Duality, Descent & Defects II

"Toying with Principaloid Bundles"

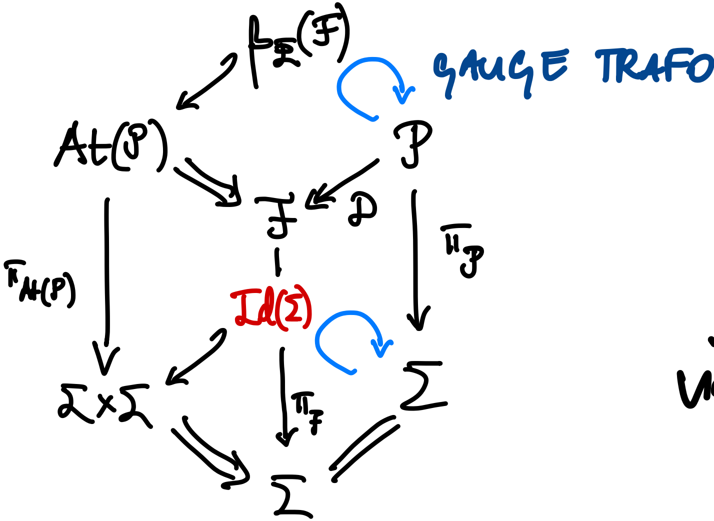
9 V 2025

I OUR STARTING POINT IS THE PRINCIPAL $\mathcal{G}$-BUNDLE with CONNECTION

$$
\begin{tikzcd}
 & \mathcal{P}, \Theta \arrow[dl, "D"'] \arrow[dr, "\mu_{\mathcal{P}}"] & & \mathcal{G} \arrow[dl, Rightarrow] \\
\mathcal{F}, \Theta_{\mathcal{F}} \arrow[d, "\pi_{\mathcal{F}}"'] & & M & \\
\Sigma & & &
\end{tikzcd}
$$

OVER SPACETIME Σ

ITS GAUGE SYMMETRIES ARE MODELLED ON π_Σ-PROJECTABLE BISECTIONS of the ATTENDANT EHRESMANN–ATIYAH GROUPOID – THOSE COVERING id_Σ, i.e.,



, $b_\Phi \in \mathrm{Bisec}_{\pi_{\mathcal{P}}}(At(\mathcal{P}) \Rightarrow \mathcal{F})$

$\rightsquigarrow \Phi \in \mathrm{Aut}(\mathcal{P})$ vertical

via INDUCTION

$$
\begin{tikzcd}
\mathcal{P} \arrow[rrr, "\Phi"] \arrow[dr, "\partial"] \arrow[dd] & & & \mathcal{P} \arrow[dl] \arrow[dd] \\
& \mathcal{F} \arrow[r, "\mathcal{F}_*(\Phi)"] \arrow[dl] & \mathcal{F} \arrow[dr] & \\
\Sigma \arrow[rrr, equal] & & & \Sigma
\end{tikzcd}
$$

WHEN CONSTRUCTING (GAUGE) FIELD THEORIES with SYMMETRY MODEL $\mathcal{G} \Rightarrow M$, WE CONSIDER OBJECTS with SIMPLE GAUGE-TRANSFORMATION PROPERTIES,

SUCH AS, E.G.,

COVARIANT DERIVATIVE

$$\Gamma(\mathcal{F}) \ni \varphi \longmapsto \boxed{H_{\mathcal{F}} \circ T\varphi =: \nabla^{H}\varphi}$$

MATTER FIELD

INDEED, UPON TAKING INTO ACCOUNT THE GAUGE TRANSFORMATIONS:

$$H \longmapsto H^{\Phi} := T\Phi \circ H \circ T\Phi^{-1}$$

$$\left(H_{\mathcal{F}} \circ T\mathcal{D} = T\mathcal{D} \circ H\right)$$

$$\varphi \longmapsto \varphi^{\Phi} := \mathcal{F}_*(\Phi) \circ \varphi$$

$$H_{\mathcal{F}} \longmapsto H^{\Phi}_{\mathcal{F}} = T\mathcal{F}_*(\Phi) \circ H_{\mathcal{F}} \circ T\mathcal{F}_*(\Phi)^{-1},$$

WE OBSERVE COVARIANCE of $\nabla^{H}\varphi$: 4

$$\nabla^{H^{\Phi}}\varphi^{\Phi} \equiv T\mathcal{F}_{*}(\Phi) \circ H_{\mathcal{F}} \circ T\mathcal{F}_{*}(\Phi)^{-1} \circ T\left(\mathcal{F}_{*}(\Phi) \circ \varphi\right)$$

$$= T\mathcal{F}_{*}(\Phi) \circ \nabla^{H}\varphi$$

IN A LOCAL TRIVIALISATION $\mathcal{F}\tau_i : \pi_{\mathcal{F}}^{-1}(O_i) \xrightarrow{\simeq} \underset{\Sigma}{\overset{\cup}{O_i}} \times M$,

WE FIND $T\mathcal{F}\tau_i^{-1} \circ \left(\nabla^{H^{\Phi}}\varphi^{\Phi}\right) = T_{\varphi_i(\sigma)}\left(t_*(\gamma_i(\sigma))\right) \circ D^{A_i}\varphi_i(\sigma)$

for $\mathcal{F}\tau_i \circ \mathcal{F}_*(\Phi) \circ \mathcal{F}\tau_i^{-1}(\sigma, m) = (\sigma, t_*(\gamma_i(\sigma))(m))$

LOCAL DATA of Φ :
$\gamma_i : O_i \to B$

$\mathcal{F}\tau_i \circ \varphi(\sigma) = (\sigma, \varphi_i(\sigma))$ & $\boxed{D^{A_i}\varphi_i(\sigma) = T_\sigma\varphi_i - \alpha \circ A_i(\sigma, \varphi_i(\sigma))}$

WHERE $\begin{tikzcd} E \arrow[rr, "\alpha"] \arrow[dr, "\pi_E"'] & & TM \arrow[dl] \\ & M & \end{tikzcd}$ IS THE ANCHOR OF THE TANGENT LIE ALGEBROID $E \xrightarrow{\pi_E} M$ OF $\mathcal{G} \rightrightarrows M$, (5)

$\mathbb{B} \equiv \mathrm{Bisec}(\mathcal{G} \rightrightarrows M)$ ARE GLOBAL BISECTIONS OF $\mathcal{G} \rightrightarrows M$,

& $A_i \in \Gamma(\mathrm{pr}_1^* T^* \mathcal{O}_i \otimes \mathrm{pr}_2^* E)$ IS THE LOCAL GAUGE FIELD.

THUS, IT SEEMS NATURAL TO CONTRACT THE LOCAL OBJECTS $\mathcal{D}^{A_i} \varphi_i$ WITH $t_*(\mathbb{B})$-INVARIANT TENSORS AS, E.G.,

$$\boxed{G_M\left(\mathcal{D}^{A_i}\varphi_i, \mathcal{D}^{A_i}\varphi_i\right)}$$

KLEIN-GORDON COUPLING

METRIC ON M, S.TH. $(t_*\beta)^* G_M = G_M \quad \forall \beta \in \mathbb{B}$.

A SERIOUS PROBLEM with THIS IDEA IS ILLUSTRATED BY THE FOLLOWING REASONING:

[FERNANDES, DEL HOYO]

well, almost...

ASSUME M ADMITS A METRIC G_M:

$$\forall \beta \in B : (t_* \beta)^* G_M = G_M,$$

i.e., $B \subset \mathrm{Isom}(M, G_M)$. WE SHALL DEMONSTRATE THAT THEN, NECESSARILY, $\forall m \in M : \dim(g \triangleright m) = 0$, FOR Id-REDUCIBLE $g \Rightarrow M$!

WHICH IS A PATHOLOGY IF WE WANT TO THINK OF GAUGING AS A MATHEMATICAL MODEL OF REDUCTION OF CONFIGURATIONAL DoFs: $M \searrow M /\!/ g$.

A.a. ASSUME $\dim(\mathcal{G} \rhd M) > 0$, SO THAT (7)

$$\exists\, x, y \in \mathcal{G} \rhd M : \Big(x \neq y \ \wedge\ \exists\, g \in \mathcal{G} : \begin{cases} s(g) = x \\ t(g) = y \end{cases}$$

$$\wedge\ \exists\, v \in T_x M \setminus \{0\},\ w \in T_y M \setminus \{0\} \Big)$$

NOW, $s^{-1}(\{x\})$ & $t^{-1}(\{y\})$ SUBMERSE $\mathcal{G} \rhd M$, HENCE

$$\exists\, \hat{v} \in T s^{-1}(\{x\}) : Tt(\hat{v}) = v,$$

& - SIMILARLY -

$$\exists\, \hat{w} \in T t^{-1}(\{y\}) : Ts(\hat{w}) = w.$$

IN OTHER WORDS, THERE EXIST PATHS OF - RESP. -

CO-TERMINAL (for x) & CO-INITIAL (for y) 8

ARROWS WHICH REPRESENT $\vec{v}$ RESP. $\vec{w}$

& PUSH DOWN TO PATHS IN M REPRESENTING

v RESP. w. BUT $y = g \blacktriangleright x$, SO WE MAY

(SMOOTHLY) CONNECT THE TWO PATHS in $\mathcal{G}$

BY g, WHEREBY WE OBTAIN A PATH

THROUGH g REPRESENTING $V \in T_g \mathcal{G}$ s.th.

$$T_g s(V) = v \quad , \quad T_g t(V) = w.$$

CLEARLY, V BELONGS TO THE COMMON COMPLETION

of $\operatorname{Ker} T_g s$ & $\operatorname{Ker} T_g t$ in $T_g \mathcal{G}$. ANY SUCH (9)
COMPLETION CAN BE VIEWED AS THE TANGENT SPACE of A <u>LOCAL</u> BISECTION through g, BECAUSE THE LATTER IS AN ARBITRARY SUBMANIFOLD $\mathcal{U}$ in $\mathcal{G}$ DIFFEOMORPHIC TO $s(\mathcal{U})$ & $t(\mathcal{U})$ in M.

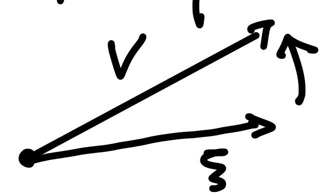
BUT $\mathcal{G} \Rightarrow M$ IS Id-REDUCIBLE, & SO $\exists\, \beta \in \mathbb{B}$: β PASSES THROUGH g.
WE SHALL ARGUE THAT β CAN BE DEFORMED

IN SUCH A MANNER THAT THE DEFORMED (10)
BISECTION $\tilde{\beta}$ (A SUBMANIFOLD in S) HAS $T_g \tilde{\beta} \ni V$.

NOTE THAT $\exists\, \xi \in T_g \tilde{\beta} : T_g s(\xi) = v$

(BECAUSE $s(\tilde{\beta}) = M$ DIFFEOMORPHICALLY).

HENCE, $V - \xi \in \operatorname{Ker} T_g s$. THE IDEA THEN
IS TO DEFORM $\tilde{\beta}$ DIFFEOMORPHICALLY
SO THAT ITS TANGENT IS TILTED AS



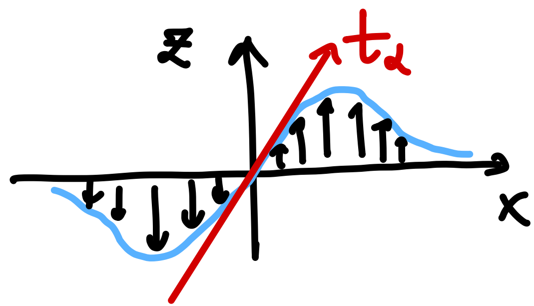
THIS IS A LOCAL PROBLEM, WHICH CAN BE (11)
CONSIDERED IN A LOCAL MODEL

$$\mathbb{R}^m \times \mathbb{R}^{g-m} \quad \text{of} \quad \mathcal{G},$$

with THE FIRST FACTOR REPRESENTING COÖRDS ALONG $\tilde{f}_0$, & THE SECOND – THOSE ALONG THE S-FIBRES TRANSVERSE TO IT.

REDUCING IT FURTHER, WE MAY ASK:
GIVEN TWO VECTORS $t=\partial_x$ & $t_\alpha=\partial_z+\alpha\partial_x$, $\alpha\neq 0$ IN $\mathbb{R}^{3+}$, CAN WE DIFFEOMORPHICALLY DEFORM

THE PLANE $z=0$ SO THAT THE DEFORMED (12)
PLANE HAS t_α IN ITS TANGENT @ 0 ?
THE ANSWER IS "YES": WE TAKE A COMPACTLY
SUPPORTED "PULSE" FIELD ALONG $\hat{x}$:



& DAMP IT SMOOTHLY OVER A COMPACT
SUPPORT IN ALL TRANSVERSE DIRECTIONS.
(CONSIDER, e.g., A TOY MODEL: $x e^{-x^2} e^{-y^2} \partial_z$

AS A GENERATOR OF A (NON-COMPACTLY) TRANSVERSALLY DAMPED PULSE FIELD DEFORMING $z=0$ AROUND 0!).

CONCLUSION: $\exists\ \beta \in \mathbb{B}$ THROUGH g : $T_g\beta \ni V$.

BUT THEN $g : x \longmapsto y$ IMPLEMENTS THE ISOMETRY $t_*\beta : x \longmapsto y$, & SO — NECESSARILY —

$$\|v\|_{GM} = \|w\|_{GM},$$

WHICH IS ABSURD (WE MAY TAKE v & w ARBITRARY!).

↯

HOW CAN WE CIRCUMNAVIGATE THE PROBLEM (14)
IN A MANNER APPLICABLE under ALL CIRCUMSTANCES
(i.e., INDEPENDENTLY of Id-REDUCIBILITY)?

This WE ACHIEVE by RESTRICTING
THE GAUGE-SYMMETRY GROUP as

$$t_*(\mathbb{B}) \searrow t_*(\mathbb{B}) \cap \mathrm{Isom}(M, G_M)$$

for GIVEN METRIC G_M!

$\underline{\mathrm{II}}$ NEXT, WE MAY CONTEMPLATE NATURAL MODELS OF DYNAMICS OF THE GAUGE FIELD $A_{(i)}$. AS BEFORE, WE LOOK IN THE DIRECTION OF GAUGE TENSORS... (15)

A CANONICAL SOURCE OF SUCH A TENSOR IS THE CURVATURE OF Θ:

$$\boxed{R(\Theta) := \Theta \circ [\cdot,\cdot]_{\Gamma(TP)} \circ \big((\mathrm{id}_{TP} - \Theta) \wedge (\mathrm{id}_{TP} - \Theta)\big)} : \Gamma(TP) \wedge \Gamma(TP) \to \Gamma(VP),$$

WHICH – CLEARLY – QUANTIFIES THE FAILURE OF THE INTEGRABILITY OF THE HORIZONTAL DISTRIBUTION HP...

WE SHALL FIRST INVESTIGATE ITS (GLOBAL) BEHAVIOUR UNDER GAUGE TRANSFORMATIONS, (16)

Prop 1.: $\forall \Phi \in \mathrm{Aut}(\mathcal{P})_{\mathrm{Vert}} : R(\omega^{\Phi}) \circ (T\Phi \wedge T\Phi) = T\Phi \circ R(\omega)$

Proof:

$$R(\omega^{\Phi}) \equiv \omega^{\Phi} \circ [\cdot,\cdot]_{\Gamma(T\mathcal{P})} \circ \left(\left(\mathrm{id}_{T\mathcal{P}} - \omega^{\Phi}\right) \wedge \left(\mathrm{id}_{T\mathcal{P}} - \omega^{\Phi}\right)\right)$$

$$= T\Phi \circ \omega \circ T\Phi^{-1} \circ [\cdot,\cdot]_{\Gamma(T\mathcal{P})} \circ (T\Phi \wedge T\Phi) \circ \left(\left(\mathrm{id}_{T\mathcal{P}} - \omega\right) \wedge \left(\mathrm{id}_{T\mathcal{P}} - \omega\right)\right) \circ (T\Phi \wedge T\Phi)^{-1}$$

$$= T\Phi \circ \omega \circ [\cdot,\cdot]_{\Gamma(T\mathcal{P})} \circ \left(\left(\mathrm{id}_{T\mathcal{P}} - \omega\right) \wedge \left(\mathrm{id}_{T\mathcal{P}} - \omega\right)\right) \circ (T\Phi \wedge T\Phi)^{-1}$$

INVARIANCE of COMMUTATOR under PUSHFORWARD

$$\equiv T\Phi \circ R(\omega) \circ (T\Phi \wedge T\Phi)^{-1} \quad \square$$

& SUBSEQUENTLY PASS TO A LOCAL DESCRIPTION ...

DENOTE

$$\mathcal{P}\tau_i^{-1*}\Theta(\sigma,g) = \mathrm{id}_{T_gG} - T_{\mathrm{Id}_{t(g)}} r_g \circ A_i(\sigma, t(g)) =: \mathrm{id}_{T_gG} - \Gamma_i(\sigma,g)$$

CHRISTOFFEL 1-FORMS

TO OBTAIN – FOR ALL $(v_A, V_A) \in \Gamma(T\Sigma|_{\mathcal{O}_i}) \times \Gamma(TG)$, $A \in \{1,2\}$ –

$$\left(\mathcal{P}\tau_i^{-1*} R(\Theta)\right)\left((v_1,V_1),(v_2,V_2)\right)$$

$$= \mathcal{P}\tau_i^{-1*}\Theta\left(\left[(\mathrm{id}-\mathcal{P}\tau_i^{-1*}\Theta)(v_1,V_1), (\mathrm{id}-\mathcal{P}\tau_i^{-1*}\Theta)(v_2,V_2)\right]_{\Gamma(pr_1^*T\Sigma \oplus pr_2^*TG)}\right)$$

On THE TOTAL TANGENT!

$$= \mathcal{P}\tau_i^{-1*}\Theta\left(\left[(v_1, v_1 \lrcorner \Gamma_i), (v_2, v_2 \lrcorner \Gamma_i)\right]_{\Gamma(pr_1^*T\Sigma \oplus pr_2^*TG)}\right)$$

CLEARLY, THIS is d_Σ!

$$= pr_i^{-1\,*} \circledast \left([v_1, v_2]_{\Gamma(T\Sigma)}, v_1 \lrcorner\, d(v_2 \lrcorner\, \Gamma_i) - v_2 \lrcorner\, d(v_1 \lrcorner\, \Gamma_i) + [v_1 \lrcorner\, \Gamma_i, v_2 \lrcorner\, \Gamma_i]_{\Gamma(T\mathfrak{g})} \right)$$ (18)

$$\equiv v_1 \lrcorner\, d(v_2 \lrcorner\, \Gamma_i) - v_2 \lrcorner\, d(v_1 \lrcorner\, \Gamma_i) + [v_1 \lrcorner\, \Gamma_i, v_2 \lrcorner\, \Gamma_i]_{\Gamma(T\mathfrak{g})} - [v_1, v_2]_{\Gamma(T\Sigma)} \lrcorner\, \Gamma_i$$

CARTAN FORMULA for $d\Gamma_i$

$$= d_\Sigma \Gamma_i (v_1, v_2) + [v_1 \lrcorner\, \Gamma_i, v_2 \lrcorner\, \Gamma_i]_{\Gamma(T\mathfrak{g})} \equiv \left(d_\Sigma \Gamma_i + \tfrac{1}{2} [\cdot, \cdot]_{\Gamma(T\mathfrak{g})} \circ \Gamma_i \wedge \Gamma_i \right)(v_1, v_2)$$

THUS, IN THE LOCAL PICTURE, THE CURVATURE IS REPRESENTED by THE $pr_1^* TO_i$-FOLIATED $pr_2^* E$-VALUED 2-FORMS

$$\left([d_\Sigma, T r_{\mathfrak{g}}] = 0 \right)$$

$$\boxed{F_i = d_\Sigma A_i + \tfrac{1}{2} [\cdot, \cdot]_E \circ (A_i \wedge A_i)}$$

LOCAL CURVATURE 2-FORMS

Prop. 2.: OVER $O_{ij} = O_i \cap O_j \ni \sigma'$, THE LOCAL CURVATURE 2-FORMS ARE IDENTIFIED AS

$$F_i(\sigma', t_*(\beta_{ij}(\sigma'))(m)) = T_{Id_m} C_{\beta_{ij}(\sigma')} \circ F_j(\sigma', m)$$

IN TERMS of THE TRANSITION 1-COCYCLE

$$\beta_{ij} : O_{ij} \to \mathbb{B} \text{ of } \mathcal{P}.$$

BY THE SAME TOKEN, $\forall\, \Phi \in \mathrm{Aut}(\mathcal{P})_{vert}$: AS BEFORE

$$F_i^{\Phi}(\sigma, t_*(\gamma_i(\sigma))(m)) = T_{Id_m} C_{\gamma_i(\sigma)} \circ F_i(\sigma, m), \quad \sigma \in O_i.$$

ABOVE, $C : \mathbb{B} \to \mathrm{Diff}(\mathcal{G})$ IS THE CANONICAL ADJOINT ACTION of $\mathbb{B}$ on $\mathcal{G}$.

Proof: We shall derive the latter formula. To this end, we pull back the identity from Prop. 1. along the local trivialisation $\wp\tau_i^{-1}$, whereby we obtain:

THE CURVATURE IS A VECTOR-VALUED 2-FORM!!!

$$\big(\wp\tau_i^{-1\,*}R(\mathfrak{A}^{\Phi})\big)\circ\big(T(\wp\tau_i\circ\Phi\circ\wp\tau_i^{-1})\wedge T(\wp\tau_i\circ\Phi\circ\wp\tau_i^{-1})\big)=T(\wp\tau_i\circ\Phi\circ\wp\tau_i^{-1})\circ\big(\wp\tau_i^{-1\,*}R(\mathfrak{A})\big)$$

THIS ACTS TRIVIALLY ON THE DISTRIBUTION $\mathrm{pr}_1^*TO_i$

$$= T_{\mathrm{Id}_{t(\check{\gamma}_i(\sigma,g))}}r_{\check{\gamma}_i(\sigma,g)}\circ F_i^{\Phi}\big(\sigma,t(\check{\gamma}(\sigma,g))\big)\circ T_{(\sigma,g)}\tilde{\gamma}_i$$

$$= T_{\mathrm{Id}_{t(\check{\gamma}_i(\sigma,g))}}r_{\check{\gamma}_i(\sigma,g)}\circ F_i^{\Phi}\big(\sigma,t(\check{\gamma}(\sigma,g))\big)$$

THIS ACTION IS EFFECTIVELY IN TG

LOCAL MODEL OF VP

$$= T_{(\sigma,g)}\tilde{\gamma}_i\circ T_{\mathrm{Id}_{t(g)}}r_g\circ F_i(\sigma,t(g))$$

$$= T_g L_{\gamma_i(\sigma)}\circ T_{\mathrm{Id}_{t(g)}}r_g\circ F_i(\sigma,t(g)),$$

written in terms of $\tilde{\gamma}_i : O_i\times G\circlearrowleft,\ (\sigma,g)\longmapsto\big(\sigma,L_{\gamma_i(\sigma)}(g)\big)=:\big(\sigma,\check{\gamma}_i(\sigma,g)\big).$

USING $r_{\check{\gamma}_i(\sigma, g)} = r_{\gamma_i(\sigma)(t(g))\cdot g} = r_g \circ r_{\gamma_i(\sigma)(t(g))}$

AND THE COMMUTATIVITY OF LEFT & RIGHT TRANSLATIONS, AS WELL AS THE INVERTIBILITY OF Tr_g, WE FIND:

$$F_i^{\Phi}\big(\sigma, t(\gamma_i(\sigma)(t(g)))\big) = T_{\check{\gamma}_i(\sigma,g)} r_{\gamma_i(\sigma)(t(g))^{-1}} \circ T_{\mathrm{Id}_{t(g)}} L_{\gamma_i(\sigma)} \circ F_i(\sigma, t(g))$$

$$= T_{\gamma_i(\sigma)(t(g))^{-1}} L_{\gamma_i(\sigma)} \circ T_{\mathrm{Id}_{t(g)}} r_{\gamma_i(\sigma)(t(g))^{-1}} \circ F_i(\sigma, t(g))$$

AT THIS STAGE, WE INVOKE

LEMMA: $\forall g \in \mathcal{G} \ \forall B_g \in \mathbb{B} : g = B_g(s(g))$
$h \in s^{-1}(\{t(g)\})$
$$\Longrightarrow \begin{cases} r_g = R_{B_g}\big|_{s^{-1}(\{t(g)\})} \\ T_h r_g = T_h R_{B_g}\big|_{\ker T_h s} \end{cases}$$

Proof: 

TO REWRITE THE ABOVE IN THE DESIRED FORM:

$$F_i^{\Phi}(\sigma, t(\gamma_i(\sigma)(t(g)))) = T_{\gamma_i(\sigma)(t(g))^{-1}} L_{\gamma_i(\sigma)} \circ T_{\mathrm{Id}_{t(g)}} R_{\gamma_i(\sigma)^{-1}} \circ F_i(\sigma, t(g))$$

THE STATEMENT OF THE PROPOSITION NOW FOLLOWS BY THE SURJECTIVITY OF t. $\square$

PRIOR TO PROCEEDING TO A PROPOSAL FOR A GAUGE-FIELD THEORY BASED ON THE DEFINITION OF CURVATURE, WE REWRITE THE FORMULA FOR LOCAL CURVATURE 2-FORMS IN A WAY WHICH HELPS TO AVOID CONFUSION, TO WIT:

$$\boxed{F_i = D_1 A_i + \tfrac{1}{2}[\cdot,\cdot]_E \circ (A_i \wedge A_i)}$$

EXTERIOR DERIVATIVE IN THE 1st ARGUMENT

THE PROBLEM with A DEFINITION of A GFT ACTION FUNCTIONAL STEMS from THE FACT THAT F_i IS A 2-FORM on $O_i \times M$, & NOT on O_i. HENCE, IN ANY FIELD THEORY over SPACETIME Σ, IT NEEDS to BE LOCALISED — ALL OVER O_i — IN THE FIBRE M of $\mathcal{F}$... THUS CAN BE ACHIEVED AS FOLLOWS: CONSIDER

$$\varphi \in \Gamma(\mathcal{F})$$ HIGGS BACKGROUND,

AND SUBSEQUENTLY PUT φ in THE TOTAL SPACE of $Ad(\mathcal{P})$ (with TYPICAL FIBRE $\mathfrak{g}$) AS

$$\boxed{\mathcal{I}d_\varphi := j_{\mathrm{Ad}(\mathscr{P})}^{-1} \circ \mathrm{I} \circ \varphi} : \Sigma \longrightarrow \mathrm{Ad}(\mathscr{P}),$$

WHERE $\mathrm{I} : \mathcal{F} \longrightarrow \mathrm{At}(\mathscr{P})$ IS THE IDENTITY POSSECTION of $\mathrm{At}(\mathscr{P}) \Longrightarrow \mathcal{F}$, & $j_{\mathrm{Ad}(\mathscr{P})} : \mathrm{Ad}(\mathscr{P}) \hookrightarrow \mathrm{At}(\mathscr{P})$ IS THE EMBEDDING (CLEARLY, $\mathrm{I}(\mathcal{F}) \subset j_{\mathrm{Ad}(\mathscr{P})}(\mathrm{Ad}(\mathscr{P}))$).

WE NOW DEFINE THE (LOCAL) HIGGSED CURVATURE 2-FORM

$$\boxed{F_i[\varphi] := \mathrm{pr}_2 \circ \mathscr{P}\tau_i^{-1\,*} R(\Theta) \circ \left(T(\mathscr{A}\tau_i \circ \mathcal{I}d_\varphi) \wedge T(\mathscr{A}\tau_i \circ \mathcal{I}d_\varphi) \right)}$$

IN TERMS of THE (INDUCED) LOCAL TRIVIALISATION

$$\mathscr{A}\tau_i : \pi_{\mathrm{Ad}(\mathscr{P})}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times \mathfrak{g}.$$

WE READILY DERIVE – @ $\sigma \in \mathcal{O}_i$ & for φ_i AS BEFORE –

F_i is $\mathrm{pr}_1^* T\mathcal{O}_i$ - FOLIATED!

$$F_i[\varphi](\sigma) \equiv \mathrm{pr}_2 \circ T_{\mathrm{Id}_{\varphi_i(\sigma)}} r_{\mathrm{Id}_{\varphi_i(\sigma)}} \circ F_i(\sigma, t(\mathrm{Id}_{\varphi_i(\sigma)})) \circ \left((\mathrm{id}_{T_\sigma \mathcal{O}_i}, T_\sigma \varphi_i) \wedge (\mathrm{id}_{T_\sigma \mathcal{O}_i}, T_\sigma \varphi_i) \right)$$

$$= \mathrm{pr}_2 \circ F_i(\sigma, t(\mathrm{Id}_{\varphi_i(\sigma)})) = \mathrm{pr}_2 \circ F_i(\sigma, \varphi_i(\sigma)) \equiv \mathrm{pr}_2 \circ F_i((\mathcal{F}_{\tau_i} \circ \varphi)(\sigma))$$

Thus,

$$\boxed{F_i[\varphi](\sigma) = \left(\mathcal{D}_1 A_i + [\cdot, \cdot]_E \circ (A_i \wedge A_i) \right) ((\mathcal{F}_{\tau_i} \circ \varphi)(\sigma))}$$

Prop. 3: OVER $\mathcal{O}_{ij} \ni \sigma'$, THE HIGGSED CURVATURE 2-FORMS ARE IDENTIFIED AS $F_i[\varphi](\sigma') = T_{\mathrm{Id}_{\varphi_j(\sigma')}} C_{\beta_{ij}(\sigma')} \circ F_j[\varphi](\sigma')$.

BY THE SAME TOKEN, $\forall\, \Phi \in \mathrm{Aut}(\mathcal{P})_{\mathrm{vert}}$ AS BEFORE:

$$F_i^{\Phi}[\varphi^{\Phi}](\sigma) = T_{\varphi_i(\sigma)} C_{\gamma_i(\sigma)} \circ F_i[\varphi](\sigma), \quad \sigma \in \mathcal{O}_i.$$

Proof: AGAIN, WE FOCUS ON THE SECOND IDENTITY, (26)

$$F_i^{\Phi}[\varphi^{\Phi}](\sigma) \equiv F_i^{\Phi}\Big(\sigma, t_*(\gamma_i(\sigma))(\varphi_i(\sigma))\Big) = T_{\mathrm{Id}_{\varphi_i(\sigma)}} C_{\gamma_i(\sigma)} \circ F_i(\sigma, \varphi_i(\sigma))$$

$$= T_{\mathrm{Id}_{\varphi_i(\sigma)}} C_{\gamma_i(\sigma)} \circ F_i[\varphi](\sigma) \quad \text{BY Prop. 2.} \quad \square$$

THUS, WE MAY CONTEMPLATE THE **YANG-MILLS-HIGGS ACTION FUNCTIONAL**!

$$\boxed{-\frac{1}{2g_{\mathrm{YMH}}^2} \kappa_E \circ \big(F_i[\varphi] \wedge *_\eta F_i[\varphi]\big)}$$

OVER SPACETIME (Σ, η)

AS LONG AS THERE EXISTS A $C(B)$-INVARIANT RIEMANNIAN METRIC κ_E ON E, $\forall \beta \in B$: $\kappa_E \circ (TC_\beta \otimes TC_\beta) = \kappa_E$.

REMARKS: (*) KOTOV & STROBL HAVE PUT A LOT of EFFORT IN DEFINING AN ADJOINT ACTION of $\mathcal{G}$ on ITSELF — WE DO NOT NEED IT!

(**) IT IS NOT CLEAR HOW TO DEAL with NONTRIVIAL BUNDLES in THIS APPROACH (POTENTIALLY OVER-CONSTRAINED SYSTEMS).

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OUR ANALYSIS SEEMS TO SUGGEST THE NECESSITY of FURTHER REDUCTION of THE GAUGE GROUP. SUCH REDUCTION HAS, INDEED, BEEN CONSIDERED IN 'PHYSICALLY' RELEVANT MODELS (ACTION & SYMPLECTIC GROUPOIDS)

ONE POSSIBILITY for A CONTROLLED REDUCTION, 28
WHICH OUGHT TO BE EXPLORED ANYWAY
(FOR MATHEMATICAL COMPLETENESS, BUT ALSO
TO ACCOUNT for THE PHYSICALLY NATURAL
SCENARIO of MANY SPECIES of MATTER FIELDS
CHARGED under A SINGLE GAUGE FIELD),
IS DESCRIBED by THE FOLLOWING ADAPTATION
of CARTAN'S MIXING PROCEDURE...

HOMEWORK: CONSIDER A LIE GROUPOID $\mathcal{G} \rightrightarrows M$, & ITS (LEFT) MODULE

$$\begin{tikzcd}
\mathcal{G} \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, phantom, "\overset{\lambda_X}{\circlearrowright}"] & & X \arrow[dl, "\mu_X"] \\
& M &
\end{tikzcd}$$

ACTION

MOMENT

1) FOR A GIVEN PRINCIPALOID $\mathcal{G}$-BUNDLE $\mathcal{G} \curvearrowright \mathcal{P} \to \Sigma$, with CONNECTION

$X \curvearrowright \Xi \to \Sigma$

DEFINE A FIBRE BUNDLE ASSOCIATED TO $\mathcal{P}$ by ACTION λ_X, IN ANALOGY with CARTAN'S MIXING CONSTRUCTION of A BUNDLE $P \times_\lambda Z$ ASSOCIATED with A PRINCIPAL G-BUNDLE P (FOR LIE GROUP G).

AS THE POINT of DEPARTURE TAKE A SUITABLE EXTENSION 30

$\mathcal{P}X$ of $\mathcal{P}$ by X, with TYPICAL FIBRE GIVEN by THE ARROW MANIFOLD of THE ACTION GROUPOID

$$\mathcal{G}_{\lambda_X}\ltimes X : \mathcal{G}_s\times_{\mu_X} X \Longrightarrow X.$$

IDENTIFY A PRINCIPAL ACTION of $\mathcal{G}_{\lambda_X}\ltimes X$ on $\mathcal{P}X$, WHICH ENABLES US TO IDENTIFY

$$\Xi \simeq \mathcal{P}X /\!\!/ \mathcal{G}_{\lambda_X}\ltimes X$$

GODEMENT QUOTIENT

2) FIND A CANONICAL HOMEOMORPHISM

$$\tilde{\mu}_X : \mathrm{Bisec}(\mathcal{G}\Rightarrow M) \longrightarrow \mathrm{Bisec}(\mathcal{G}_{\lambda_X}\ltimes X),$$

& USE IT TO INTERPRET THE PREVIOUS CONSTRUCTION (31) of THE PAIR $(\mathscr{P}X, \Xi)$ AS AN INSTANTIATION of THE CONSTRUCTION of A PAIR PRINCIPALOID-SHADOW BUNDLE. WHAT IS THE STRUCTURE GROUP of $(\mathscr{P}X, \Xi)$? HOW DOES IT RELATE to THAT of $(\mathscr{P}, \mathscr{F})$ via $\widetilde{\mu}_X$? FORMULATE YOUR CONCLUSION AS A STATEMENT of REDUCTION of STRUCTURE GROUP of PRINCIPALOID BUNDLE.

3) INDUCE A CONNECTION $\hat{\Theta}_X$ on $\mathscr{P}X$ COMPATIBLE with ACTION of $\mathcal{G}_{\lambda_X} \ltimes X$. DESCEND IT TO A CONNECTION Θ_X on Ξ, AND GIVE ITS LOCAL DESCRIPTION.

4) CONSIDER AN ACTION of THE EHRESMANN-ATIYAH GROUPOID $At(\mathcal{P})$ of $\mathcal{P}$ ON $\mathcal{P}X$ & DESCEND IT TO Ξ. USE IT TO DEFINE (REDUCED) AUTOMORPHISMS of BOTH BUNDLES. FIND CANONICAL GROUP HOMOMORPHISMS $\mathrm{Ext}_X : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P}X)$ & $\Xi_* : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\Xi)$

5) STUDY THE BEHAVIOUR of THE CONNECTION 1-FORMS $\widehat{\Theta}_X$ & Θ_X UNDER THE CORRESPONDING REDUCED GAUGE TRANSFORMATIONS.

6) UPON FINDING A CANONICAL MAP $\Upsilon_X : \Xi \to \mathcal{F}$, COLLECT YOUR FINDINGS ...

IN AN EXTENDED TRIDENT DIAGRAM

$$
\begin{tikzcd}
 & & \mathcal{P}X \arrow[dl] \arrow[dd] \arrow[dr] & & \\
 & \mathcal{E} \arrow[dd, "\gamma_x"'] & & X \arrow[dd] & \\
\mathrm{At}(\mathcal{P}) \arrow[dr, Rightarrow] & & \mathcal{P} \arrow[dl] \arrow[dr] & & \mathcal{G} \arrow[dl, Rightarrow] \\
 & \mathcal{F} & & M &
\end{tikzcd}
$$

F) POSTULATE A NATURAL GAUGE FIELD THEORY FOR MATTER FIELDS $\Gamma(\mathcal{E})$. CONTEMPLATE CONCEPTUAL CHALLENGES IN A DYNAMICAL DETERMINATION of THE CORRESPONDING LOCALISED GAUGE FIELD IN RELATION TO ITS CANONICAL HIGGSING. PROPOSE A MECHANISM WHICH CIRCUMNAVIGATES THE REEFS...