Duality, Descent & Defects I

LECTURES II & III

"GROUPOIDS in ACTION"

2024/25


A LATINO PAIR GROUPOID — IN ACTION

GROUPOID'S ANATOMY:

ON LIE GROUPS, LOCAL & GLOBAL DIFFERENTIAL STRUCTURE IS ENCODED BY THE LEFT RESP. RIGHT REGULAR ACTION ℓ. RESP. ρ. ON LIE GROUPOIDS, THINGS GET SUBTLER AS THEY ACT ON THEMSELVES ONLY FIBREWISE...

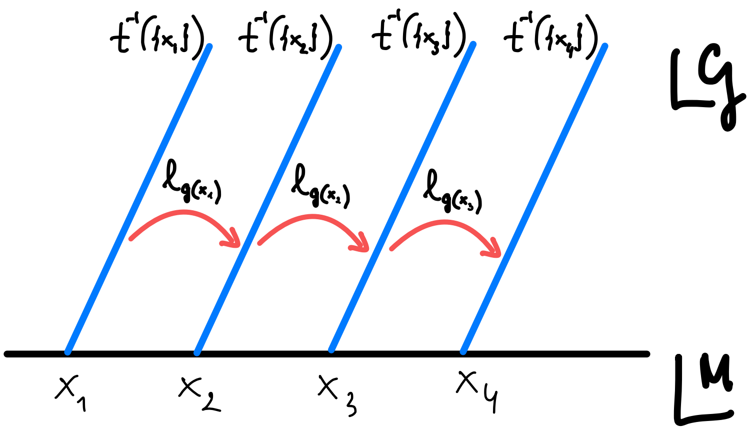
DEF. 31. FOR ANY LIE GROUPOID $(M, \mathcal{G}, s, t, Id, Inv, m)$, & FOR ANY $g \in s^{-1}(\{x\}) \cap t^{-1}(\{y\})$, THE LEFT-TRANSLATION BY g IS THE SMOOTH MAP

$$\ell_g : t^{-1}(\{x\}) \longrightarrow t^{-1}(\{y\}) : h \longmapsto g.h .$$

THE RIGHT-TRANSLATION BY g IS THE SMOOTH MAP

$$\rho_g : s^{-1}(\{y\}) \longrightarrow s^{-1}(\{x\}) : h \longmapsto h.g .$$

IN PHYSICAL APPLICATIONS, EXISTENCE of A FIBREWISE ACTION IS NOT GOOD ENOUGH, & SO WE LOOK for GENERALISATIONS of, SAY, ℓ....



REPLACE: $g \longmapsto g(\cdot) : t(\mathcal{G}) \equiv M \to \mathcal{G}$

with $s(g(x)) = t(t^{-1}(\{x\})) \equiv x$

THE REQUIREMENT THAT SUCH AN ACTION BE A DIFFEOMORPHISM IMPLIES THAT ITS RESTRICTION TO **M** HAS THIS PROPERTY, i.e., $t \circ g \overset{!}{\in} \mathrm{Diff}(M)$. THIS LEADS to...

Definition 32 ([MMr03]). *Let* $\mathbf{Gr} = (M, \mathscr{G}, s, t, \mathrm{Id}, \mathrm{Inv}, .)$ *be a Lie groupoid. A* (**global**) **bisection of Gr** *is a section* $\sigma : M \longrightarrow \mathscr{G}$ *of the surjective submersion* $s : \mathscr{G} \longrightarrow M$ *such that the induced map*

$$t_*\sigma \equiv t \circ \sigma : M \longrightarrow M$$

is a diffeomorphism. Equivalently, it is a submanifold $S \subset \mathscr{G}$ *with the property that both restrictions:* $s|_S$ *and* $t|_S$ *are diffeomorphisms. We shall denote the set of bisections as* $\mathrm{Bisec}(\mathbf{Gr})$.

A **local bisection of Gr** *is a local section* $\sigma : O \longrightarrow \mathscr{G}$ *of* s *over an open subset* $O \subset M$ *such that the induced map*

$$t_*\sigma \equiv t \circ \sigma : O \longrightarrow t \circ \sigma(O)$$

is a diffeomorphism. We shall denote the set of local bisections as $\mathrm{Bisec}_{\mathrm{loc}}(\mathbf{Gr})$.

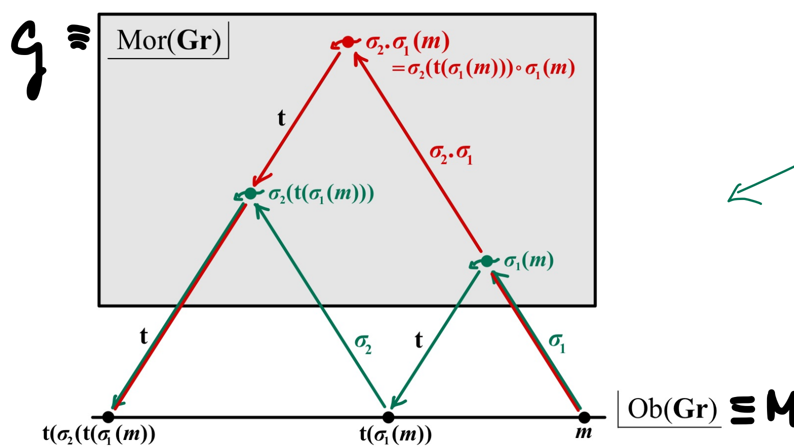
≡ B

Definition 33. *The* **group of bisections of Gr** *is the canonical structure of a group on* $\mathrm{Bisec}(\mathbf{Gr})$. *Its binary operation is defined as*

$$\cdot : \mathrm{Bisec}(\mathbf{Gr}) \times \mathrm{Bisec}(\mathbf{Gr}) \longrightarrow \mathrm{Bisec}(\mathbf{Gr}) : (\sigma_2, \sigma_1) \longmapsto \sigma_2\big(t \circ \sigma_1(\cdot)\big).\sigma_1(\cdot) \equiv \sigma_2 \cdot \sigma_1 .$$

The neutral element is Id, *termed the* **unit bisection** *in the present context, and the corresponding inverse is*

$$\mathrm{Inv} : \mathrm{Bisec}(\mathbf{Gr}) \longrightarrow \mathrm{Bisec}(\mathbf{Gr}) : \beta \longmapsto \mathrm{Inv} \circ \beta \circ (t_*\beta)^{-1} \equiv \beta^{-1} .$$



GEOMETRIC PICTURE

E.g.,

Ex. 34. FOR $\mathrm{Gr} \equiv \hat{G}$, WE FIND $\mathrm{Bisec}(\hat{G}) \simeq G$.

Ex. 35. FOR $\mathrm{Gr} \equiv \mathrm{Pair}(M)$, WE FIND $\mathrm{Bisec}(\mathrm{Pair}(M)) \simeq \mathrm{Diff}(M)$.

Ex. 36. FOR $\mathrm{Gr} \equiv \mathrm{Pair}_\Sigma(M)$, WE FIND

$$\mathrm{Bisec}(\mathrm{Pair}_\Sigma(M)) \simeq \mathrm{Aut}_{\mathrm{Bun}(\Sigma)}(M \mid \mathrm{id}_\Sigma) =: \mathrm{Aut}_{\mathrm{Bun}(\Sigma)}(M)_{\mathrm{vert.}}$$

Ex. 37. FOR $\mathrm{Gr} \equiv G \ltimes_\lambda M$, WE FIND

$$\mathrm{Bisec}(G \ltimes_\lambda M) \simeq \{ f : M \to G \mid (m \mapsto \lambda_{f(m)}(m)) \in \mathrm{Diff}(M) \}$$

Ex. 38. FOR $\mathrm{Gr} \equiv \hat{M}$, WE FIND $\mathrm{Bisec}(\hat{M}) = \{\mathrm{id}_M\} \simeq 1$.

$\mathbb{B}$-ACTIONS :

Definition 39. *The* **left regular action of $\mathbb{B}$** *(***on itself***) is*

$$\ell : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B} : (\gamma, \beta) \longmapsto \gamma \cdot \beta \equiv \ell_\gamma(\beta) ,$$

and the **right regular action of $\mathbb{B}$** *(***on itself***) is*

ON ITSELF

$$\wp : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B} : (\beta, \gamma) \longmapsto \beta \cdot \gamma \equiv \wp_\gamma(\beta) .$$

The **adjoint action of $\mathbb{B}$ on itself** *is*

$$c : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B} : (\gamma, \beta) \longmapsto \gamma \cdot \beta \cdot \vec{\gamma}^{-1} \equiv c_\gamma(\beta) .$$

Definition 40. *The* **shadow action of $\mathbb{B}$ on M** *is*

$$t_* : \mathbb{B} \times M \longrightarrow M : (\sigma, m) \longmapsto t\big(\sigma(m)\big) .$$

By the usual abuse of the notation, we shall refer by the same name and use the same symbol for the group homomorphism

ON M

$$t_* : \mathbb{B} \longrightarrow \mathrm{Diff}(M) .$$

Definition 41. *The* **left-multiplication of $\mathscr{G}$ by $\mathbb{B}$** *is the left action*

$$L : \mathbb{B} \times \mathscr{G} \longrightarrow \mathscr{G} : (\sigma, g) \longmapsto \sigma\big(t(g)\big).g \equiv L_\sigma(g) \equiv \sigma \rhd g .$$

The **right-multiplication of $\mathscr{G}$ by $\mathbb{B}$** *is the right action*

ON $\mathcal{G}$

$$R : \mathscr{G} \times \mathbb{B} \longrightarrow \mathscr{G} : (g, \sigma) \longmapsto g.\big(\sigma^{-1}\big(s(g)\big)\big)^{-1} \equiv R_\sigma(g) \equiv g \lhd \sigma .$$

The **conjugation of $\mathscr{G}$ by $\mathbb{B}$** *is the left action*

$$C : \mathbb{B} \times \mathscr{G} \longrightarrow \mathscr{G} : (\sigma, g) \longmapsto \sigma\big(t(g)\big).g.\sigma\big(s(g)\big)^{-1} \equiv C_\sigma(g) \equiv \sigma \rhd g \lhd \sigma^{-1} .$$

IT IS NOT HARD TO SEE THAT GENERICALLY THERE EXIST ARROWS WITH NO GLOBAL BISECTIONS through THEM (SEE: Rem. 45.). HENCE,

Definition 42. *A Lie groupoid $\mathscr{G}$ is called* **Id-reducible** *if for each $g \in \mathscr{G}$ there exists $\beta \in \mathbb{B}$ such that $g = \beta(s(g))$, i.e., if there is a global bisection through every arrow.*

Remark 43. *The name is justified by the following simple observation: The condition $g = \beta(s(g))$ is satisfied iff $g = R_\beta(\mathrm{Id}_{t(g)})$. Note, e.g., that the action groupoid of Ex. 25 is manifestly* Id*-reducible.*

HOWEVER,

Theorem 44. [ZCL09, Thm. 3.1] *Every Lie groupoid with connected fibres of the source map is* Id*-reducible.*

Remark 45. *The significance of the assumption of s-connectedness of $\mathscr{G}$ is emphasised by the following counter-example, which we borrow from Ref.* [SWo16, Rem. 2.18 b)]. *Take any two non-diffeomorphic manifolds M and N, and consider the pair groupoid $\mathrm{Pair}(M \sqcup N) \equiv \mathbf{Gr}$ of their disjoint union, with $\mathrm{Bisec}(\mathbf{Gr}) \cong \mathrm{Diff}(M \sqcup N)$. Pick arbitrary points $m \in M$ and $n \in N$. Clearly, there is no global bisection through $(n,m) \in \mathrm{Mor}\,\mathbf{Gr}$ (here, we view M and N as submanifolds in $M \sqcup N$) as there is no (global) diffeomorphism $M \longrightarrow N$, which could map $m \longmapsto n$.*

THE SITUATION CHANGES DRAMATICALLY, AND CONSEQUENTIALLY, TOO, WHEN WE PASS from GLOBAL to LOCAL BISECTIONS...

PROP. 46. FOR ANY LIE GROUPOID $\mathcal{G}r = (M, \mathcal{G}, s, t, Id, Inv, m)$ & ANY ARROW $g \in \mathcal{G}$, THERE EXISTS A LOCAL BISECTION $\beta \in Bis_{loc}(\mathcal{G}r)$ ON A NEIGHBOURHOOD of $s(g)$ s.t. $g = \beta(s(g))$.

PROOF: WE CONSIDER THE TANGENTS of s & t at g. BOTH MAPS ARE SUBMERSIVE, & SO WE CAN USE THE FOLLOWING

LEMMA 47. LET $V, W_1, W_2 \in Vect_{\mathbb{K}}^{<\infty}$ with $W_1 \xrightarrow[\omega]{\cong} W_2$, & LET $\chi_A \in Hom_{\mathbb{K}}(V, W_A)$, $A \in \{1,2\}$ BE EPI. THERE EXISTS A SUBSPACE $\Delta \subset V$ with PROPERTY $\chi_A|_\Delta : \Delta \xrightarrow{\cong} W_A$, $A \in \{1,2\}$.

PROOF of LEMMA: WITHOUT ANY LOSS of GENERALITY, WE MAY ASSUME $W_1 = W_2 \equiv W$ (IT SUFFICES to CONSIDER $\tilde{\chi}_2 := \omega \circ \chi_2$ instead of χ_2).

DENOTE $D = \dim_{\mathbb{K}} W$. PICK ANY $\{v_i\}_{i \in \overline{1,D}}$ s.t.

$$W = \langle \chi_1(v_i) \mid i \in \overline{1,D} \rangle_{\mathbb{K}} .$$

IF THE $\chi_2(v_i)$ ARE LINEARLY INDEPENDENT, THEN $I := \langle v_i \mid i \in \overline{1,D} \rangle_{\mathbb{K}}$ IS THE SOUGHT-AFTER SUBSPACE, i.e., $\Delta = I$.

IF NOT, ASSUME – without loss of generality – THAT

$$\chi_2(I) \equiv \langle \chi_2(v_j) \mid j \in \overline{1,K} \rangle_{\mathbb{K}}$$ (POSSIBLY $K=0$).

WE HAVE $V = I \oplus \operatorname{Ker} \chi_1$, & SO THERE EXIST VECTORS $\xi_a \in \operatorname{Ker} \chi_1$, $a \in \overline{K+1,D}$ s.t. $\langle \chi_2(v_j), \chi_2(\xi_a) \mid j \in \overline{1,K} \wedge a \in \overline{K+1,D} \rangle_{\mathbb{K}} = W$.

WE MAY THEN TAKE $\delta_l := \begin{cases} v_l & \text{for } l \in \overline{1,K} \\ v_l + \xi_l & \text{for } l \in \overline{K+1,D} \end{cases}$ to OBTAIN

$\Delta = \langle \delta_l \mid l \in \overline{1,D} \rangle_{\mathbb{K}}$. $\boxed{L}$

IN VIRTUE of **LEMMA 47.**, THERE EXISTS $\Delta \subset T_g \mathcal{G}$ s.t.

$$\Delta \oplus \operatorname{Ker} T_g t = T_g \mathcal{G} = \Delta \oplus \operatorname{Ker} T_g s.$$

CONSIDER NEIGHBOURHOODS of $g \in \mathcal{G}$ & $s(g) \in M$ with RESPECTIVE COÖRDS s.t. THE CORRESPONDING COORDINATE PRESENTATION of s is

$$pr_1 : \mathbb{R}^{\dim M} \oplus \mathbb{R}^{\dim \mathcal{G} - \dim M} \equiv \mathbb{R}^{\dim M} \longrightarrow \mathbb{R}^{\dim M},$$

with THE COORDINATE DERIVATIONS COINCIDING with THE BASIS of Δ & $\operatorname{Ker} T_g s$ (i.e., COÖRDS ADAPTED to THE SPLITTING $\Delta \oplus \operatorname{Ker} T_g s$)

TAKE A LOCAL SECTION σ of s with THE CANONICAL PRESENTATION in THE CHOSEN COÖRDS. By CONSTRUCTION $T_g(t \circ \sigma)$ is ISO, & so – by THE INVERSE-FUNCTION THEOREM – $t \circ \sigma$ is DIFFEO on SOME NEIGHBOURHOOD $\mathcal{U}$ of $s(g)$. WE THEN TAKE $\beta \equiv \sigma|_{\mathcal{U}}$. $\square$

PROP. 48. FOR ANY LIE GROUPOID $Gr = (M, G, s, t, Id, Inv, m)$ & ANY $m \in M$, THE RESTRICTION $t\restriction_{s^{-1}(\{m\})}$ of t to THE SOURCE FIBRE HAS CONSTANT RANK.

PROOF: CONSIDER ANY TWO POINTS $g, h \in s^{-1}(\{m\})$.
AS $t(g^{-1}) = s(g) = m \equiv s(h)$, THE ARROW $h.g^{-1}$ IS WELL-DEFINED, & SO THERE EXISTS A LOCAL BISECTION $\beta \in Bisec(Gr)_{loc}$ with THE PROPERTY $\beta(t(g)) \equiv \beta(s(h.g^{-1})) = h.g^{-1}$, WHICH IMPLIES $\tilde{t}_*\beta(t(g)) = t(h.g^{-1}) = t(h)$ for (SEE: **DEF. 39.**)

$$\tilde{t}_* : Bisec(Gr)_{loc} \longrightarrow Diff(M)_{loc}$$

$$: \beta \longmapsto t\circ\beta$$

DENOTE $U := Dom(\beta) \subset M$ & $V := \tilde{t}_*\beta(U) \subset M$,

SO THAT WE OBTAIN THE DIFFEOMORPHISM (SEE: DEF. 40.)

$$\widetilde{L}_\beta : t^{-1}(U) \xrightarrow{\simeq} t^{-1}(V) : k \longmapsto \beta(t(k)).k.$$

NOTE THAT $\widetilde{L}_\beta(g) \equiv \beta(t(g)).g = h.g^{-1}.g = h.$

AS $s \circ \widetilde{L}_\beta = s$, WE SEE THAT $\widetilde{L}_\beta$ RESTRICTS to A DIFFEO on EACH s-FIBRE within $t^{-1}(U)$. (THE STATEMENT MAKES SENSE in VIRTUE of THE CONSTANT-RANK LEVEL-SET THEOREM [Lee 2012, Th^m 5.12] AS the s-FIBRES ARE PREIMAGES of POINTS in M along THE SUBMERSION s.) MOREOVER,

$$t \circ \widetilde{L}_\beta = t \circ \beta \circ t \equiv t_* \beta \circ t,$$

AND SO WE HAVE A COMMUTATIVE DIAGRAM

$$
\begin{tikzcd}
t^{-1}(U) \cap s^{-1}(\{m\}) \arrow[r, "\tilde{L}_\beta \restriction_{s^{-1}(\{m\})}"] \arrow[d, "t\restriction_{s^{-1}(\{m\})}"'] \arrow[dr, phantom, "\circlearrowleft"] & t^{-1}(V) \cap s^{-1}(\{m\}) \arrow[d, "t\restriction_{s^{-1}(\{m\})}"] \\
U \arrow[r, "\tilde{t}_* \beta"'] & V
\end{tikzcd}
$$

THE HORIZONTAL ARROWS in IT REPRESENT DIFFEOS, of WHICH THE UPPER ONE TAKES g to h. HENCE,

$$\operatorname{rk} t(g) = \operatorname{rk} t(h). \quad \square$$

THE LAST RESULT HAS IMPORTANT CONSEQUENCES...

DEF. 49. LET $\mathrm{Gr} = (M, \mathcal{G}, s, t, \mathrm{Id}, \mathrm{Inv}, m)$ BE A LIE GROUPOID, & LET $m \in M$ BE ARBITRARY. THE ISOTROPY GROUP of m IS THE SUBSET

$$\mathcal{G}_m := s^{-1}(\{m\}) \cap t^{-1}(\{m\}) \subset \mathcal{G}$$

with the MULTIPLICATION & INVERSE MAPS of $\mathcal{G}$ RESTRICTED to IT, & with $\bullet \longmapsto \mathrm{Id}_m$ AS THE GROUP UNIT.

PROP. 50. THE ISOTROPY GROUP of ANY POINT in THE OBJECT MANIFOLD of A LIE GROUPOID IS A LIE GROUP.

PROOF: FIX $m \in M$. THE ISOTROPY GROUP $\mathcal{G}_m$ IS THE PREIMAGE of $\{m\}$ along THE RESTRICTION of t to THE s-FIBRE $s^{-1}(\{m\})$. BUT by **PROP. 48.**, $t|_{s^{-1}(\{m\})}$ HAS CONSTANT RANK, & SO —

In virtue of the constant-rank level-set theorem [Lee 2012, Thm 5.12] $\mathcal{G}_m$ is a submanifold in $S^{-1}(\{m\})$, i.e., it is, in particular, a manifold. Now, the restrictions of m & Inv to $\mathcal{G}_m$ are smooth, & so the quadruple

$$\left(\mathcal{G}_m, m\restriction_{\mathcal{G}_m\times\mathcal{G}_m}, \mathrm{Inv}\restriction_{\mathcal{G}_m}, \bullet \mapsto \mathrm{Id}_m\right)$$

is a group object in $\mathbf{Man}$, i.e., a Lie group. $\square$

PROP. 51. LET $\mathrm{Gr} = (M, \mathcal{G}, s, t, \mathrm{Id}, \mathrm{Inv}, m)$ BE A LIE GROUPOID, & LET $m \in M$ BE ARBITRARY. THE ISOTROPY GROUP $\mathcal{G}_m$ OF m ACTS SMOOTHLY, FREELY & PROPERLY FROM THE RIGHT ON THE s-FIBRE $s^{-1}(\{m\})$.

PROOF: THE ACTION OF INTEREST IS

$$\underline{P} : s^{-1}(\{m\}) \times \mathcal{G}_m \longrightarrow s^{-1}(\{m\}) : (g, h) \longmapsto m(g, h).$$

THIS MAKES SENSE AS $t(h) = m \equiv s(g)$. SMOOTHNESS OF $\underline{P}$ IS INHERITED FROM m. ITS FREENESS IS IMPLIED BY

$$\underline{P}_h(g) = g \iff h = g^{-1} \cdot g = \mathrm{Id}_{s(g)} = \mathrm{Id}_m.$$

FINALLY, CONSIDER A CONVERGENT SEQUENCE $g_\cdot : \mathbb{N} \to s^{-1}(\{m\})$, with $g \equiv \lim_{n\to\infty} g_n \in S^{-1}(\{m\})$, & A SEQUENCE $h_\cdot : \mathbb{N} \to \mathfrak{g}_m$

s.t. THE PRODUCT SEQUENCE $\mu_\cdot : \mathbb{N} \to S^{-1}(\{m\}) : n \mapsto \mathrm{P}_{h_n}(g_n)$

CONVERGES to $\tilde{\mu} \equiv \lim_{n\to\infty} \mu_n$. TAKING into ACCOUNT CONTINUITY

of m & Inv (IMPLIED by SMOOTHNESS of THESE MAPS),

WE ESTABLISH THE IDENTITY

$$\lim_{n\to\infty} h_n \equiv \lim_{n\to\infty}\big((g_n^{-1}\cdot g_n)\cdot h_n\big) = \lim_{n\to\infty}(g_n^{-1}\cdot \mu_n) = \Big(\lim_{n\to\infty} g_n\Big)^{-1}\cdot \lim_{n\to\infty}\mu_n$$

$$= g^{-1}\cdot\tilde{\mu},$$

WHICH DOCUMENTS CONVERGENCE of $h_\cdot$. $\square$

COR. 52. THE SPACE OF ORBITS $s^{-1}(\{m\})/\mathcal{G}_m$ CARRIES THE STRUCTURE OF A SMOOTH MANIFOLD S.T. THE ORBIT PROJECTION $\pi : s^{-1}(\{m\}) \longrightarrow s^{-1}(\{m\})/\mathcal{G}_m : g \longmapsto g \cdot \mathcal{G}_m$ IS A SURJECTIVE SUBMERSION.

PROOF: FOLLOWS DIRECTLY FROM **THM. 21**. $\square$

DEF. 53. LET $\mathcal{G}r = (M, \mathcal{G}, s, t, Id, Inv, m)$ BE A LIE GROUPOID, & LET $m \in M$ BE ARBITRARY. THE ORBIT OF m IS THE SUBSET

$$\mathcal{G} \triangleright m := t(s^{-1}(\{m\})) \left(\equiv \{ x \in M \mid \exists\, g \in s^{-1}(\{m\}) \cap t^{-1}(\{x\}) \} \right) \subset M .$$

PROP. 54. THE RELATION ON M:

$$m_1 \underset{\mathcal{G}}{\sim} m_2 \;\overset{\text{ex}}{\underset{\text{def}}{\Longleftrightarrow}}\; \exists\, g \in s^{-1}(\{m_1\}) \cap t^{-1}(\{m_2\})$$

IS AN EQUIVALENCE RELATION.

PROOF: TRIVIAL. $\square$

PROP. 55. THE ORBIT OF AN ARBITRARY POINT $m \in M$ IN THE OBJECT MANIFOLD M OF A LIE GROUPOID IS AN IMMERSED SUBMANIFOLD OF M.

PROOF: FIX $m \in M$ & CONSIDER A MAP

$$\imath : s^{-1}(\{m\})/\mathcal{G}_m \longrightarrow M : g \cdot \mathcal{G}_m \longmapsto t(g).$$

IT IS MANIFESTLY WELL-DEFINED AS $t(g \cdot h) = t(g)$. MOREOVER,

IT CLOSES THE COMMUTATIVE DIAGRAM

$$\begin{tikzcd} & & M \\ s^{-1}(\{m\}) \arrow[rru, "t"] \arrow[rr, "\pi"'] & & s^{-1}(\{m\})/\mathcal{G}_m \arrow[u, "\iota"'] \end{tikzcd},$$

in WHICH π IS A SURJECTIVE SUBMERSION & t IS SMOOTH. THEREFORE, ι IS SMOOTH in VIRTUE of THE THEOREM on QUASI-UNIVERSALITY of SURJECTIVE SUBMERSIONS.

LET $g_A \in s^{-1}(\{m\})$, $A \in \{1,2\}$ BE s.t. $t(g_1) = t(g_2)$. THEN, $g_1^{-1} \cdot g_2$ IS WELL-DEFINED, & $s(g_1^{-1} \cdot g_2) = s(g_2) = m \equiv s(g_1) = t(g_1^{-1}) \equiv t(g_1^{-1} \cdot g_2)$, WHICH IMPLIES $g_1^{-1} \cdot g_2 \in \mathcal{G}_m$. THUS, $g_2 \equiv g_1 \cdot (g_1^{-1} \cdot g_2) \in g_1 \cdot \mathcal{G}_m$,

& so $g_1 \cdot G_m = g_2 \cdot G$. WE CONCLUDE THAT ι IS INJECTIVE.

NEXT, NOTE THAT $t\restriction_{s^{-1}(\{m\})} \equiv \iota \circ \pi$, HENCE DUE TO SUBMERSIVITY OF π, WE FIND $\mathrm{rk}\,\iota = \mathrm{rk}\, t\restriction_{s^{-1}(\{m\})} = \mathrm{const}$, i.e., ι IS A SMOOTH INJECTION OF CONSTANT RANK. ITS IMAGE

$$\iota\left(s^{-1}(\{m\})/G_m\right) = t\left(s^{-1}(\{m\})\right) \equiv G \triangleright m$$

IS AN IMMERSED SUBMANIFOLD OF M IN VIRTUE OF THE GLOBAL-RANK THEOREM [Lee 2012, Thm 5.12]. $\square$

DEF. 56. THE SPACE

$$M /\!/ \mathcal{G} := \{ \mathcal{G} \rhd m \mid m \in M \}$$

of ORBITS of A LIE GROUPOID $\mathbf{Gr} = (M, \mathcal{G}, s, t, \mathrm{Id}, \mathrm{Inv}, m)$, ENDOWED with QUOTIENT TOPOLOGY IS CALLED THE ORBISPACE of $\mathbf{Gr}$.

THE DECOMPOSITION of M into CONNECTED COMPONENTS of (PAIRWISE DISJOINT) ORBITS IS CALLED THE CHARACTERISTIC FOLIATION of $\mathbf{Gr}$.

OFF on THE TANGENT:

Definition 57. *Let M be a smooth manifold. A* (**real**) **Lie algebroid over** M (**of rank** $N \in \mathbb{N}^\times$) *is a quintuple* $(\mathcal{E}, M, \mathbb{R}^{\times N}, \pi_\mathcal{E}, \alpha_\mathcal{E}, [\cdot,\cdot]_\mathcal{E})$ *composed of*

- *a vector bundle* $(\mathcal{E}, M, \mathbb{R}^{\times N}, \pi_\mathcal{E})$ *(of rank* $N \in \mathbb{N}^\times$*);*
- *a vector-bundle morphism*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\alpha_\mathcal{E}"] \arrow[d, "\pi_\mathcal{E}"'] & \mathrm{T}M \arrow[d, "\pi_{\mathrm{T}M}"] \\ M \arrow[r, equal, "\mathrm{id}_M"'] & M \end{tikzcd} ,$$

  termed the **anchor** (**map**)*;*
- *a binary operation* $[\cdot,\cdot]_\mathcal{E} \,:\, \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E})$,

satisfying the following conditions:

- $[\cdot,\cdot]_\mathcal{E}$ *is a Lie bracket;*
- $\forall_{\varepsilon_1,\varepsilon_2 \in \Gamma(\mathcal{E})} \forall_{f \in C^\infty(M;\mathbb{R})} \,:\, [\varepsilon_1, f \triangleright \varepsilon_2]_\mathcal{E} = f \triangleright [\varepsilon_1, \varepsilon_2] + \alpha_\mathcal{E}(\varepsilon_1)(f) \triangleright \varepsilon_2$ *(the* **Leibniz** PROPERTY*).*

E.g.,

Ex. 58. A LIE ALGEBROID OVER $M=\{*\}$ IS A LIE ALGEBRA.

Example 59. The **tangent Lie algebroid of** M is the canonical structure of a Lie algebroid on the tangent bundle $\pi_{TM} : TM \longrightarrow M$ with the identity anchor $\alpha_{TM} = \mathrm{id}_{TM}$, and the standard Lie bracket $[\cdot,\cdot]_{TM}$ of vector fields on M.

Ex. 60. GIVEN $M \in \mathrm{Man}$ & $\omega \in \Omega^2(M)$, THERE EXISTS A CANONICAL STRUCTURE of A LIE ALGEBROID on

$$\begin{array}{c} \mathcal{E} = TM \times \mathbb{R} \equiv TM \times_M (M \times \mathbb{R}) \\ \pi_{TM} \circ \mathrm{pr}_1 \Big\downarrow \\ M \end{array}$$

with $\alpha_{\mathcal{E}} = \mathrm{pr}_1$

$$[(X,f),(Y,g)] = \left([X,Y]_{\Gamma(TM)}, X(g) - Y(f) + \omega(X,Y)\right)$$

$X,Y \in \Gamma(TM)$; $f,g \in C^\infty(M;\mathbb{R})$

IFF $d\omega = 0$.

Ex. 61. THE LIE ALGEBROID of THE LIE GROUPOID, WHICH WE DISCUSS BELOW.

Prop. 62. IN EVERY LIE ALGEBROID, THE ANCHOR INDUCES A LIE-ALGEBRA HOMOMORPHISM on SECTIONS.

Proof: CONSIDER ARBITRARY $X, Y, Z \in \Gamma(\mathcal{E})$ & $f \in C^\infty(M, \mathbb{R})$. WE CALCULATE, with the HELP of THE JACOBI & LEIBNIZ IDENTITIES,

$$[[X,Y]_\mathcal{E}, f \triangleright Z]_\mathcal{E} \overset{(L)}{=} f \triangleright [[X,Y]_\mathcal{E}, Z] + \alpha_\mathcal{E}([X,Y]_\mathcal{E})(f) \triangleright Z$$

$$(J) \parallel$$

$$[[X, f \triangleright Z]_\mathcal{E}, Y]_\mathcal{E} - [[Y, f \triangleright Z]_\mathcal{E}, X]_\mathcal{E}$$

$$\overset{L}{=} [f \triangleright [X,Z]_\mathcal{E} + \alpha_\mathcal{E}(X)(f) \triangleright Z, Y]_\mathcal{E} - [f \triangleright [Y,Z]_\mathcal{E} + \alpha_\mathcal{E}(Y)(f) \triangleright Z, X]_\mathcal{E}$$

$$= f \triangleright \left([[X,Z]_{\mathcal{E}}, Y]_{\mathcal{E}} - [[Y,Z]_{\mathcal{E}}, X]_{\mathcal{E}} \right) - \alpha_{\mathcal{E}}(Y)(f) \triangleright [X,Z]_{\mathcal{E}} + \alpha_{\mathcal{E}}(X)(f) \triangleright [Z,Y]_{\mathcal{E}}$$

$$- \alpha_{\mathcal{E}}(Y) \circ \alpha_{\mathcal{E}}(X)(f) \triangleright Z + \alpha_{\mathcal{E}}(X)(f) \triangleright [Y,Z]_{\mathcal{E}} - \alpha_{\mathcal{E}}(Y)(f) \triangleright [Z,X]_{\mathcal{E}}$$

$$+ \alpha_{\mathcal{E}}(X) \circ \alpha_{\mathcal{E}}(Y)(f) \triangleright Z,$$ WHENCE

$$\left(\alpha_{\mathcal{E}}([X,Y]_{\mathcal{E}}) - [\alpha_{\mathcal{E}}(X), \alpha_{\mathcal{E}}(Y)]_{\Gamma(TM)} \right)(f) \triangleright Z = 0.$$

In VIEW of ARBITRARINESS of Z & f, WE THUS OBTAIN THE DESIRED IDENTITY

$$\alpha_{\mathcal{E}}([X,Y]_{\mathcal{E}}) - [\alpha_{\mathcal{E}}(X), \alpha_{\mathcal{E}}(Y)]_{\Gamma(TM)} = 0$$

$$\forall\, X,Y \in \Gamma(\mathcal{E}). \qquad \square$$

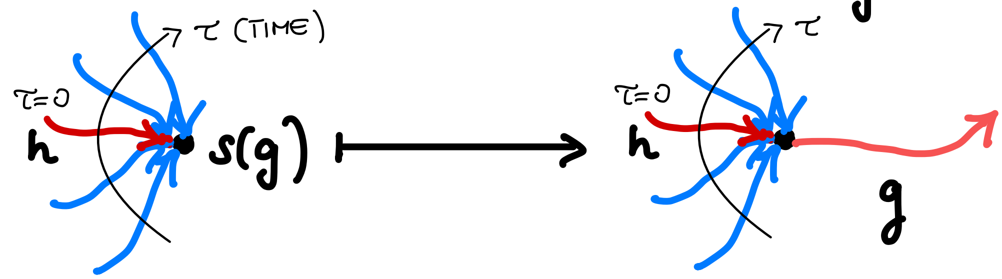
WE ARE NOW READY to APPROACH THE QUESTION of A DIFFERENTIAL CALCULUS on $\mathcal{G}$ COMPATIBLE with, SAY, LEFT-TRANSLATIONS of **DEF. 31**. RECALL

$$l_g : t^{-1}(\{s(g)\}) \longrightarrow t^{-1}(\{t(g)\})$$

to CONCLUDE THAT THE ONLY WAY to MAKE SENSE of THE NOTION of LEFT-INVARIANCE of A VECTOR FIELD on $\mathcal{G}$ IS to TAKE IT from THE DISTRIBUTION TANGENT to t-FIBRES. ACTUALLY, in VIRTUE of SUBMERSIVITY of t (IMPLYING $\mathrm{rk}\, t \equiv \dim M = \mathrm{const}$), THE DISTRIBUTION IS REGULAR (by THE CONSTANT-RANK THEOREM for VECTOR BUNDLES), i.e., IT IS A VECTOR SUBBUNDLE $\operatorname{Ker} Tt \subset T\mathcal{G}$.

WE HAVE $T_h \ell_g : (\operatorname{Ker} Tt)_h \xrightarrow{\simeq} (\operatorname{Ker} Tt)_{g.h}$

$t(h) = s(g)$



(PATH of ARROWS through h)

(PATH of ARROWS through $g.h$)

DEF. 63. LET $\mathrm{Gr} = (M, \mathcal{G}, s, t, \mathrm{Id}, \mathrm{Inv}, m)$ BE A LIE GROUPOID.

A LEFT-INVARIANT VECTOR FIELD on $\mathcal{G}$ IS A SECTION

$$\mathcal{L} \in \Gamma(\operatorname{Ker} Tt) \subset \Gamma(T\mathcal{G})$$

with THE PROPERTY $\forall g \in \mathcal{G}\ \forall h \in t^{-1}(\{s(g)\}) : T_h \ell_g (\mathcal{L}(h)) = \mathcal{L}(g.h)$.

WE SHALL DENOTE THE VECTOR SPACE of LI VECTOR FIELDS on $\mathcal{G}$ AS $\mathfrak{X}_L(\mathcal{G})$.