\beta_{ij}\}_{(i,j) \in I^{\times 2}_{\mathcal{O}}}$. An *automorphism* of $\mathscr{P}$ is a bundle map (Φ, f) with base component $f \in \mathrm{Diff}(B)$ covered by a diffeomorphism $\Phi \in \mathrm{Diff}(\mathscr{P})$, as in the commutative diagram

$$\begin{tikzcd} \mathscr{P} \arrow[r, "\Phi"] \arrow[d, "\pi_{\mathscr{P}}"'] & \mathscr{P} \arrow[d, "\pi_{\mathscr{P}}"] \\ \mathbb{B} \arrow[r, "f"'] & B \end{tikzcd}\,,$$

which admits local restrictions

$$\Phi\restriction \;:\; \pi_{\mathscr{P}}^{-1}\big(\mathcal{O}^f_{(j,i)}\big) \xrightarrow{\cong} \pi_{\mathscr{P}}^{-1}\big(f\big(\mathcal{O}^f_{(j,i)}\big)\big) \;:\; \mathscr{P}\tau_i^{-1}(x,g) \longmapsto \mathscr{P}\tau_j^{-1}\big(f(x), L_{\gamma_{(j,i)}(x)}(g)\big)\,,$$

written for the refined cover $\mathcal{O}^f := \{O^f_{(j,i)} \equiv f^{-1}(O_j) \cap O_i\}_{(j,i) \in I^{\times 2}_{f,\mathcal{O}}}$ of Σ with $I^{\times 2}_{f,\mathcal{O}} := \{\ (j,i) \in I^{\times 2} \mid O^f_{(j,i)} \neq \varnothing\ \}$ in terms of a family of smooth maps

$$\gamma_{(j,i)} \;:\; O^f_{(j,i)} \longrightarrow \mathbb{B}\,, \qquad (j,i) \in I^{\times 2}_{f,\mathcal{O}}$$

subject to the gluing relations

$$\gamma_{(l,k)}\restriction_{\mathcal{O}^f_{(j,i)(l,k)}} = \big(f^*\beta_{lj} \cdot \gamma_{(j,i)} \cdot \beta_{ik}\big)\restriction_{\mathcal{O}^f_{(j,i)(l,k)}} \tag{10}$$

over the $\mathcal{O}^f_{(j,i)(l,k)} \equiv O^f_{(j,i)} \cap O^f_{(l,k)}$. Whenever $f = \mathrm{id}_\Sigma$, we call the automorphism **vertical**, or a **gauge transformation** of $\mathscr{P}$.

The **group of automorphisms** of $\mathscr{P}$ shall be denoted as

$$\mathrm{Aut}(\mathscr{P})\,,$$

and its subgroup composed of vertical automorphisms, also to be referred to as the **gauge group** of $\mathscr{P}$, as

$$\mathrm{Gauge}(\mathscr{P}) \equiv \mathrm{Aut}_{\mathrm{vert}}(\mathscr{P})\,.$$

Remark 3. The gluing relations (10) are determined uniquely by the requirement that the local presentations glue to a globally smooth bundle map.

We have the obvious

Proposition 2. Automorphisms of a principaloid bundle $\mathscr{P}$ are equivariant with respect to the right action of its structure Lie groupoid $\mathscr{G}$ of Thm. 1, and so also with respect to the induced right $\mathbb{B}$-action of Cor. 2. For $\mathscr{G}$ Id-reducible, the converse is also true: Every $\mathbb{B}$-equivariant bundle (self-)map $\Phi \in \mathrm{Diff}_{\mathbb{B}}(\mathscr{P})$ is an automorphism of the principaloid bundle $\mathscr{P}$.

Proof: The claim is a straightforward consequence of the associativity of the groupoid multiplication (which ensures the commutativity of the left and right multiplications), and—in its second part—of Thm. 2. □

As desired, automorphisms of the 'mother' bundle $\mathscr{P}$ give rise to distinguished automorphisms of its shadow—the 'matter' bundle $\mathscr{F}$.

Proposition 3. There exists a canonical group homomorphism

$$\mathscr{F}_* \;:\; \mathrm{Aut}(\mathscr{P}) \longrightarrow \mathrm{Aut}(\mathscr{F})\,,$$

satisfying, for all $\Phi \in \mathrm{Aut}(\mathscr{P})$ and $\Phi_{\rm v} \in \mathrm{Gauge}(\mathscr{P})$, the equivariance identity

$$\mathscr{D} \circ \Phi = \mathscr{F}_*(\Phi) \circ \mathscr{D}\,. \tag{11}$$

Proof: In the light of Thm. 1 and Def. 3, we find, for $(x,g) \in O^f_{(j,i)} \times \mathscr{G}$,

$$(\mathscr{D} \circ \Phi)(x,g) = \big(f(x), t\big(L_{\gamma_{(j,i)}(x)}(g)\big)\big) = \big(f(x), t_*(\gamma_{(j,i)}(x))(t(g))\big)\,,$$

where the last equality uses Prop. 6 (ii). This provides us with the smooth candidate

$$\mathscr{F}_*(\Phi)_{(j,i)} \;:\; \pi_{\mathscr{F}}^{-1}\big(O^f_{(j,i)}\big) \longrightarrow \pi_{\mathscr{F}}^{-1}\big(f\big(O^f_{(j,i)}\big)\big) \;:\; \mathscr{F}\tau_i^{-1}(x,m) \longmapsto \mathscr{F}\tau_j^{-1}\big(f(x), t_*\big(\gamma_{(j,i)}(x)\big)(m)\big)$$

for a local presentation of $\mathscr{F}_*(\Phi)$. We readily check the globality of the thus presented bundle automorphism through a direct calculation (carried out for $x' \in O^f_{(j,i)(l,k)}$):

$$\begin{aligned}
&\mathscr{F}_*(\Phi)_{(l,k)}\big(\mathscr{F}\tau_i^{-1}(x',m)\big) = \mathscr{F}_*(\Phi)_{(l,k)}\big(\mathscr{F}\tau_k^{-1}\big(x', \beta_{ki}(x') \trianglerighteq m\big)\big)\\
= \;\; &\mathscr{F}\tau_l^{-1}\big(f(x'), \gamma_{(l,k)}(x') \cdot \beta_{ki}(x') \trianglerighteq m\big) = \mathscr{F}\tau_j^{-1}\big(x', f^*\beta_{jl}(x') \cdot \gamma_{(l,k)}(x') \cdot \beta_{ki}(x') \trianglerighteq m\big)\\
= \;\; &\mathscr{F}\tau_j^{-1}\big(x', \gamma_{(j,i)}(x') \trianglerighteq m)\big)\big)\big) \equiv \mathscr{F}_*(\Phi)_{(j,i)}\big(\mathscr{F}\tau_i^{-1}(x',m)\big)\,,
\end{aligned}$$

in which the homomorphic nature of t_* has been taken into account, see Def. II-III.40. The latter also ensures homomorphicity of $\mathscr{F}_*$. □

Definition 4. Let $\mathscr{F}$ be the shadow of a principaloid bundle $\mathscr{P}$, and let $\varphi \in$ gauge transform of $\varphi \in \Gamma(\mathscr{F})$ be a global section of $\mathscr{F}$. Its pushforward by an automorphism $\mathscr{F}_*(\Phi)$ of $\mathscr{F}$ induced from a gauge transformation $\Phi \in \mathrm{Gauge}(\mathscr{P})$,

$$\varphi^\Phi \equiv \mathscr{F}_*(\Phi) \circ \varphi\,,$$

shall be called a **gauge transform** of φ.

3. The Gauge Trident: Ehresmann, momentum, the sitting duck and... Action!

In Lectures VII, VIII and IX, we encoded the action of the automorphism group of the principal bundle on the associated bundle $\mathsf{P}\times_\lambda M$ in the structure of a left module with respect to the Atiyah–Ehresmann groupoid $\mathrm{At}(\mathsf{P})$ on the latter bundle, descended from an analogous structure on $\mathsf{P} \times M$ (inherited from P). Below, we want to replicate that successful approach for a generic structure groupoid. The 'only' impediment that we encounter on our way is the absence of a principal $\mathbb{B}$-bundle and the inadequacy of the various actions—that of $\mathscr{G}$ and that of $\mathbb{B}$, engendered by it—on its 'tame' structural substitute $\mathscr{P}$ as candidates for a quotienting procedure. Therefore, we take as the basis of our present generalisation a judiciously chosen *presentation* of the Atiyah–Ehresmann groupoid from the proof of Prop. VII-VIII-IX.11., to wit,

$$\mathrm{At}(\mathsf{P}) = \bigsqcup_{j\in I}\big(\mathsf{P} \times O_j\big)/\sim_{r_{\mathrm{Invo}g_{\cdot\cdot}}} \cong \bigsqcup_{i,j\in I}\big(O_i^{(1)} \times \mathrm{G} \times O_j^{(2)}\big)/\sim_{\ell_{g^{(1)}_{\cdot\cdot}}\circ\wp_{\mathrm{Invo}g^{(2)}_{\cdot\cdot}}}\,.$$

With this in mind, we give

Definition 5. We shall call the fibre bundle $\pi_{\mathrm{At}(\mathscr{P})}\ :\ \mathrm{At}(\mathscr{P})\longrightarrow\Sigma$ with typical fibre $\mathscr{P}$ and model

$$\pi_{\mathrm{At}(\mathscr{P})}\ :\ \mathrm{At}(\mathscr{P})\equiv\bigsqcup_{i\in I}\big(\mathscr{P}\times\pi_{\mathrm{At}(\mathscr{P})}\ :\ O_i\big)/\sim_{\mathscr{R}_{\beta_{..}^{-1}}}\ :\ [(p,x)]\longmapsto x\,,$$

the **Atiyah bundle of** $\mathscr{P}$. We shall also denote, in the above model,

$$\pi_1\ :\ \bigsqcup_{i\in I}\big(\mathscr{P}\times O_i\big)/\sim_{\mathscr{R}_{\beta_{..}^{-1}}}\longrightarrow\Sigma\ :\ [(p,x)]\longmapsto\pi_{\mathscr{P}}(p)\,.$$

Remark 4. $\mathrm{At}(\mathscr{P})$ is canonically induced by the principaloid bundle $\mathscr{P}$ through its transition 1-cocycle $\{\beta_{ij}\}_{i,j\in I_{\mathcal{O}}^{\times 2}}$ realised by the defining action (8).

Theorem 3. The pair $(\mathrm{At}(\mathscr{P}),\mathscr{F})$ carries a canonical structure of a Lie groupoid, fitting into the following short exact sequence:

$$(12)\qquad\begin{tikzcd}[column sep=large, row sep=large]
\mathrm{Ad}(\mathscr{P}) \arrow[r, hook, "j_{\mathrm{Ad}(\mathscr{P})}"] \arrow[d, shift right, "\mathsf{T}|"'] \arrow[d, shift left, "\mathsf{S}|"] & \mathrm{At}(\mathscr{P}) \arrow[r, two heads, "\pi"] \arrow[d, shift right, "\mathsf{T}"'] \arrow[d, shift left, "\mathsf{S}"] & \Sigma\times\Sigma \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\mathrm{pr}_2"] \\
\mathscr{F} \arrow[r, equal] & \mathscr{F} \arrow[r, "\pi_{\mathscr{F}}"'] & \Sigma
\end{tikzcd}\,.$$

Here, $\pi=(\pi_1,\pi_{\mathrm{At}(\mathscr{P})})$ and $j_{\mathrm{Ad}(\mathscr{P})}$ is the embedding of $\mathrm{Ad}(\mathscr{P})=\pi^{-1}(\mathrm{Id}(\Sigma))$, where $\mathrm{Id}(\Sigma)\subset\Sigma\times\Sigma$ is the identity bisection.

The proof of Theorem 3 bases upon a pair of propositions and a lemma stated below.

Proposition 4. The pair $(\mathrm{At}(\mathscr{P}),\mathscr{F})$ composes, in a canonical way, a Lie groupoid, with the following structure maps, modelled on those of the Lie groupoid $\mathscr{G}$ in local trivialisations,

$$(13)\qquad\begin{tikzcd}
\mathrm{At}(\mathscr{P})\times_{\mathscr{F}}\mathrm{At}(\mathscr{P}) \arrow[r, "\mathsf{M}"] & \mathrm{At}(\mathscr{P}) \arrow[r, "\mathsf{J}"] & \mathrm{At}(\mathscr{P}) \arrow[r, shift left, "\mathsf{S}"] \arrow[r, shift right, "\mathsf{T}"'] & \mathscr{F} \arrow[l, bend right=50, "\mathsf{I}"']
\end{tikzcd}\,.$$

<u>*Proof:*</u> We begin by noting that $\mathrm{At}(\mathscr{P})$ admits further (model) resolution given by

$$(14)\qquad\mathrm{At}(\mathscr{P})\cong\bigsqcup_{i,j\in I}\big(O_i^{(1)}\times\mathscr{G}\times O_j^{(2)}\big)/\sim_{L^{(1)}_{\beta_{..}}\circ R^{(2)}_{\beta_{..}^{-1}}}\,,$$

with local charts glued by the identifications

$$(x_1,g,x_2,k,l)\sim\big(x_1,\beta_{ik}(x_1)\triangleright g\triangleleft\beta_{lj}(x_2),x_2,i,j\big)\,.$$

Accordingly, we define the structure maps as follows

$$(15)\qquad\begin{aligned}
\mathsf{S}\ &:\ \mathrm{At}(\mathscr{P})\longrightarrow\mathscr{F},\ [(x_1,g,x_2,i,j)]\longmapsto[(x_2,s(g),j)]\,,\\
\mathsf{T}\ &:\ \mathrm{At}(\mathscr{P})\longrightarrow\mathscr{F},\ [(x_1,g,x_2,i,j)]\longmapsto[(x_1,t(g),i)]\,,\\
\mathsf{I}\ &:\ \mathscr{F}\longrightarrow\mathrm{At}(\mathscr{P}),\ [(x,m,i)]\longmapsto[(x,\mathrm{Id}_m,x,i,i)]\,,\\
\mathsf{J}\ &:\ \mathrm{At}(\mathscr{P})\longrightarrow\mathrm{At}(\mathscr{P}),\ [(x_1,g,x_2,i,j)]\longmapsto[(x_2,g^{-1},x_1,j,i)]
\end{aligned}$$

and

$$\mathsf{M}\ :\ \mathrm{At}(\mathscr{P})_{\mathsf{S}}\times_{\mathsf{T}}\mathrm{At}(\mathscr{P})\longrightarrow\mathrm{At}(\mathscr{P}),\ \big([(x_1,g,x_2,i,j)],[(x_2,h,x_3,k,l)]\big)\longmapsto[(x_1,g.\big(\beta_{jk}(x_2)\triangleright h\big),x_3,i,l)]\,.$$

The well-definedness of the above maps is ensured by the identifications

- for the source map, at $x_1\in O_{ik}$ and $x_2\in O_{jl}$,

  $$\mathsf{S}\big([(x_1,\beta_{ki}(x_1)\triangleright g\triangleleft\beta_{jl}(x_2),x_2,k,l)]\big)=[(x_2,\beta_{lj}(x_2)\trianglerighteq s(g),l)]\equiv[(x_2,s(g),j)]\,,$$

  see Eq. (24) in Appendix A;

- for the target map, at $x_1 \in O_{ik}$ and $x_2 \in O_{jl}$,

$$\mathsf{T}\big([(x_1, \beta_{ki}(x_1) \rhd g \lhd \beta_{jl}(x_2), x_2, k, l)]\big) = [(x_1, \beta_{ki}(x_1) \unrhd t(g), k)] \equiv [(x_1, t(g), i)]\,,$$

  see Eq. (25);
- for the identity map, at $x \in O_{ij}$,

$$\mathsf{I}\big([(x, \beta_{ji}(x) \unrhd m, j)]\big) = [(x, C_{\beta_{ji}(x)}(\mathrm{Id}_m), x, j, j)] \equiv [(x, \mathrm{Id}_m, x, i, i)]\,,$$

  see Eq. (26);
- for the inverse map, at $x_1 \in O_{ik}$ and $x_2 \in O_{jl}$,

$$\mathsf{J}[(x_1, \beta_{ki}(x_1) \rhd g \lhd \beta_{jl}(x_2), x_2, k, l)] = [(x_2, \beta_{lj}(x_2) \rhd g^{-1} \lhd \beta_{ik}(x_2), x_1, l, k)] \equiv [(x_2, g^{-1}, x_1, j, i)]\,,$$

  see Eq. (27);
- for the multiplication map, at $x_1 \in O_{im}, x_2 \in O_{jkno}$ and $x_3 \in O_{lp}$,

$$\begin{aligned} &\mathsf{M}\big([(x_1, \beta_{mi}(x_1) \rhd g \lhd \beta_{jn}(x_2), x_2, m, n)], [(x_2, \beta_{ok}(x_2) \rhd h \lhd \beta_{lp}(x_3), x_3, o, p)]\big) \\ = \;& [(x_1, (\beta_{mi}(x_1) \rhd g \lhd \beta_{nj}(x_2)^{-1}).(\beta_{no}(x_2) \rhd (\beta_{ok}(x_2) \rhd h) \lhd \beta_{lp}(x_3)), x_3, m, p)] \\ = \;& [(x_1, (\beta_{mi}(x_1) \rhd g).((\beta_{jn}(x_2) \rhd (\beta_{no}(x_2) \cdot \beta_{ok}(x_2)) \rhd h) \lhd \beta_{lp}(x_3)), x_3, m, p)] \\ = \;& [(x_1, (\beta_{mi}(x_1) \rhd g).((\beta_{jn}(x_2) \cdot \beta_{nk}(x_2)) \rhd h \lhd \beta_{lp}(x_3)), x_3, m, p)] \\ = \;& [(x_1, \beta_{mi}(x_1) \rhd (g.(\beta_{jk}(x_2) \rhd h)) \lhd \beta_{lp}(x_3), x_3, m, p)] \equiv [(x_1, g.(\beta_{jk}(x_2) \rhd h), x_3, i, l)]\,, \end{aligned}$$

  see Eq. (29).

The constitutive relations between the structure maps are implied by the same relations for their local models. The only seemingly non-obvious ones are those involving the multiplication map M, but in the light of the above consistency check and the simple relation $\mathrm{At}(\mathscr{P})\,{}_{\mathsf{S}}{\times}_{\mathsf{T}}\,\mathrm{At}(\mathscr{P}) \subset \mathrm{At}(\mathscr{P}) \times_\Sigma \mathrm{At}(\mathscr{P})$, we may rewrite the definition of M as

$$\mathsf{M} \;:\; \mathrm{At}(\mathscr{P})_{\mathsf{S}}{\times}_{\mathsf{T}}\mathrm{At}(\mathscr{P}) \longrightarrow \mathrm{At}(\mathscr{P}),\ \big([(x_1, g_1, x_2, i, j)], [(x_2, g_2, x_3, j, l)]\big) \longmapsto [(x_1, g_1.g_2, x_3, i, l)]\,. \tag{16}$$

□

Definition 6. We shall call $\mathrm{At}(\mathscr{P}) \rightrightarrows \mathscr{F}$ the **Atiyah–Ehresmann groupoid** of $\mathscr{P}$. The short exact sequence (12) of Lie groupoids shall be referred to as the **Atiyah sequence** for $\mathscr{P}$.

Lemma 1. The pair $(\pi, \pi_{\mathscr{F}})$ is an epimorphism from the Lie groupoid $\mathrm{At}(\mathscr{P})$ to the Lie groupoid $\mathrm{Pair}(\Sigma)$.

Proof: The statement of the Lemma follows straightforwardly from the fact that the structure maps (15) and (16) cover the corresponding structure maps of $\mathrm{Pair}(\Sigma)$. □

Proof of Theorem 3. The existence of the structure of a Lie groupoid on $(\mathrm{At}(\mathscr{P}), \mathscr{F})$ is stated in Prop. 4. Exactness of sequence (12) at its node $\mathrm{Pair}(\Sigma)$ then follows from Lem. 1. As a set, $\mathrm{Ad}(\mathscr{P})$ fits into the short exact sequence by definition, and the only thing that remains to be proven is the embedding of the pair $(\mathrm{Ad}(\mathscr{P}), \mathscr{F})$ in the Atiyah–Ehresmann groupoid $\mathrm{At}(\mathscr{P}) \rightrightarrows \mathscr{F}$ as a Lie subgroupoid. Since $(\pi, \pi_{\mathscr{F}})$ is an epimorphism of Lie groupoids in that category, its arrow component π is automatically transverse to $\mathrm{Id}(\Sigma) \subset \Sigma \times \Sigma$ (as a submersion), and so we conclude the present proof by invoking Prop. VII-VIII-IX.9. □

Definition 7. We shall call $\mathrm{Ad}(\mathscr{P})$ the **adjoint bundle** of $\mathscr{P}$, and $\mathrm{Ad}(\mathscr{P}) \rightrightarrows \mathscr{F}$ the **adjoint groupoid** of $\mathscr{P}$. The latter shall be denoted as $C(\mathscr{P})$.

Proposition 5. The adjoint fibre bundle $\mathrm{Ad}(\mathscr{P})$ has the model

$$\mathrm{Ad}(\mathscr{P}) \cong \bigsqcup_{i\in I} \big(O_i \times \mathscr{G}\big)/ \sim_{C_{\beta_{..}}}\,, \tag{17}$$

written in terms of the transition 1-cocycle $\{\beta_{ij}\}_{(i,j)\in I_{\mathcal{O}}^{\times 2}}$ of $\mathscr{P}$ which we realise by conjugation of $\mathscr{G}$ by $\mathbb{B}$.

Proof: By definition, we have—in the notation of Eq. (14)—

$$\mathrm{Ad}(\mathscr{P}) \equiv \pi^{-1}(\mathrm{Id}(\Sigma)) = \bigsqcup_{i,j\in I} \big(O_i^{(1)} \times_\Sigma \big(\mathscr{G} \times O_j^{(2)}\big)\big)/ \sim_{L^{(1)}_{\beta_{..}} \circ R^{(2)}_{\beta_{..}^{-1}}}\,.$$

Thus, a point in $\mathrm{Ad}(\mathscr{P})$ is an equivalence class $[(x,g,x,i,j)] \equiv [(x, g \triangleleft \beta_{ji}(x), x, i, i)] \equiv [(x, \beta_{ji}(x) \triangleright g, x, j, j)]$. We define (smooth) maps

$$\jmath_{\mathrm{Ad}(\mathscr{P})} \;:\; \bigsqcup_{i\in I} \big(O_i \times \mathscr{G}\big)/ \sim_{C_{\beta_{..}}} \longrightarrow \mathrm{Ad}(\mathscr{P}) \;:\; [(x,g,i)] \longmapsto [(x,g,x,i,i)]$$

and

$$\iota_{\mathrm{Ad}(\mathscr{P})} \;:\; \mathrm{Ad}(\mathscr{P}) \longrightarrow \bigsqcup_{i\in I} \big(O_i \times \mathscr{G}\big)/ \sim_{C_{\beta_{..}}} \;:\; [(x,g,x,i,j)] \longmapsto [(x, g \triangleleft \beta_{ji}(x), i)]\,,$$

which are readily checked to be each other's inverses. Their well-definedness is a consequence of the identities, written for $x \in O_{ij}$ and $x' \in O_{ijkl}$,

$$\jmath_{\mathrm{Ad}(\mathscr{P})}\big([(x, C_{\beta_{ji}(x)}(g), j)]\big) \equiv [(x, C_{\beta_{ji}(x)}(g), x, j, j)] = [(x, \beta_{ji}(x) \triangleright g \triangleleft \beta_{ji}(x)^{-1}, x, j, j)] = [(x,g,x,i,i)]\,,$$

and

$$\begin{aligned}
&\iota_{\mathrm{Ad}(\mathscr{P})}\big([(x', \beta_{ki}(x') \triangleright g \triangleleft \beta_{lj}(x')^{-1}, x', k, l)]\big) \equiv [(x', \big(\beta_{ki}(x') \triangleright g \triangleleft \beta_{lj}(x')^{-1}\big) \triangleleft \beta_{lk}(x'), k)]\\
=\;& [(x', \beta_{ki}(x') \triangleright g \triangleleft \beta_{jl}(x') \cdot \beta_{lk}(x'), k)] = [(x', \beta_{ki}(x') \triangleright g \triangleleft \beta_{jk}(x'), k)]\\
=\;& [(x', \beta_{ki}(x') \triangleright \big(g \triangleleft \beta_{jk}(x') \cdot \beta_{ki}(x')\big) \triangleleft \beta_{ki}(x')^{-1}, k)] = [(x', g \triangleleft \beta_{jk}(x') \cdot \beta_{ki}(x'), i)]\\
=\;& [(x', g \triangleleft \beta_{ji}(x'), i)]\,,
\end{aligned}$$

where in the second, third and fourth lines, we have used commutativity of the left and right actions of $\mathbb{B}$ on $\mathscr{G}$, alongside the 1-cocycle condition satisfied by the $\beta_{ij}(x')$. □

Remark 5. The π-fibres of $\mathrm{At}(\mathscr{P})$ are isomorphic to the Lie groupoid $\mathscr{G}$. We can thus read the exact sequence (12) also as saying that $(\mathrm{At}(\mathscr{P}), \mathrm{Pair}(B), \mathscr{G}, \pi)$ is a fibre-bundle object in the category of Lie groupoids:

$$\begin{tikzcd}
\mathscr{G} \arrow[r, hook, dotted] & \mathrm{At}(\mathscr{P}) \arrow[d, "\pi"] \\
& \mathrm{Pair}(B)
\end{tikzcd}\,.$$

This is a general feature noted already in the classic setting, on the special examples captured by Diags. VI.9. and VII-VIII-IX.11.

Our considerations are crowned by

Theorem 4. The Atiyah–Ehresmann groupoid acts on the principaloid bundle and its shadow in the following way:

- On every principaloid bundle $\mathscr{P}$, there exists a canonical structure of a left $\mathrm{At}(\mathscr{P})$-module, with momentum $\mu_{\mathscr{P}} = \mathscr{D} \;:\; \mathscr{P} \longrightarrow \mathscr{F}$ and action

$$\lambda_{\mathscr{P}} \;:\; \mathrm{At}(\mathscr{P})\,{}_{\mathsf{S}}\!\times_{\mathscr{D}} \mathscr{P} \longrightarrow \mathscr{P} \;:\; \big([(x_1, g, x_2, i, j)], [(x_2, h, k)]\big) \longmapsto [(x_1, g.(\beta_{jk}(x_2) \triangleright h), i)]\,, \tag{18}$$

written in the local models: (14) for $\mathrm{At}(\mathscr{P})$ and (1) for $\mathscr{P}$, for $x_1 \in O_i$ and $x_2 \in O_{jk}$.

- This structure covers the canonical structure of a left $\mathrm{Pair}(B)$-module on B, with momentum $\mu_B \equiv \mathrm{id}_B$ and action

$$\lambda_B \;:\; (B \times B)\,{}_{\mathrm{pr}_2}\!\times_{\mathrm{id}_B} B \longrightarrow B\,,\ \big((x_1, x_2), x_2\big) \longmapsto x_1\,.$$

- On the corresponding shadow bundle $\mathscr{F}$, there exists a canonical structure of a left $\mathrm{At}(\mathscr{P})$-module, with momentum $\mu_{\mathscr{F}} = \mathrm{id}_{\mathscr{F}}$ and action

$$\lambda_{\mathscr{F}} \;:\; \mathrm{At}(\mathscr{P})_{\mathsf{S}}\!\times_{\mathrm{id}_{\mathscr{F}}} \mathscr{F} \longrightarrow \mathscr{F}\,,\ \big([(x_1, g, x_2, i, j)], [(x_2, m, k)]\big) \longmapsto [(x_1, t(g), i)]\,. \tag{19}$$

- The action $\lambda_{\mathscr{P}}$ is intertwined with $\lambda_{\mathscr{F}}$ by the sitting-duck map $\mathscr{D}$, as reflected in the identities

$$\mu_{\mathscr{P}} = \mu_{\mathscr{F}} \circ \mathscr{D}\,, \qquad\qquad \lambda_{\mathscr{F}} \circ \big(\mathrm{id}_{\mathrm{At}(\mathscr{P})} \times \mathscr{D}\big) = \mathscr{D} \circ \lambda_{\mathscr{P}}\,. \tag{20}$$

Proof: Let us first analyse the triple $(\mathscr{P}, \mu_{\mathscr{P}}, \lambda_{\mathscr{P}})$. Composability of the two arrows: g and $\beta_{jk}(x_2) \rhd h$ in Eq. (18) is ensured by the identity

$$[(x_2, s(g), j)] \equiv \mathsf{S}\big([(x_1, g, x_2, i, j)]\big) = \mathscr{D}\big([(x_2, h, k)]\big) \equiv [(x_2, t(h), k)] = [(x_2, t_*(\beta_{jk}(x_2))(t(h)), j)]\,.$$

Using the 1-cocycle condition for the β_{ij} alongside identities (29), we readily establish, for any $x_1 \in O_{il}$ and $x_2 \in O_{jkmn}$, the identity

$$\begin{aligned}
&\lambda_{\mathscr{P}}\big([(x_1, \beta_{li}(x_1) \rhd g \lhd \beta_{jm}(x_2), x_2, l, m)], [(x_2, \beta_{nk}(x_2) \rhd h, n)]\big)\\
\equiv\;& [(x_1, (\beta_{li}(x_1) \rhd g \lhd \beta_{jm}(x_2)).(\beta_{mn}(x_2) \rhd (\beta_{nk}(x_2) \rhd h)), l)]\\
\equiv\;& [(x_1, (\beta_{li}(x_1) \rhd g \lhd \beta_{jm}(x_2)).(\beta_{mk}(x_2) \rhd h), l)]\\
=\;& [(x_1, (\beta_{li}(x_1) \rhd g).(\beta_{jm}(x_2) \rhd (\beta_{mk}(x_2) \rhd h)), l)] = [(x_1, (\beta_{li}(x_1) \rhd g).(\beta_{jk}(x_2) \rhd h), l)]\\
=\;& [(x_1, \beta_{li}(x_1) \rhd (g.(\beta_{jk}(x_2) \rhd h)), l)] = [(x_1, g.(\beta_{jk}(x_2) \rhd h), i)]\,,
\end{aligned}$$

which demonstrates that $\lambda_{\mathscr{P}}$ is a well-defined globally smooth action map sought after. Its constitutive properties are readily verified in the following calculations: Axiom (GlM1) of Def. IV.75. is checked in

$$\begin{aligned}
&(\mu_{\mathscr{P}} \circ \lambda_{\mathscr{P}})\big([(x_1, g, x_2, i, j)], [(x_2, h, k)]\big) \equiv \mathscr{D}\big([(x_1, g.(\beta_{jk}(x_2) \rhd h), i)]\big)\\
=\;& [(x_1, t(g.(\beta_{jk}(x_2) \rhd h)), i)] = [(x_1, t(g), i)] \equiv \mathsf{T}\big([(x_1, g, x_2, i, j)]\big)\,.
\end{aligned}$$

Axiom (GlM2) is seen to hold true in virtue of

$$\begin{aligned}
&\lambda_{\mathscr{P}}\big(\mathsf{I} \circ \mu_{\mathscr{P}}\big([(x, g, i)]\big), [(x, g, i)]\big) \equiv \lambda_{\mathscr{P}}\big([(x, \mathrm{Id}_{t(g)}, x, i, i)], [(x, g, i)]\big) \equiv [(x, \mathrm{Id}_{t(g)}.(\beta_{ii}(x) \rhd g), i)]\\
=\;& [(x, g, i)]\,.
\end{aligned}$$

Axiom (GlM3) is established, with the help of the first of identities (28), in

$$\begin{aligned}
&\lambda_{\mathscr{P}}\big([(x_1, g_1, x_2, i, j)], \lambda_{\mathscr{P}}\big([(x_2, g_2, x_3, k, l)], [(x_3, h, m)]\big)\big)\\
=\;& \lambda_{\mathscr{P}}\big([(x_1, g_1, x_2, i, j)], [(x_2, g_2.(\beta_{lm}(x_3) \rhd h), k)]\big) = [(x_1, g_1.(\beta_{jk}(x_2) \rhd (g_2.(\beta_{lm}(x_3) \rhd h))), i)]\\
=\;& [(x_1, (g_1.(\beta_{jk}(x_2) \rhd g_2)).(\beta_{lm}(x_3) \rhd h))), i)] \equiv \lambda_{\mathscr{P}}\big([(x_1, g_1.(\beta_{jk}(x_2) \rhd g_2), x_3, i, l)], [(x_3, h, m)]\big)\\
\equiv\;& \lambda_{\mathscr{P}}\big(\mathsf{M}\big([(x_1, g_1, x_2, i, j)], [(x_2, g_2, x_3, k, l)]\big), [(x_3, h, m)]\big)\,.
\end{aligned}$$

Projectability of the above structure of a left $\mathrm{At}(\mathscr{P})$-module on $\mathscr{P}$ onto the canonical structure of a left $\mathrm{Pair}(B)$-module on B (along $(\pi, \pi_{\mathscr{F}})$) is self-evident.

Passing to the triple $(\mathscr{F}, \mu_{\mathscr{F}}, \lambda_{\mathscr{F}})$, we note well-definedness of $\lambda_{\mathscr{F}}$ and manifest projectability of the entire structure onto $(B, \mathrm{id}_B, \lambda_B)$. The first of identities (20) is trivially satisfied, and so we are left with the second one to prove. First of all, note the obvious identity

$$\big(\mathrm{id}_{\mathrm{At}(\mathscr{P})} \times \mathscr{D}\big)\big(\mathrm{At}(\mathscr{P})_{\mathsf{S}}\times_{\mathscr{D}}\mathscr{P}\big) \equiv \mathrm{At}(\mathscr{P})_{\mathsf{S}}\times_{\mathrm{id}_{\mathscr{F}}}\mathscr{F}\,,$$

which ensures meaningfulness of that $\mathrm{At}(\mathscr{P})$-equivariance condition. The latter is checked in a direct computation:

$$\begin{aligned} &\big(\lambda_{\mathscr{F}} \circ \big(\mathrm{id}_{\mathrm{At}(\mathscr{P})} \times \mathscr{D}\big)\big)\big([(x_1, g, x_2, i, j)], [(x_2, h, k)]\big) \equiv \lambda_{\mathscr{F}}\big([(x_1, g, x_2, i, j)], [(x_2, t(h), k)]\big) \\ \equiv\ \ & [(x_1, t(g), i)] = [(x_1, t(g.(\beta_{jk}(x_2) \triangleright h)), i)] \equiv \mathscr{D}\big([(x_1, g.(\beta_{jk}(x_2) \triangleright h), i)]\big) \\ \equiv\ \ & \big(\mathscr{D} \circ \lambda_{\mathscr{P}}\big)\big([(x_1, g, x_2, i, j)], [(x_2, h, k)]\big)\,. \end{aligned}$$

$\square$

Remark 6. Note that the independence of $\lambda_{\mathscr{P}}$ of the choice of local charts permits us to simplify Eq. (18) as

$$\lambda_{\mathscr{P}}\ :\ \mathrm{At}(\mathscr{P})_{\mathsf{S}}\times_{\mathscr{D}}\mathscr{P} \longrightarrow \mathscr{P},\ \big([(x_1, g, x_2, i, j)], [(x_2, h, j)]\big) \longmapsto [(x_1, g.h, i)]\,. \tag{21}$$

Our findings are most concisely expressed in

Theorem 5. For every principaloid bundle $\mathscr{P}$, the quintuple $(\mathrm{At}(\mathscr{P}), \mathscr{P}, \mathscr{G}; B, \mathscr{G})$ is a trident (in the sense of Def. VII-VIII-IX.3.), captured by the following diagram

$$\begin{tikzcd} \mathscr{G} \arrow[dr, shift left] \arrow[dr, shift right] \arrow[dd, hook, dotted] & & \mathscr{G} \arrow[dl, "t"'] \arrow[dd, hook', dotted] & & \\ & M \arrow[dd, hook, dotted] & & & \\ \mathrm{At}(\mathscr{P}) \arrow[dr, shift left] \arrow[dr, shift right] \arrow[dd, "\pi"'] & & \mathscr{P} \arrow[dl, "\mathscr{D}"'] \arrow[dr, "\mu"] \arrow[dd, "\pi_{\mathscr{P}}"] & & \mathscr{G} \arrow[dl, shift left] \arrow[dl, shift right] \\ & \mathscr{F} \arrow[dd, "\pi_{\mathscr{F}}"'] & & M & \\ B \times B \arrow[dr, shift left] \arrow[dr, shift right] & & B \arrow[dl, "\mathrm{id}_B"] & & \\ & B & & & \end{tikzcd} \tag{22}$$

<u>*Proof:*</u> The structure of a right principal $\mathscr{G}$-module on $(\mathscr{P}, \mathscr{F}, \mathscr{D})$ was established in Thm. 1. In particular, the corresponding right $\mathscr{G}$-action was shown to preserve $\pi_{\mathscr{P}}$-fibres. Furthermore, the identification of $\mathrm{At}(\mathscr{P})$ as a fibre-bundle object in the category of Lie groupoids, with base $\mathrm{Pair}(\Sigma)$ and typical fibre $\mathscr{G}$, was made in Rem. 5. Therefore, it remains to investigate the left action of $\mathrm{At}(\mathscr{P})$ on $\mathscr{P}$, as identified in Thm. 4.

We begin by noting that $\lambda_{\mathscr{P}}\ :\ \mathrm{At}(\mathscr{P})_{\mathsf{S}}\times_{\mathscr{D}}\mathscr{P} \longrightarrow \mathscr{P}$ preserves μ-fibres. Indeed, the action restricts to fibres of $\mathscr{P}$ as the *left*-multiplication, and so leaves the moment map μ, locally modelled on s, invariant.

The action gives rise to a smooth map

$$(\lambda_{\mathscr{P}}, \mathrm{pr}_2)\ :\ \mathrm{At}(\mathscr{P})_{\mathsf{S}}\times_{\mathscr{D}}\mathscr{P} \longrightarrow \mathscr{P}\,{}_{\mu}\times_{\mu}\mathscr{P}\,,$$

whose well-definedness is ensured by the following equality (see Eq. (21)):

$$\mu\big([(x_1, g.h, i)]\big) \equiv s(g.h) = s(h) \equiv \mu\big([(x_2, h, j)]\big).$$

Consider the division map

$$\psi_{\mathscr{P}} \;:\; \mathscr{P}_\mu{\times}_\mu\mathscr{P} \longrightarrow \mathrm{At}(\mathscr{P}),\ \big([(x_1, g, i)], [(x_2, h, j)]\big) \longmapsto [(x_1, g.h^{-1}, x_2, i, j)].$$

It is well-defined since $s(g) \equiv \mu([(x_1, g, i)]) = \mu([(x_2, h, j)]) \equiv s(h) = t(h^{-1})$. It now remains to verify the identity

$$(\psi_{\mathscr{P}}, \mathrm{pr}_2) = (\lambda_{\mathscr{P}}, \mathrm{pr}_2)^{-1}. \tag{23}$$

We have

$$\mathsf{S}\big([(x_1, g.h^{-1}, x_2, i, j)]\big) \equiv [x_2, s(g.h^{-1}), j] = [x_2, t(h), j] \equiv \mathscr{D}\big([(x_2, h, j)]\big),$$

so that $(\psi_{\mathscr{P}}, \mathrm{pr}_2)$ maps $\mathscr{P}_\mu{\times}_\mu\mathscr{P}$ to $\mathrm{At}(\mathscr{P})_{\mathsf{S}}{\times}_{\mathscr{D}}\mathscr{P}$. Identity (23) can now be checked in a direct calculation. This establishes $\lambda_{\mathscr{P}}$ as a principal action.

That $\lambda_{\mathscr{P}}$ covers the canonical left action of $\mathrm{Pair}(B)$ on B is part of Thm. 4. □

Remark 7. In the approach to the theory of principal G-bundles due to Ehresmann, we encounter triples $(\mathsf{P}, \mathrm{G}, \mathrm{At}(\mathsf{P}))$ consisting of a principal G-bundle P, its structure *group* G, and its structure *groupoid* $\mathrm{At}(\mathsf{P})$, which we introduced in Thm. VI.2. Already in this classic setting, one can generalise the structure as we did in Thm. VII-VIII-IX.3. In the above ultimate generalisation, the right wing of the W-diagram is a general groupoid. This aligns with an idea of Pradines[3] [Pra77] (see also [Pra07]), whose goal was to symmetrise the structure as in Def. VI.6, except that here M_1 cannot remain to be Σ, but it is to be replaced, in an essential way, by the bundle $\mathscr{F}$, whose fibre is $M_2 = M$. Thus, in distinction to Pradines, there emerges the fully fledged Trident Diagram (22).

4. The three-floor groupoidal Atiyah sequence

Appendix A. Useful properties of Lie groupoids, bisections, and Lie algebroids

Proposition 6. The left- and right-multiplications of $\mathscr{G}$ by $\mathbb{B}$ from Def. II-III.41. and the shadow action of $\mathbb{B}$ on M from Def. II-III.40. have the following properties relative to the structure maps of $\mathscr{G}$ (written for arbitrary $(\beta, g) \in \mathbb{B} \times \mathscr{G}$):

- (i) $s \circ L_\beta = s$, $\quad s \circ R_\beta = t_*(\beta^{-1}) \circ s$ (s intertwines L with the trivial action and R with $t_* \circ \mathrm{Inv}$);
- (ii) $t \circ L_\beta = t_*\beta \circ t$, $\quad t \circ R_\beta = t$ (t intertwines L with t_* and R with the trivial action);
- (iii) $L_\beta \circ \mathrm{Id} = \beta$, $\quad R_\beta \circ \mathrm{Id} = \beta \circ t_*\big(\beta^{-1}\big)$;
- (iv) $\mathrm{Inv} \circ L_\beta = R_{\beta^{-1}} \circ \mathrm{Inv}$, $\quad \mathrm{Inv} \circ R_\beta = L_{\beta^{-1}} \circ \mathrm{Inv}$ (Inv intertwines L with $R \circ \mathrm{Inv}$);
- (v) $L_\beta \circ r_g = r_g \circ L_\beta$, $\quad R_\beta \circ l_g = l_g \circ R_\beta$;
- (vi) $r_g \circ R_\beta = r_{L_\beta(g)}$, $\quad l_g \circ L_\beta = l_{R_\beta(g)}$.

Proof:

Ad (i) The first identity is trivial. For the second one, we compute explicitly, for any $h \in \mathscr{G}$,

$$\begin{aligned} s \circ R_\beta(h) &\equiv s\big(h.\big(\beta^{-1}\big(s(h)\big)\big)^{-1}\big) = t\big(\beta^{-1}\big(s(h)\big)\big) \equiv t\big(\big(\beta \circ (t_*\beta)^{-1}\big(s(h)\big)\big)^{-1}\big) = s \circ \beta \circ (t_*\beta)^{-1}\big(s(h)\big) \\ &= (t_*\beta)^{-1} \circ s(h) = t_*\big(\beta^{-1}\big) \circ s(h). \end{aligned}$$

where the last equality follows from the homomorphicity of t_*.

Ad (ii) The second identity is trivial. For the first one, we compute explicitly, for any $h \in \mathscr{G}$,

$$t \circ L_\beta(h) \equiv t\big(\beta\big(t(h)\big).h\big) = t \circ \beta\big(t(h)\big) \equiv t_*\beta \circ t(h).$$

[3] The idea has resurfaced in various incarnations in the study of foliations and generalised (Morita) morphisms of groupoids, in particular in the works of Hilsum and Skandalis [HS83, HS87] (leading to the related notion of Hilsum–Skandalis maps), and Haefliger [Hae84].

Ad (iii) We find, for an arbitrary $m \in M$,

$$L_\beta \circ \mathrm{Id}(m) \equiv \beta\big(t(\mathrm{Id}_m)\big).\mathrm{Id}_m = \beta(m)$$

and

$$R_\beta \circ \mathrm{Id}(m) \equiv \mathrm{Id}_m.\big(\beta^{-1}\big(s(\mathrm{Id}_m)\big)\big)^{-1} = \big(\beta^{-1}(m)\big)^{-1} \equiv \beta \circ (t_*\beta)^{-1}(m) = \beta \circ t_*\big(\beta^{-1}\big)(m)\,.$$

Ad (iv) For the first identity, we compute explicitly, for any $h \in \mathscr{G}$,

$$\mathrm{Inv} \circ L_\beta(h) \equiv \big(\beta\big(t(h)\big).h\big)^{-1} = h^{-1}.\beta\big(t(h)\big)^{-1} \equiv h^{-1}.\big(\beta^{-1}\big)^{-1}\big(s\big(h^{-1}\big)\big)^{-1} \equiv R_{\beta^{-1}} \circ \mathrm{Inv}(h)\,.$$

The second identity now follows by replacing β with β^{-1} in the one just proved, and subsequently sandwiching both sides of it between two copies of Inv.

Ad (v) Take an arbitrary arrow $h \in s^{-1}(\{t(g)\})$ and calculate directly:

$$L_\beta \circ r_g(h) \equiv \beta\big(t(h.g)\big).(h.g) = \beta\big(t(h)\big).h.g \equiv r_g\big(\beta\big(t(h)\big).h\big) \equiv r_g \circ L_\beta(h)\,.$$

The proof of the second identity is fully analogous.

Ad (vi) For the first identity, take an arbitrary arrow $h \in s^{-1}(t_*\beta(t(g)))$ and calculate directly

$$r_g \circ R_\beta(h) \equiv \big(h.\big(\beta^{-1}\big(s(h)\big)\big)^{-1}\big).g = h.\big(\beta^{-1}\big(t_*\beta\big(t(g)\big)\big)\big)^{-1}.g \equiv h.\beta\big(t(g)\big).g \equiv h.\big(L_\beta(g)\big) \equiv r_{L_\beta(g)}(h)\,.$$

The second identity follows by replacing β with β^{-1} and g with g^{-1} in the one just proved, and subsequently using (iv).

□

Remark 8. Upon evaluation on the respective arguments $h \in \mathscr{G},\ m \in M,\ u \in s^{-1}(\{t(g)\}),\ v \in t^{-1}(\{s(g)\}),\ w \in s^{-1}(t_*\beta(t(g))),\ y \in t^{-1}((t_*\beta)^{-1}(s(g)))$ and in the shorthand notation $L_\beta(h) \equiv \beta \triangleright h,\ R_\beta(h) \equiv h \triangleleft \beta$ and $t_*\beta(m) \equiv \beta \,\underline{\triangleright}\, m$, the functional identities of Prop. 6 take the following form:

$$s(\beta \triangleright h) = s(h)\,, \qquad s(h \triangleleft \beta) = \beta^{-1} \,\underline{\triangleright}\, \big(s(h)\big)\,, \tag{24}$$

$$t(\beta \triangleright h) = \beta \,\underline{\triangleright}\, t(h)\,, \qquad t(h \triangleleft \beta) = t(h)\,, \tag{25}$$

$$\beta \triangleright \mathrm{Id}_m = \beta(m)\,, \qquad \mathrm{Id}_m \triangleleft \beta = \beta\big(\beta^{-1} \,\underline{\triangleright}\, m\big)\,, \tag{26}$$

$$(\beta \triangleright h)^{-1} = h^{-1} \triangleleft \beta^{-1}\,, \qquad (h \triangleleft \beta)^{-1} = \beta^{-1} \triangleright h^{-1}\,, \tag{27}$$

$$\beta \triangleright (u.g) = (\beta \triangleright u).g\,, \qquad (g.v) \triangleleft \beta = g.(v \triangleleft \beta)\,, \tag{28}$$

$$(w \triangleleft \beta).g = w.(\beta \triangleright g)\,, \qquad g.(\beta \triangleright y) = (g \triangleleft \beta).y\,. \tag{29}$$

Corollary 3. The conjugation of $\mathscr{G}$ by $\mathbb{B}$ from Def. II-III.41. has the following properties relative to the structure maps of $\mathscr{G}$ (written for arbitrary $(\beta, g) \in \mathbb{B} \times \mathscr{G}$):

(i) $s \circ C_\beta = t_*\beta \circ s$ (s intertwines C with t_*);
(ii) $t \circ C_\beta = t_*\beta \circ t$ (t intertwines C with t_*);
(iii) $C_\beta \circ \mathrm{Id} = \mathrm{Id} \circ t_*\beta$ (Id intertwines C with t_*);
(iv) $\mathrm{Inv} \circ C_\beta = C_\beta \circ \mathrm{Inv}$ (Inv intertwines C with itself);
(v) $C_\beta \circ \mathrm{m} = \mathrm{m} \circ (C_\beta \times C_\beta)$ (C distributes over m).

Proof of Corollary:

Ad (i) Point (i) of Prop. 6 implies $s \circ C_\beta = s \circ R_{\beta^{-1}} = t_*\beta \circ s$.
Ad (ii) Point (ii) of Prop. 6 implies $t \circ C_\beta = t \circ L_\beta = t_*\beta \circ t$.
Ad (iii) Point (iii) of Prop. 6 implies $C_\beta \circ \mathrm{Id} \equiv R_{\beta^{-1}} \circ (L_\beta \circ \mathrm{Id}) = R_{\beta^{-1}} \circ \beta = R_{\beta^{-1}} \circ (R_\beta \circ \mathrm{Id} \circ t_*\beta) = R_{\beta.\beta^{-1}} \circ \mathrm{Id} \circ t_*\beta = R_{\mathrm{Id}} \circ \mathrm{Id} \circ t_*\beta = \mathrm{Id} \circ t_*\beta$.

Ad (iv) Point (iv) of Prop. 6 implies $\mathrm{Inv} \circ C_\beta \equiv (\mathrm{Inv} \circ L_\beta) \circ R_{\beta^{-1}} = R_{\beta^{-1}} \circ (\mathrm{Inv} \circ R_{\beta^{-1}}) = (R_{\beta^{-1}} \circ L_\beta) \circ \mathrm{Inv} \equiv C_\beta \circ \mathrm{Inv}$.

Ad (v) We calculate directly, in the notation of Rem. 8 and using identities (28) and (29) along the way,

$$C_\beta \circ \mathrm{m}(g_2, g_1) \equiv \big(\beta \triangleright (g_2.g_1)\big) \triangleleft \beta^{-1} = \big((\beta \triangleright g_2).g_1\big) \triangleleft \beta^{-1} = \Big(\big((\beta \triangleright g_2) \triangleleft \beta^{-1}\big).(\beta \triangleright g_1)\Big) \triangleleft \beta^{-1}$$

$$\equiv \quad \big((\beta \triangleright g_2) \triangleleft \beta^{-1}\big).\big(((\beta \triangleright g_1) \triangleleft \beta^{-1}\big) \equiv C_\beta(g_2).C_\beta(g_1) \equiv \mathrm{m} \circ (C_\beta \times C_\beta)(g_2, g_1)\,.$$

■

Proposition 7. Let (X, μ, ϱ) be a right-$\mathscr{G}$-module. The action ϱ canonically induces a (right) action $\mathbb{B}\varrho \;:\; X \times \mathbb{B} \longrightarrow X$ of $\mathbb{B}$ on X, as determined by the following commutative diagram:

$$\begin{tikzcd} X\,{}_\mu\!\times_{\mathrm{id}_M} M \times \mathbb{B} \arrow[r, "\mathrm{id}_X \times \mathrm{Inv}\circ\mathrm{ev}"] & X\,{}_\mu\!\times_t \mathscr{G} \arrow[d, "\varrho"] \\ X \times \mathbb{B} \arrow[u, "(\mathrm{id}_X,\mu)\times\mathrm{Inv}"] \arrow[r, "\mathbb{B}\varrho"'] & X \end{tikzcd}\;,$$

where $\mathrm{ev} \;:\; M \times \mathbb{B} \longrightarrow \mathscr{G},\; (m, \beta) \longmapsto \beta(m)$ is the canonical **evaluation map**.

An analogous statement holds true for left-$\mathscr{G}$-modules.

Proof: The diagram yields the map

$$\mathbb{B}\varrho \;:\; X \times \mathbb{B} \longrightarrow X \;:\; (x, \beta) \longmapsto x \blacktriangleleft \beta^{-1}\big(\mu(x)\big)^{-1}\,,$$

whose well-definedness follows from the identity

$$t\big(\beta^{-1}\big(\mu(x)\big)^{-1}\big) \equiv t\Big(\beta\Big((t_*\beta)^{-1}\big(\mu(x)\big)\Big)\Big) = \mu(x)\,.$$

It now suffices to check the axioms of a group action, invoking those of the groupoid module along the way. We have—for $x \in X$—

$$\mathbb{B}\varrho(x, \mathrm{Id}) \equiv x \blacktriangleleft \big(\mathrm{Id}^{-1}_{\mu(x)}\big)^{-1} = x \blacktriangleleft \mathrm{Id}_{\mu(x)} = x$$

(by (GrM2)), and—for $\beta_1, \beta_2 \in \mathbb{B}$—

$$\mathbb{B}\varrho\big(\mathbb{B}\varrho(x, \beta_1), \beta_2\big) \equiv \big(x \blacktriangleleft \beta_1^{-1}\big(\mu(x)\big)^{-1}\big) \blacktriangleleft \beta_2^{-1}\Big(\mu\big(x \blacktriangleleft \beta_1^{-1}\big(\mu(x)\big)^{-1}\big)\Big)^{-1}$$

$$= \quad \big(x \blacktriangleleft \beta_1^{-1}\big(\mu(x)\big)^{-1}\big) \blacktriangleleft \beta_2^{-1}\Big(s\big(\beta_1^{-1}\big(\mu(x)\big)^{-1}\big)\Big)^{-1} = \big(x \blacktriangleleft \beta_1^{-1}\big(\mu(x)\big)^{-1}\big) \blacktriangleleft \beta_2^{-1}\Big(\big(t_*\beta_1^{-1}\big)\big(\mu(x)\big)\Big)^{-1}$$

$$= \quad x \blacktriangleleft \Big(\beta_2^{-1}\Big(\big(t_*\beta_1^{-1}\big)\big(\mu(x)\big)\Big).\beta_1^{-1}\big(\mu(x)\big)\Big)^{-1} \equiv x \blacktriangleleft \big(\beta_2^{-1} \cdot \beta_1^{-1}\big)\big(\mu(x)\big)^{-1} = x \blacktriangleleft (\beta_1 \cdot \beta_2)^{-1}\big(\mu(x)\big)^{-1}$$

$$\equiv \quad \mathbb{B}\varrho\big(x, \beta_1 \cdot \beta_2\big)\,.$$

□

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