TRANSFORMATA FOURIERA FUNKCJI

„POTOCZNIE”

$$C^{\infty}(\mathbb{R};\mathbb{R}) \ni f \longmapsto \mathcal{F}f \ : \ \mathbb{R} \ni k \longmapsto \int_{-\infty}^{\infty} dx \begin{matrix} \sin(kx) \\ \cos(kx) \end{matrix} f(x)$$

TYPY KLASYCZNE :

OGÓLNIEJ : $\mathcal{F}f(k) = \int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} f(x)$

TYP I : f JEST (NAJCZĘŚCIEJ) WYRAŻENIEM WYMIERNYM

$$f(x) = \frac{P(x)}{Q(x)}$$

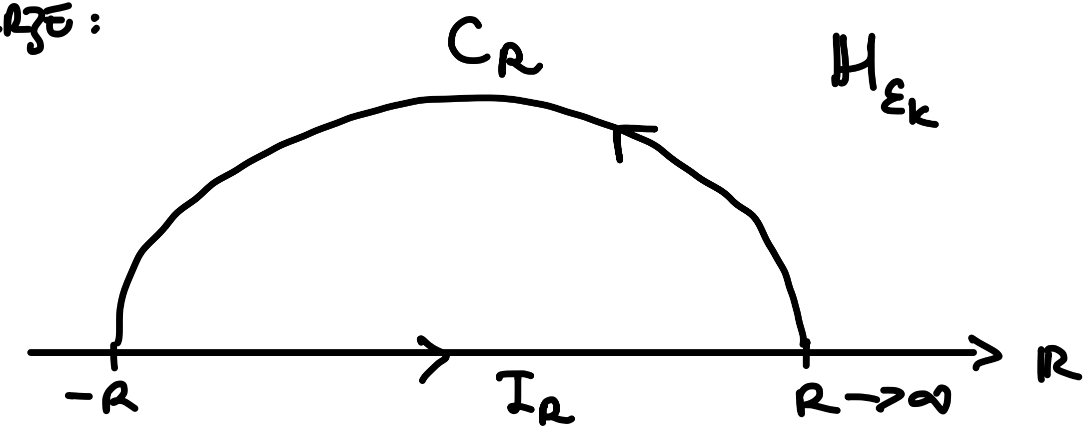
P, Q – F-CJE WIELOMIANOWE

$\deg Q \geq \deg P + 1$

ROZWAŻAMY $g(z) := e^{-2\pi i k z} f(z)$, $z \in \mathbb{H}_{\varepsilon_k}$, $\varepsilon_k = \begin{cases} + & \text{DLA } k<0 \\ - & \text{DLA } k>0 \end{cases}$

(RACHUNEK DLA $k=0$ – NIEZALEŻNIE)

NA KONTURZE:



WYBÓR OBSZARU, NA KTÓRYM ROZPATRUJEMY g, JEST PODYKTOWANY PRZEZ $\underline{\text{II}}$ LEMAT JORDANA (POTRZEBUJEMY CZYNNIKA UZBIEŻNIAJĄCEGO – TAKIM JEST

$$e^{-2\pi i k z} = \underbrace{e^{-2\pi i k x}}_{\text{OSCYLACJA}} \cdot \underbrace{e^{2\pi k y}}_{\text{TŁUMIENIE}}$$

$k>0 \Rightarrow y<0$ $(\mathbb{H}_-)$

$k<0 \Rightarrow y>0$ $(\mathbb{H}_+)$)

ZAKŁADAMY PRZY TYM, ŻE Q <u>NIE</u> MA PIERWIASTKÓW $\in \mathbb{R}$.
(PATRZ: NIŻEJ)

PRZYKŁADY :

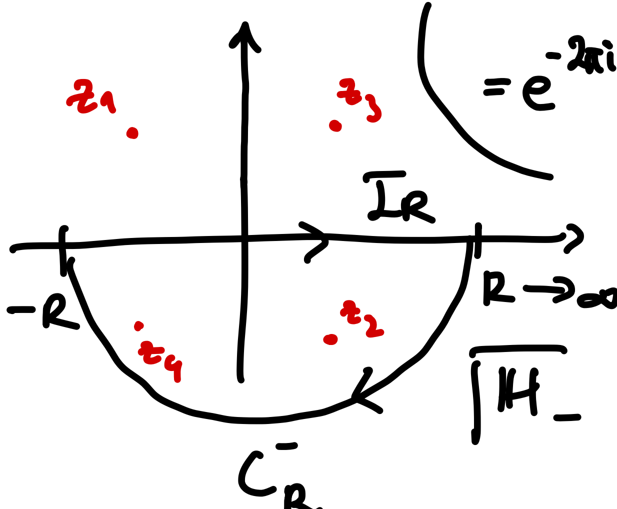
(i) $f(x) = \dfrac{1}{x^4+16}$ ← PARZYSTA, WIĘC $\cos(2\pi k x)\dfrac{1}{x^4+16}$

$$g(z) = e^{-2\pi i k z}\frac{1}{z^4+16} = e^{-2\pi i k z}\frac{1}{\left(z^2-4e^{i\frac{3\pi}{2}}\right)\left(z^2-4e^{i\frac{\pi}{2}}\right)}$$

$$= e^{-2\pi i k z}\frac{1}{\underbrace{\left(z-2e^{i\frac{3\pi}{4}}\right)}_{z-z_1}\underbrace{\left(z+2e^{i\frac{3\pi}{4}}\right)}_{z-z_2}\underbrace{\left(z-2e^{i\frac{\pi}{4}}\right)}_{z-z_3}\underbrace{\left(z+2e^{i\frac{\pi}{4}}\right)}_{z-z_4}}$$

$z_1 = \sqrt{2}(-1+i)$, $z_2 = \sqrt{2}(1-i)$, $z_3 = \sqrt{2}(1+i)$, $z_4 = -\sqrt{2}(1+i)$

$k > 0$:



$\overline{\mathbb{H}_-}$ $(\mathrm{Im}\, z < 0)$

(4)

$$\lim_{R\to\infty} \int_{I_R \cup C_R^-} dz\, g(z) = -2\pi i \left(\mathrm{Res}(g; z_2) + \mathrm{Res}(g; z_4) \right)$$

ORIENTACJA KONTURY !!!

$$= -2\pi i e^{-2\sqrt{2}\pi k} \left(\frac{e^{-i2\sqrt{2}\pi k}}{-4e^{i\frac{3\pi}{4}} \cdot \left(-8e^{i\frac{\pi}{2}}\right)} + \frac{e^{i2\sqrt{2}\pi k}}{8e^{i\frac{\pi}{2}} \cdot \left(-4e^{i\frac{\pi}{4}}\right)} \right)$$

$$= +\frac{\pi i}{16} e^{-2\sqrt{2}\pi k} \left(e^{i2\sqrt{2}\pi k} e^{-i\frac{3\pi}{4}} - e^{-i2\sqrt{2}\pi k} e^{-i\frac{5\pi}{4}} \right)$$

$$-i\frac{3\pi}{4} = \pi - \frac{\pi}{4}, \qquad \frac{5\pi}{4} = \pi + \frac{\pi}{4}$$

$$= -\frac{\pi i}{16} e^{-2\sqrt{2}\pi k} \frac{e^{i\left(2\sqrt{2}\pi k + \frac{\pi}{4}\right)} - e^{-i\left(2\sqrt{2}\pi k + \frac{\pi}{4}\right)}}{2i} \cdot 2i$$

$$+\frac{\pi}{8} e^{-2\sqrt{2}\pi k} \sin\left(2\sqrt{2}\pi k + \frac{\pi}{4}\right) =$$

JEST ZATEM

$$\int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} \frac{1}{x^4+16} + \lim_{R\to\infty} \underbrace{\int_0^{-\pi} d\varphi\, i R e^{i\varphi} e^{-2\pi i k R(\cos\varphi + i\sin\varphi)} \frac{1}{R^4 e^{i4\varphi}+16}}_{I_R}$$

$$\hookrightarrow + \frac{\pi}{8} e^{-2\sqrt{2}\pi k} \sin\left(2\sqrt{2}\pi k + \frac{\pi}{4}\right)$$

SZACUJEMY :

$$|I_R| \le \frac{R}{R^4-16} \left| \int_0^{-\pi} d\varphi\, e^{2\pi k R \sin\varphi} \right| \overset{\psi=-\varphi}{=} \frac{R}{R^4-16} \int_0^{\pi} d\psi\, e^{-2\pi k R \sin\psi}$$

$(R \gg 1!)$

$$= \frac{2R}{R^4-16} \int_0^{\pi/2} d\psi\, e^{-2\pi k R \sin\psi}$$

, ALE $\psi \le \frac{\pi}{2} \Rightarrow \sin\psi \ge \frac{2}{\pi}\psi$

$\Downarrow$

$-\sin\psi \le -\frac{2}{\pi}\psi$

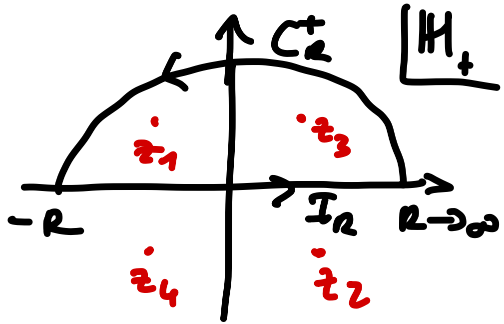
$$\leq \frac{2R}{R^4-16}\int_0^{\pi/2} d\varphi\, e^{-4kR\varphi} = \frac{2R}{R^4-16}\left(-\frac{1}{4kR}\right) e^{-4kR\varphi}\Big|_{\varphi=0}^{\varphi=\pi/2}$$

$$= \frac{1}{2k(R^4-16)}\left(1-e^{-2\pi kR}\right) \xrightarrow[R\to\infty]{} 0$$

<u>WNIOSEK</u>: $\mathcal{F}f(k) = +\frac{\pi}{8} e^{-2\sqrt{2}\pi k}\sin\left(2\sqrt{2}\pi k + \frac{\pi}{4}\right)$ DLA $k>0$

ANALOGICZNY RACHUNEK DLA $k<0$ PRZEPROWADZAMY

NA KONTURZE



$\mathbb{H}_+$ $(\operatorname{Im} z>0)$

$$\lim_{R\to\infty}\int_{I_R\cup C_R^+} dz\, g(z) = +2\pi i\left(\mathrm{Res}(g; z_1) + \mathrm{Res}(g; z_3)\right)$$

$$= \mathcal{F}f(k)$$

SZACOWANIE PODOBNE JAK POPRZEDNIO

$$= +2\pi i e^{2\sqrt{2}\pi k}\left(\frac{e^{i2\sqrt{2}\pi k}}{4e^{i\frac{3\pi}{4}}\cdot\left(-8e^{i\frac{\pi}{2}}\right)} + \frac{e^{-i2\sqrt{2}\pi k}}{8e^{i\frac{\pi}{2}}\cdot 4e^{i\frac{\pi}{4}}}\right)$$

$$= -\frac{\pi i}{16}e^{2\sqrt{2}\pi k}\left(e^{i2\sqrt{2}\pi k}e^{-i\frac{5\pi}{4}} - e^{-i2\sqrt{2}\pi k}e^{-i\frac{3\pi}{4}}\right)$$

$$= -\frac{\pi}{8}e^{2\sqrt{2}\pi k}\sin\left(2\sqrt{2}\pi k - \frac{\pi}{4}\right) = \frac{\pi i}{16}e^{2\sqrt{2}\pi k}\,\frac{e^{i\left(2\sqrt{2}\pi k - \frac{\pi}{4}\right)} - e^{-i\left(2\sqrt{2}\pi k - \frac{\pi}{4}\right)}}{2i}\cdot 2i$$

$$\stackrel{k}{=} \frac{\pi}{8}e^{-2\sqrt{2}\pi(-k)}\sin\left(2\sqrt{2}\pi(-k) + \frac{\pi}{4}\right)$$

WNIOSEK: $\mathcal{F}f(k) = \frac{\pi}{8} e^{-2\sqrt{2}\pi(-k)} \sin\left(2\sqrt{2}\pi(-k) + \frac{\pi}{4}\right)$ DLA $-k > 0$ (8)

WARTOŚĆ TRANSFORMATY W $k=0$ MOŻEMY POLICZYĆ WPROST (TEN SAM KONTUR DLA $g(z) = \frac{1}{z^4+16}$) LUB POPRZEZ $\lim_{k \to 0}$.

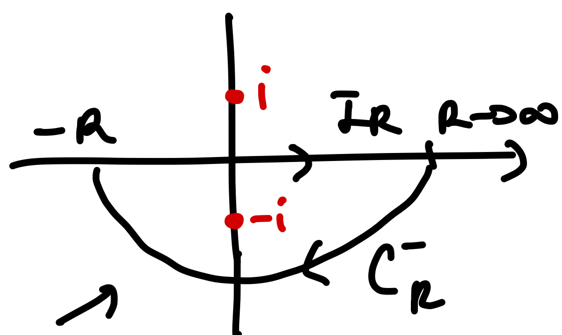
WYNIK:

$$\boxed{\mathcal{F}f(k) = \frac{\pi}{8} e^{-2\sqrt{2}\pi|k|} \sin\left(2\sqrt{2}\pi|k| + \frac{\pi}{4}\right)}$$

(ii) $f(x) = \dfrac{x}{x^2+1}$ ← NIEPARZYSTA, więc $\sim \sin(2\pi kx)\,\dfrac{x}{x^2+1}$

$$g(z) = e^{-2\pi i k z}\,\frac{1}{(z+i)(z-i)}$$

$k>0$:



ANTY-ORIENTACJA!

$$\int_{-\infty}^{\infty} dx\,\frac{e^{-2\pi i k x}}{x^2+1} + \lim_{R\to\infty}\int_{C_R^-} dz\, g(z) = -2\pi i \,\mathrm{Res}(g; -i)$$

$$= -2\pi i\, e^{-2\pi k}\,\frac{1}{-2i} = +\pi e^{-2\pi k}$$

SZACUJEMY:

$$\left|\int_{C_R^-} dz\, g(z)\right| \leq \frac{R}{R^2-1}\left|\int_0^{-\pi} d\varphi\, e^{-2\pi i k R(\cos\varphi + i\sin\varphi)}\right|$$

$$\leq \frac{R}{R^2-1}\int_0^{\pi} d\psi\, e^{-2\pi k R \sin\psi} \qquad (\psi = -\varphi)$$

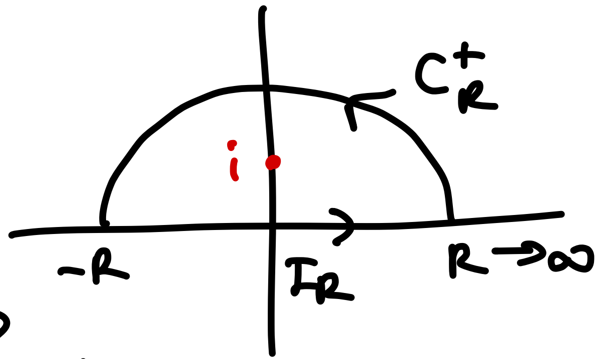
$$\Big\downarrow R\to\infty \quad \text{J/W}$$

$$0$$

WNIOSEK: $\mathcal{F}f(k) = \pi e^{-2\pi k}$ DLA $k>0$

ANALOGICZNIE:

$k<0$



ORIENTACJA!

$$\int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} \frac{1}{x^2+1} + \lim_{R\to\infty} \int_{C_R^+} dz\, g(z) = 2\pi i \operatorname{Res}(g;i) = 2\pi i\, e^{2\pi k} \frac{1}{2i} = \pi e^{2\pi k}$$

$\lim_{R\to\infty} \int_{C_R^+} dz\, g(z) = 0$ J/W

WNIOSEK: $\mathcal{F}f(k) = \pi e^{-2\pi(-k)}$ DLA $-k>0$

$$\mathcal{F}f(0) = \pi \quad \left(\text{WPROST LUB } \lim_{k \to 0} \right)$$

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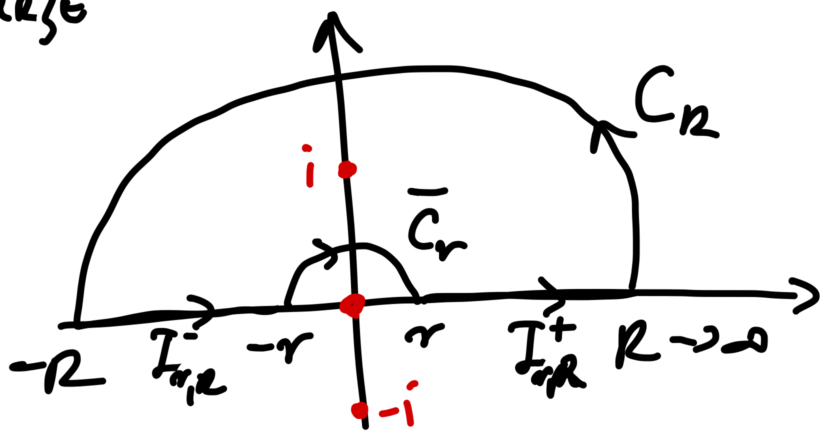
WYNIK: $$\mathcal{F}f(k) = \pi e^{-2\pi|k|}$$

ZAGADNIENIE POKREWNE

$$I = \int_{-\infty}^{\infty} dx \, \frac{\sin(\pi x)}{x^3 - 3x^2 + 4x - 2} = ?$$

ROZWAŻAMY $\quad g(z) = e^{i\pi z} \dfrac{1}{z(z^2+1)}$

NA KONTURZE



: $\Gamma_{r,R}$

ISTOTNIE,

$$I = -\int_{-\infty}^{\infty} dx \, \frac{\sin \pi(x-1)}{(x-1)\left[(x-1)^2+1\right]} = -\int_{-\infty}^{\infty} dy \, \frac{\sin(\pi y)}{y(y^2+1)}$$

$$= -\frac{1}{2i} \lim_{\substack{R\to\infty \\ r\to 0^+}} \left(\int_{-R}^{-r} + \int_{r}^{R} \right) dy \, \frac{e^{i\pi y} - e^{-i\pi y}}{y(y^2+1)} = -\frac{1}{i} \lim_{\substack{R\to\infty \\ r\to 0^+}} \left(\int_{-R}^{-r} + \int_{r}^{R} \right) dy \, \frac{e^{i\pi y}}{y(y^2+1)}$$

$$= -\frac{1}{i} \lim_{\substack{R\to\infty \\ r\to 0^+}} \int_{I_{r,R}^+ \cup I_{r,R}^-} dz \, g(z) = i \lim_{\substack{R\to\infty \\ r\to 0^+}} \left(\int_{\Gamma_{r,R}} - \int_{\bar{C}_r} - \int_{C_R} \right) dz \, g(z)$$

$$= i \lim_{\substack{R\to\infty \\ r\to 0^+}} \left(2\pi i \operatorname{Res}(g; i) - \int_{\bar{C}_r} dz \, g(z) - \int_{C_R} dz \, g(z) \right)$$

$$= -2\pi \,\mathrm{Res}(g;i) - i \lim_{r\to 0^+} \int_{\bar{C}_r} dz\, g(z) - i \lim_{R\to\infty} \int_{C_R} dz\, g(z)$$

szacujemy :

$$\left|\int_{C_R} dz\, g(z)\right| \le \frac{R}{R(R^2-1)} \int_0^{\pi} d\varphi \left|e^{i\pi R(\cos\varphi + i\sin\varphi)}\right|$$

$$= \frac{1}{R^2-1}\int_0^{\pi} d\varphi\, e^{-\pi R\sin\varphi} = \frac{2}{R^2-1}\int_0^{\pi/2} d\varphi\, e^{-\pi R\sin\varphi}$$

$$\le \frac{2}{R^2-1}\int_0^{\pi/2} d\varphi\, e^{-2R\varphi} = \frac{1}{R(R^2-1)}\left(1-e^{-\pi R}\right) \xrightarrow[R\to\infty]{} 0$$

OBLICZAMY

$$\lim_{r\to 0^+}\int_{\overline{C}_r} dz\, g(z) = \lim_{r\to 0^+}\int_{\overline{C}_r} dz\left[\overbrace{\frac{e^{i\pi z}-1}{z(z^2+1)}}^{=:\,h(z)} + \frac{1}{z(z^2+1)}\right] = *$$

REGULARNE w $z=0$

„WYŁOWIONA" OSOBLIWOŚĆ

$$\left|\int_{\overline{C}_r} dz\, h(z)\right| \le r\int_0^\pi d\varphi\, \left|h(re^{i\varphi})\right| \le r\max_{\varphi\in[0,\pi]}\left|h(re^{i\varphi})\right|\pi \xrightarrow[r\to 0^+]{} 0$$

$\varphi \mapsto \left|h(re^{i\varphi})\right|$ ciągła i regularna na zwartym $[0,\pi]$

$\Downarrow$

ograniczona

$$* = \lim_{r\to 0^+}\int_{\overline{C}_r} dz\, \frac{1}{z(z^2+1)} = \lim_{r\to 0^+}\int_{\overline{C}_r} dz\, \frac{1}{z}\left[\left(\frac{1}{z^2+1}-1\right)+1\right]$$

$$= \lim_{r\to 0^+} \int_{\overline{C}_r} dz\, \frac{1}{z}\left(\frac{-z^2}{1+z^2}+1\right) = \lim_{r\to 0^+} \int_{\overline{C}_r} dz\, \frac{-z}{1+z^2} + \lim_{r\to 0^+} \int_{\overline{C}_r} dz\, \frac{1}{z} = *$$

$$\left|\int_{\overline{C}_r} dz\, \frac{-z}{1+z^2}\right| \le r\int_0^{\pi} d\varphi\; r\,\frac{1}{|r^2 e^{i2\varphi}+1|} = \int_0^{\pi} d\varphi\, \frac{1}{|r^{-2}+e^{i2\varphi}|}$$

$$|r^{-2}+e^{i2\varphi}| \ge r^{-2}-1$$

$$r \approx 0$$

$$\le \frac{1}{r^{-2}-1}\int_0^{\pi} d\varphi = \frac{r^2}{1-r^2}\pi \xrightarrow[r\to 0]{} 0$$

OSTATECZNIE JATEM $\quad * = \lim_{r\to 0^+}\int_{\overline{C}_r} dz\,\frac{1}{z} = i\int_{\pi}^{0} d\varphi\, re^{i\varphi}\frac{1}{re^{i\varphi}} = -i\pi$

JEST JASNE, ŻE TO

$$I = -2\pi \operatorname{Res}(g; i) - i(-i\pi) = -\pi\left(1 + 2\operatorname{Res}(g; i)\right)$$

$$\operatorname{Res}(g; i) = \lim_{z \to i} \frac{(z-i)\, e^{i\pi z}}{z(z^2+1)} = \frac{e^{-\pi}}{i \cdot 2i} = -\frac{e^{-\pi}}{2},$$

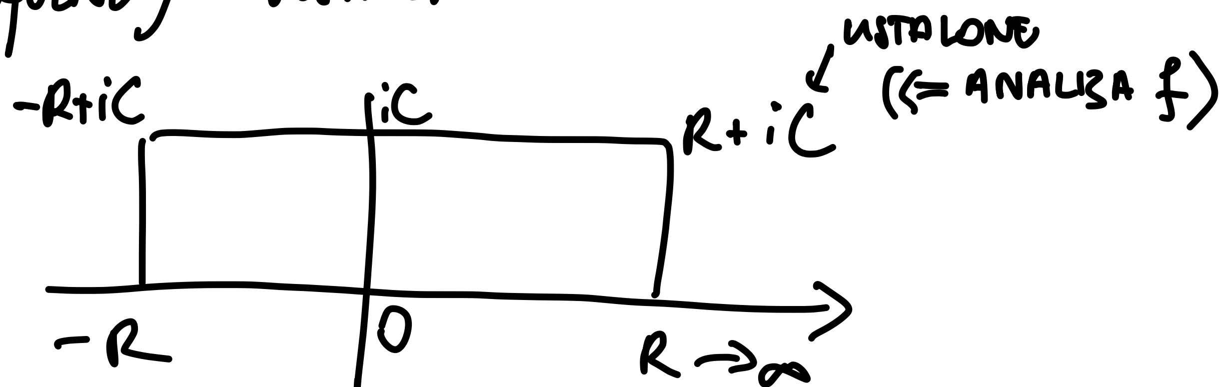
WIĘC TEŻ

$$\boxed{I = \pi\left(e^{-\pi} - 1\right)}$$

TYP II : f WYRAŻA SIĘ ZASADNICZO PRZEZ exp:

WYRAŻENIA WYMIERNE W e^x, e^{-x}; WYMIERNE OD F-CJI HIPERBOLICZNYCH I POCHODNE (PATRZ: OSTATNIE ZAJĘCIA)

ROZWAŻAMY STOSOWNE F-CJE MEROMORFICZNE WEWNĄTRZ KONTURÓW OGÓLNEJ POSTACI



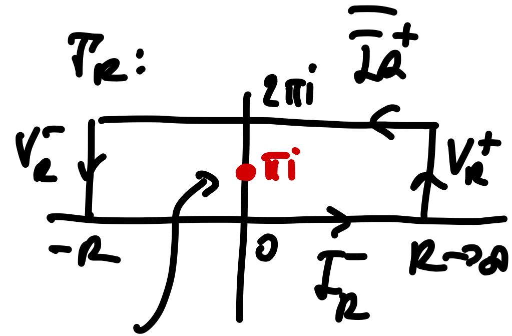
LUB EW. Z „WCIĘCIAMI” OMIJAJĄCYMI BIEGUNY (NA KONTURZE)

PRZYKŁAD (WIĘCEJ: NA NAGRANIU)

$$f(x) = \frac{x e^{ax}}{1+e^x} \qquad 0<a<1 \quad \rightsquigarrow \quad \mathcal{F}f = ?$$

ROZWAŻAMY $g(z) = \dfrac{z e^{az} e^{-2\pi i k z}}{1+e^z}$ NA

Γ_R:



$$1+e^z = 0 \iff e^z = -1 \iff e^x e^{iy}$$

$$x=0 \wedge y \in (2k+1)\pi$$

$$\lim_{R\to\infty} \int_{\Gamma_R} dz\, g(z) = 2\pi i \operatorname{Res}(g; \pi i)$$

ORIENTACJA

$$= \mathcal{F}f(k) - \int_{-\infty}^{\infty} dx \, \frac{(x+2\pi i) e^{ax} e^{-2\pi i k x} \cdot e^{2\pi i a} e^{4\pi^2 k}}{1+e^x} + \lim_{R\to\infty} \int_{V_R^+ \cup V_R^-} dz\, g(z)$$

$$= \mathcal{F}f(k)\left(1 - e^{2\pi i a}\cdot e^{4\pi^2 k}\right) - 2\pi i\, e^{2\pi i a} e^{4\pi^2 k}\mathcal{F}h(k)$$ 21

$$+ \lim_{R\to\infty} \int_{V_R^+ \cup V_R^-} dz\, g(z)$$, gdzie $h(x) = \dfrac{e^{ax}}{1+e^x}$

Szacujemy:

$$\left|\int_{V_R^+} dz\, g(z)\right| \leq \int_0^{2\pi} dy \frac{|R+iy|\, e^{aR} \left|e^{-2\pi i k(R+iy)}\right|}{\left|1+e^R\cdot e^{iy}\right|}$$

$$\leq \frac{R+2\pi}{e^R-1}\cdot e^{aR}\int_0^{2\pi} dy\, e^{2\pi k y} \xrightarrow[R\to\infty]{\overbrace{a<1}} 0$$

$$\left|\int_{V_R^-} dz\, g(z)\right| \leq \int_0^{2\pi} dy \frac{R+2\pi}{e^{-R}(e^R-1)} e^{-aR} e^{2\pi k y} = \frac{R+2\pi}{e^{aR}}\cdot\frac{1}{1-e^{-R}}\int_0^{2\pi} dy\, e^{2\pi k y} \xrightarrow[R\to\infty]{} 0$$

Liczymy

$$\operatorname{Res}(g;\pi i) = \lim_{z\to\pi i} \frac{z(z-\pi i)\, e^{az} e^{-2\pi i k z}}{1+e^{z}}$$

$$= \pi i\, e^{\pi a i} e^{2\pi^2 k} \lim_{\xi\to 0} \frac{\xi}{1+e^{\xi}\cdot e^{\pi i}}$$

$$= -\pi i\, e^{\pi a i} e^{2\pi^2 k} \frac{1}{\lim\limits_{\xi\to 0} \frac{e^{\xi}-e^{0}}{\xi-0}} = -\pi i\, e^{\pi a i} e^{2\pi^2 k} \frac{1}{e^{\xi}\big|_{\xi=0}}$$

$$= -\pi i\, e^{\pi a i} e^{2\pi^2 k}$$

JEST PRZETO

$$\mathcal{F}f(k)\left(1-e^{2\pi i a}\cdot e^{4\pi^2 k}\right) = 2\pi^2 e^{\pi a i} e^{2\pi^2 k} + 2\pi i\, e^{2\pi i a} e^{4\pi^2 k}\,\mathcal{F}h(k)$$

$$= e^{\pi(2\pi k+ia)}$$

$$\|$$

$$-2\mathcal{F}f(k)\, e^{\pi i a} e^{2\pi^2 k}\,\mathrm{sh}(2\pi^2 k+\pi i a)$$

$$\Updownarrow$$

$$\mathcal{F}f(k)\cdot\mathrm{sh}(2\pi^2 k+\pi i a) = -\pi^2 - \pi i\, e^{\pi i a} e^{2\pi^2 k}\,\mathcal{F}h(k)$$

ALE $\mathrm{sh}(\pi\xi)=0 \iff e^{2\pi\xi}=1 \iff \begin{cases} x=0\\ y\in\mathbb{Z}\end{cases}$, ZATEM ODWRACALNE

$\xi = x+iy$, $\quad e^{2\pi\xi} = e^{2\pi x}\cdot e^{i2\pi y}$

POZOSTAJE POLICZYĆ $Fh(k)$ ...

ROZWAŻAMY $\quad l(z) = \dfrac{e^{az}}{1+e^{z}} e^{-2\pi i k z}$ NA Γ_R JAK POPRZEDNIO

$$\lim_{R\to\infty} \int_{\Gamma_R} dz\, l(z) = 2\pi i \operatorname{Res}(l; \pi i)$$

$$= Fh(k)\left(1 - e^{2\pi i a} e^{4\pi^2 k}\right) + \lim_{R\to\infty} \int_{V_R^+ \cup V_R^-} dz\, l(z)$$

SZACUJEMY:

$$\left| \int_{V_R^+} dz\, l(z) \right| \leq \int_0^{2\pi} dy \frac{e^{aR} e^{2\pi k y}}{e^R - 1} \xrightarrow[R\to\infty]{} 0$$

$$\left| \int_{V_R^-} dz\, l(z) \right| \xrightarrow[R\to\infty]{} 0$$

$$-2\,\mathcal{F}h(k)\,e^{\pi i a}e^{2\pi^2 k}\,\mathrm{sh}(2\pi^2 k+i\pi a) = 2\pi i\cdot\left(-e^{\pi a i}\cdot e^{2\pi^2 k}\right)$$

$$\Downarrow$$

$$\mathcal{F}h(k) = \frac{\pi i}{\mathrm{sh}(2\pi^2 k+i\pi a)}$$

$$\Downarrow$$

$$\mathcal{F}f(k) = -\frac{\pi^2 + \pi i\, e^{\pi i a} e^{2\pi^2 k}\cdot \pi i\,\dfrac{1}{\mathrm{sh}(2\pi^2 k+i\pi a)}}{\mathrm{sh}(2\pi^2 k+i\pi a)}$$

$$= \pi^2\,\frac{e^{\pi i a}e^{2\pi^2 k} - \mathrm{sh}(2\pi^2 k+i\pi a)}{\mathrm{sh}^2(2\pi^2 k+i\pi a)} = \pi^2\,\frac{\mathrm{ch}(2\pi^2 k+i\pi a)}{\mathrm{sh}^2(2\pi^2 k+i\pi a)}$$