CLASSICAL FIELD THEORY (interaction at a distance)

In the last episode... (1)

Our considerations gave us local data:

$$(\theta_i, \chi_{ij}, \nu_{ijk}) \in \Omega^1(\mathcal{O}_i) \times C^\infty(\mathcal{O}_{ij}, \mathbb{R}) \times \mathbb{R}$$

$\mathcal{O}_i \in \mathcal{T}(\mathcal{P}_L)$, $\mathcal{O}_{ij} = \mathcal{O}_i \cap \mathcal{O}_j$ etc.

for the (pre) symplectic form $\Omega_L \in \Omega^2(\mathcal{P}_L)$ over the space of states $\mathcal{P}_L$ of the lagrangean mechanics with the lagrangian L.

Upon introducing 'local wave functions'

$$\psi_i : \mathcal{O}_i \to \mathbb{C},$$

we found consistency conditions

$$\psi_i \xmapsto[\theta_i \mapsto \theta_i + \delta f_i]{} e^{i f_i} \psi_i$$

or

$$\psi_j(x) = e^{i\chi_{ij}(x)} \psi_i(x)$$

for $x \in \mathcal{O}_{ij}$, whence

also

$$\nu_{ijk} \overset{!}{\in} \mathbb{Z}$$

$$\Downarrow$$

$$\mathrm{Per}(\Omega_L) \subset 2\pi\mathbb{Z}$$

(Dirac's quantisation cond$^{\underline{n}}$)

Thus, we may write - ②

for $g_{ij} := e^{-i\chi_{ij}} \in C^\infty(\mathcal{O}_{ij}, U(1))$ -

$$\begin{cases} \Omega_L \restriction_{\mathcal{O}_i} = \delta\theta_i & (\text{PB1}) \\ (\theta_j - \theta_i)\restriction_{\mathcal{O}_{ij}} = i\,\delta \log g_{ij} & (\text{PB2}) \\ (g_{jk} \cdot g_{ik}^{-1} \cdot g_{ij})\restriction_{\mathcal{O}_{ijk}} = 1 & (\text{PB3}) \end{cases}$$

LOCAL PRESENTATION OF A PRINCIPAL $\mathbb{C}^\times$ - BUNDLE over $\mathcal{P}_L$ ($\mathbb{C}^\times \equiv \mathbb{C} \setminus \{0\}$ is a (LIE) GROUP!)

In this context, the ψ_i acquire the interpretation of local sections of a line bundle over $\mathcal{P}_L$ associated with the aforementioned principal $\mathbb{C}^\times$-bundle.

We shall, next, geometrise these objects & subsequently ~~abstract~~ abstract from our analysis the fundamental notion of a FIBRE BUNDLE ...
(to be employed in later considerations)

First, we geometrise the data $(\mathcal{O}_i, g_{ij})$. 3

Consider $\mathcal{O}_i \times \mathbb{C}^\times \ni (x, z_i)$

$\leftarrow$ ~~global~~ local coordinate on $\mathbb{C}^\times$ (induced from $\mathbb{C} \supset \mathbb{C}^\times$), carrying the label of $\mathcal{O}_i$ to distinguish $z_i \in \mathbb{C}^\times$ over $\mathcal{O}_i$ from $z_j \in \mathbb{C}^\times$ over $\mathcal{O}_j$

& ($\equiv$ AND) define local 1-forms

$$\theta\,\cancel{A}(x, z_i) := i\,\frac{\delta z_i}{z_i} + \theta_i(x)$$

on $\mathcal{O}_i \times \mathbb{C}^\times$.

At points $x \in \mathcal{O}_{ij}$,

we have a redundancy: we may choose

$\mathcal{O}_{ij} \times \mathbb{C}^\times \ni (x, z_i)$ or $\ni (x, z_j)$

Can we have a globally smooth

$\theta\,\cancel{A} \in \Omega^1\left(\bigsqcup_{i \in I} \mathcal{O}_i \times \mathbb{C}^\times\right)$ that does <u>not</u> depend on this (arbitrary) choice?

Impose:

$$i\,\frac{\delta z_i}{z_i} + \theta_i(x) \overset{!}{=} i\,\frac{\delta z_j}{z_j} + \theta_j(x) \quad (\text{at } x \in \mathcal{O}_{ij})$$

to obtain the consistency condition:

$$(\theta_j - \theta_i)(x) = i\,\delta \log \frac{z_i}{z_j}, \text{ which}$$

$$\underset{(PB2)}{=} i\,\delta \log g_{ij}(x)$$

is solved by

$$z_i = g_{ij}(x)\, z_j \qquad (G)$$

as the gluing condition for 'fibres' ($\simeq \mathbb{C}^\times$) over a given point $x \in \mathcal{O}_{ij}$.

Thus, we may pass from the redundant

$$\bigsqcup_{i \in I} \mathcal{O}_i \times \mathbb{C}^\times$$

~~[illegible]~~ (multiple 'fibres' over points in multiple intersections)

to the non-redundant

$$\mathcal{L} := \Big(\bigsqcup_{i \in I} \mathcal{O}_i \times \mathbb{C}^\times \Big) \Big/ \sim_{g_{\cdot\cdot}}$$

← we identify points in $\bigsqcup_{i \in I} \mathcal{O}_i \times \mathbb{C}^\times$ according to the prescription (G)

$$\cup\!\!\mathsf{I}$$

with $[(x, i, z_i)]_{\sim g_{\cdot\cdot}}$

$$|||$$

$[(x, j, g_{ji}(x) z_i)]_{\sim g_{\cdot\cdot}}$ for $x \in \mathcal{O}_{ij}$ (recall: we chose $g_{ji}(x) = g_{ij}(x)^{-1}$)

We shall demonstrate shortly that the crucial feature of the g_{ij} that enables us to equip $\mathcal{L}$ with the structure of a differentiable manifold is the 1-COCYCLE CONDITION (PB3) (p. 2)

Heuristically, we see it as the consistency condition (at $x \in \mathcal{O}_{ijk}$): (5)

$$\underbrace{g_{ik}(x)}\, z_k = z_i = g_{ij}(x)\, z_j = \underbrace{g_{ij}(x)\, g_{jk}(x)}\, z_k$$
$$g_{ik}(x) \overset{!}{=} g_{ij}(x)\, g_{jk}(x)$$

<u>Notabene</u>: ~~...~~ We have (1) a surjective submersion (both $\pi_{\mathcal{L}}$ & $T\pi_{\mathcal{L}}$ are surjective):

$$\pi_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathcal{P}_L$$
$$[(x, i, z_i)]_{\sim_{g..}} \longmapsto x$$

(2) ~~...~~ local trivialisations:

$$\mathcal{L}\restriction_{\mathcal{O}_i} \cong \mathcal{O}_i \times \mathbb{C}^\times$$

restriction to $\mathcal{O}_i$

TYPICAL FIBRE

i.e., $\mathcal{L}$ locally 'looks like' the cartesian product of a neighbourhood $\mathcal{O}_i$ in the BASE $\mathcal{P}_L$ & the TYPICAL FIBRE $\mathbb{C}^\times$.

(3) smooth transition maps

$$g_{ij} : \mathcal{O}_{ij} \longrightarrow \mathbb{C}^\times \qquad \left(\mathcal{O}_{ij} \to U(1) \to \mathbb{C}^\times \right)$$

$U(1) = \{ u \in \mathbb{C} \mid |u| = 1 \}$

(4) globally smooth connection 1-form $A_L \in \Omega^1(\mathscr{L})$ ⑥

given by

$$A_L([(x,i,z_i)]_{\sim g_{..}}) := \theta(x,z_i) \quad ,$$

with the property

$$\delta A_L([(x,i,z_i)]_{\sim g_{..}}) \equiv d\theta(x,z_i)$$

$$= i\, d^2 \log z_i + \delta\theta_i(x) = \delta\theta_i(x)$$

$$\equiv \Omega_L(x) \equiv (\pi_{\mathscr{L}}^* \Omega_L)([(x,i,z_i)]_{\sim g_{..}}) \quad ,$$

or simply : $\boxed{\pi_{\mathscr{L}}^* \Omega_L = \delta A_L}$

In this context, we call Ω_L the <u>CURVATURE</u> of the connection A_L on $\mathscr{L}$.

We shall, now, ~~abstract~~ from the technical detail of our 'geometrisation' $\mathscr{L}$ of (θ_i, g_{ij}) & subsequently verify that our abstraction gives us back such data (locally) ...

In so doing, we modify the geometric description ever so slightly to have a structure amenable to generalisations

We start with a SURJECTIVE SUBMERSION (7)

$$\begin{tikzcd} YB \arrow[d, twoheadrightarrow, "\pi_{YB}"] \\ \text{BASE} \to B \end{tikzcd} \qquad \left(\text{previously } \begin{tikzcd} \bigsqcup_{i \in I} O_i \times \mathbb{C}^\times \arrow[d, "pr_1"] \ni (x, i, z_i) \\ P_L \ni x \end{tikzcd} \right)$$

over a manifold B (previously: P_L) on which there exists $\underset{(2)}{F} \in Z^2_{dR}(B)$ (closed 2-form) with $\mathrm{Per}(F) \subset 2\pi\mathbb{Z}$

On YB, we request existence of $\underset{(1)}{A} \in \Omega^1(YB)$ such that

$$\pi_{YB}^* \underset{(2)}{F} = d \underset{(1)}{A}$$

The set $\pi_{YB}^{-1}(\{x\}) =: Y_x B$ is termed the FIBRE of the surjective submersion (previously, the fibre was $\mathbb{C}^\times$)

We want to compare the values of $\underset{(1)}{A}$ of points in YB mapped to the SAME point in the base B, in a meaningful manner...

Previously, we compared the ~~Θ~~ of

$$pr_1(x,i,z_i) \equiv \overset{\mathcal{O}_{ij}}{x} = pr_1(x,j,z_j),$$

therefore, we try the <u>fibred product</u>

$$\begin{tikzcd} YB \times_B YB \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\circlearrowright"] & YB \arrow[d, "\pi_{YB}"] \\ YB \arrow[r, "\pi_{YB}"'] & B \end{tikzcd}$$

fixed as a submanifold in $YB \times YB$ by the commutativity of this diagram, i.e.,

$$YB \times_B YB := \{ (y_1,y_2) \in YB \times YB \mid \pi_{YB}(y_2) = \pi_{YB}(y_1) \}$$

More generally, given maps

$$f_i : Y_i \longrightarrow X \quad , \; i \in \{1,2\},$$

we define $\subset Y_1 \times Y_2$ the FIBRED PRODUCT

$$\begin{tikzcd} Y_1 \times_X Y_2 \equiv Y_1 \, {}_{f_1}\!\times_{f_2} Y_2 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\circlearrowright"] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

$\parallel$

$\{ (y_1,y_2) \in Y_1 \times Y_2 \mid f_2(y_2) = f_1(y_1) \}$

We shall need these...

We now impose

over $YB \times_B YB \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} YB$

the 'gluing' condition:

$$\exists\, g_{(0)} \in C^\infty(YB \times_B YB, U(1)) :$$

$$\underbrace{pr_2^* A_{(1)} - pr_1^* A_{(1)} \stackrel{!}{=} i\,d\log g_{(0)}}_{\in\ \Omega^1(YB \times_B YB)}$$

This line of thought can be extended naturally to encode geometrise the 1-COCYCLE condition:

Over $YB \times_B YB \times_B YB \equiv \{ (y_1, y_2, y_3) \in YB \times YB \times YB \mid \pi_{YB}(y_1) = \pi_{YB}(y_2) = \pi_{YB}(y_3) \}$

$$\begin{tikzcd}
 & YB \times_B YB \times_B YB \arrow[dl, "pr_1"'] \arrow[d, "pr_2"] \arrow[dr, "pr_3"] & \\
YB \arrow[r, phantom, "\circlearrowright"] \arrow[dr, "\pi_{YB}"'] & YB \arrow[r, phantom, "\circlearrowright"] \arrow[d, "\pi_{YB}"] & YB \arrow[dl, "\pi_{YB}"] \\
 & B &
\end{tikzcd}$$

we postulate require the coherence condition:

$$pr_{1,2}^* g_{(0)} \underset{\text{pointwise product}}{\cdot} pr_{2,3}^* g_{(0)} \stackrel{!}{=} pr_{1,3}^* g_{(0)}$$

where

$$YB \times_B YB \times_B YB \xrightarrow{\ pr_{i,j}\ } YB \times_B YB$$

is the canonical projection

$$pr_{i,j}(y_1, y_2, y_3) := (y_i, y_j)\,, \quad \text{e.g.,}$$

$$pr_{1,2}(y_1, y_2, y_3) = (y_1, y_2) \ \ldots$$

&

Denote

$$Y^{[n]}B := \underbrace{YB \times_B YB \times_B \cdots \times_B YB}_{n \text{ times}} \equiv \{ (y_1, y_2, \ldots, y_n) \in YB^{\times n} \mid \pi_{YB}(y_1) = \pi_{YB}(y_2) = \ldots = \pi_{YB}(y_n) \}$$

Altogether, we obtain the geometric object

$$pr_{1,2}^* \underset{(0)}{g} \cdot pr_{2,3}^* \underset{(0)}{g} = pr_{1,3}^* \underset{(0)}{g} \qquad \mathrm{id} \log \underset{(0)}{g} = (pr_2^* - pr_1^*) \underset{(1)}{A} \qquad d\underset{(1)}{A} = \pi_{YB}^* \underset{(2)}{F}$$

$$\begin{tikzcd}
Y^{[3]}B \arrow[r, shift left=2, "pr_{1,2}"] \arrow[r, "pr_{2,3}" description] \arrow[r, shift right=2, "pr_{1,3}"'] & Y^{[2]}B\,,\ \underset{(0)}{g} \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & YB\,,\ \underset{(1)}{A} \arrow[d] \\
 & & B\,,\ \underset{(2)}{F}
\end{tikzcd}$$

(this, we might call an ABELIAN BUNDLE 0-GERBE as the 0^{th} level in a natural hierarchy of geometrisations of de Rham $(p+2)$-cocycles)

$U(1)$; p

We have been rather careful in our 11

~~gen~~ abstraction, but have we not overshot? In other words, have we succeeded in encoding in our very general structure the local data $(\mathcal{O}_i, g_{ji})$??
The following proposition helps us verify ~~that~~ we have done well...

<u>Prop$^{\underline{n}}$</u>:

Let M_1, M_2 be ~~smo~~ manifolds, & let $\pi : M_1 \xrightarrow{C^\infty} M_2$ be a submersion at $x \in M_1$ (i.e., $T_x\pi$ is a surjection).
Then, there exists a neighbourhood $\mathcal{O}_{\pi(x)} \subset M_2$ of $\pi(x)$ and a C^∞-map $\sigma : \mathcal{O}_{\pi(x)} \to M_1$ with the property

$$\pi \circ \sigma = \mathrm{id}_{\mathcal{O}_{\pi(x)}}$$

& such that $\sigma \circ \pi(x) = x$. We call σ a LOCAL SECTION of π.

Proof: Consider a neighbourhood $\mathcal{O}_x \subset M_1$ (12)
of x on which there exists a local
chart $\kappa_1 : \mathcal{O}_x \xrightarrow[\text{homeo}]{\cong} \mathcal{U}_1$ for $\mathcal{U}_1$ open in $\mathbb{R}^{n_1}$,
$n_1 \equiv \dim M_1$, s.t. such that $\kappa_1(x) = 0$,
& let $\widetilde{\mathcal{O}}_{\pi(x)} \ni \pi(x)$ be the domain
of a local chart $\kappa_2 : \widetilde{\mathcal{O}}_{\pi(x)} \xrightarrow{\cong} \mathcal{U}_2$
for $\mathcal{U}_2$ open in $\mathbb{R}^{n_2}$, $n_2 \equiv \dim M_2$ in which
$\kappa_2 \circ \pi(x) = 0$. As π is submersive,
the tangent map

$$T_{\kappa_1(x)\equiv 0}(\kappa_2 \circ \pi \circ \kappa_1^{-1}) : T_{\kappa_1(x)\equiv 0}\mathbb{R}^{n_1} \cong \mathbb{R}^{n_1} \longrightarrow T_{\kappa_2\circ\pi(x)\equiv 0}\mathbb{R}^{n_2} \cong \mathbb{R}^{n_2}$$

is an epimorphism ($\equiv$ is onto) of $\mathbb{R}$-linear
spaces. ~~Hence~~ Let $V_1 \subset \mathbb{R}^{n_1}$ be an arbitrary
subspace of $\mathbb{R}^{n_1}$ isomorphically mapped
to $\mathbb{R}^{n_2}$ by $T_{\kappa_1(x)\equiv 0}(\kappa_2 \circ \pi \circ \kappa_1^{-1})\restriction_{V_1}$. Then, the tangent
of the C^∞-mapping

$$F := \kappa_2 \circ \pi \circ \kappa_1^{-1}\restriction_{\mathcal{U}_1 \cap V_1} : \mathcal{U}_1 \cap V_1 \longrightarrow \mathcal{U}_2 \subset \mathbb{R}^{n_2},$$

with a manifestly non-empty domain

(recall that V_1 is a subspace in $\mathbb{R}^{n_1}$, & so it contains $0 \in \mathbb{R}^{n_1}$, & $\mathcal{U}_1$ is a neighbourhood of $0 \in \mathbb{R}^{n_1}$), is invertible. Indeed, since $T_0 V_1 \cong V_1$, the domain of $T_0 F$ has the form $T_0 \mathcal{U}_1 \cap T_0 V_1 \equiv \mathbb{R}^{\times n_1} \cap V_1 = V_1$, & so

$$T_0 F \equiv T_0 (\kappa_2 \circ \pi \circ \kappa_1^{-1}) \restriction_{V_1},$$

which was <u>assumed</u> to be an isomorphism. As a result, in virtue of the Inverse-Function Theorem (Tw. o Lokalnej Odwracalności Odwzorowań), we infer that F has the desired C^∞-smooth inverse

$$(F\restriction_{\mathcal{U}_0})^{-1} =: \kappa_1 \circ \sigma \circ \kappa_2^{-1} \restriction_{F(\mathcal{U}_0)}$$

over some neighbourhood $\mathcal{U}_0 \subset \mathcal{U}_1 \cap V_1$ of $0 \equiv \kappa_1(x)$. The homeomorphic pre-image $\kappa_1^{-1}(\mathcal{U}_0)$ of $\mathcal{U}_0$ in M_1 is the postulated neighbourhood of x on which we have the local section σ (C^∞-smooth). $\square$

What does the last Prop$^{\underline{n}}$ give us?

Consider the surjective submersion

$$
\begin{array}{c}
YB \\
\big\downarrow \pi_{YB} \\
B
\end{array}
$$

Every point in $x \in B$ has a neighbourhood $O_x \ni x$ on which there exists a C^∞-smooth right inverse $\equiv$ section of π_{YB}. Let $\{O_i\}_{i \in I}$ be an open cover of B,

$$\bigcup_{i \in I} O_i \equiv B ,$$

such that over every O_i we have

$$\sigma_i : O_i \longrightarrow YB$$

with the property $\pi_{YB} \circ \sigma_i = \mathrm{id}_{O_i}$ (i.e., every σ_i is a (local) section of π_{YB}).

We may now define

$$A_i := \sigma_i^* \underset{(1)}{A} \quad \text{to obtain}$$

$$dA_i = \sigma_i^* d\underset{(1)}{A} = \sigma_i^* \pi_{YB}^* \underset{(2)}{F} \equiv (\pi_{YB} \circ \sigma_i)^* \underset{(2)}{F}$$

$$= \mathrm{id}_{\mathcal{O}_i}^* \underset{(2)}{F} \equiv \underset{(2)}{F} \restriction_{\mathcal{O}_i}$$, which is what we wanted!

Next, take $\sigma_{ij} := (\sigma_i, \sigma_j) : \underset{\mathcal{O}_i \cap \mathcal{O}_j}{\overset{\shortparallel}{\mathcal{O}_{ij}}} \longrightarrow YB \times_B YB$

$$x \longmapsto (\sigma_i(x), \sigma_j(x)).$$

Indeed, $\begin{cases} \pi_{YB} \circ \mathrm{pr}_1 \circ (\sigma_i, \sigma_j)(x) = \pi_{YB} \circ \sigma_i(x) = x \\ \pi_{YB} \circ \mathrm{pr}_2 \circ (\sigma_i, \sigma_j)(x) = \pi_{YB} \circ \sigma_j(x) = \overset{\shortparallel}{x} \end{cases}$ ✓,

so that $\sigma_{ij}(\mathcal{O}_{ij}) \subset Y^{[2]}B \subset YB \times YB$.

Define

$$g_{ij} := \sigma_{ij}^* g$$

to get

$$\mathrm{idlog}\, g_{ij} = \sigma_{ij}^*(\mathrm{idlog}\, g)$$

$$= \sigma_{ij}^* (\mathrm{pr}_2^* \underset{(1)}{A} - \mathrm{pr}_1^* \underset{(1)}{A})$$

$$= \sigma_j^* \underset{(1)}{A}\Big|_{\mathcal{O}_{ij}} - \sigma_i^* \underset{(1)}{A}\Big|_{\mathcal{O}_{ij}} \equiv (A_j - A_i) \restriction_{\mathcal{O}_{ij}},$$

which is – once more – the desired relation!

Finally, we set

$$\sigma_{ijk} := (\sigma_i, \sigma_j, \sigma_k) : O_{ijk} \longrightarrow Y^{[3]}B$$

$$: x \longmapsto (\sigma_i(x), \sigma_j(x), \sigma_k(x))$$

& compute

$$\begin{array}{ccc} (\sigma_{ij}^* g \cdot \sigma_{jk}^* g)\restriction_{O_{ijk}} = \sigma_{ijk}^* (pr_{1,2}^* g \cdot pr_{2,3}^* g) \equiv & \sigma_{ijk}^* pr_{1,3}^* g \\ \| & \| \\ (g_{ij} \cdot g_{jk})\restriction_{O_{ijk}} & \sigma_{ik}^* g \restriction_{O_{ijk}} \\ & \| \\ & g_{ik}\restriction_{O_{ijk}} \end{array}$$

Upon setting $i=j=k$, we find

$$g_{ii} \cdot g_{ii} = g_{ii} \implies g_{ii} = 1 \text{ (constant map)},$$

and so also – for $k=i$ –

$$g_{ij} \cdot g_{ji} = g_{ii} = 1,$$

i.e., $g_{ji} = g_{ij}^{-1}$, which is what we imposed previously.

This concludes the reconstruction of the local data from our geometric structure.

Prior to formalising our result, we fill up one more gap, to wit, the structure of the fibre ... So far, nothing has forced us to demand that $Y_x B$ be of a specific form, or that $Y_{x_1} B$ & $Y_{x_2} B$ ~~for~~ $x_1 \neq x_2$ be related in any manner.

In physics, ~~the~~ geometric ~~objects~~ constructs of the type considered above are used for a very concrete purpose: They model locally smooth distributions of 'objects' of a fixed type, e.g., vectors, spinors, scalars (numbers), scalar products, differential p-forms etc.

This leads us to assume pragmatically that every 'fibre' $Y_x B$ is isomorphic as a manifold (possibly NON-canonically)

to a <u>fixed</u> manifold F, termed the TYPICAL FIBRE, & the TOTAL SPACE YB of the surjective submersion to be locally modelled on the cartesian product of (an ~~~~ open subset of) the BASE B with the typical fibre F.
(so that, locally, sections are just F-valued ~~functions~~ maps)

Thus, we arrive at

Def$^{\underline{n}}$: A ($\mathscr{C}^\infty$-smooth) FIBRE BUNDLE is a quadruple (E, B, F, π_E) composed of

(*) a C^∞-manifold E, termed the TOTAL SPACE of the bundle

(*) — " — B, — " — the BASE

(*) — " — F, — " — the TYPICAL FIBRE

(*) a C^∞-smooth surjection $\pi_E : E \to B$, termed the PROJECTION ~~to~~ the BASE,

with the following property ~~of~~, termed LOCAL TRIVIALISABILITY:
There exists an open cover $\mathcal{O}_B := \{O_i\}_{i\in I}$ of B, $\bigcup_{i\in I} O_i = B$, (termed the TRIVIALISING COVER)

& ~~then~~ an associated family of C^∞-diffeomorphisms

$$\tau_i : \pi_E^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times F$$

(pre-image of $\mathcal{O}_i$ under π_E) typical fibre!

termed LOCAL TRIVIALISATIONS, that enter the commutative diagrams

$$\begin{tikzcd} \pi_E^{-1}(\mathcal{O}_i) \arrow[rr, "\tau_i"] \arrow[dr, "\pi_E"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{O}_i \times F \arrow[dl, "pr_1"] \\ & \mathcal{O}_i & \end{tikzcd}$$

& such that the mappings

$$g_{ij} : \mathcal{O}_{ij} \longrightarrow \mathrm{Aut}(F), \quad i,j \in I$$

(auto-diffeomorphisms of F)

determined by the superpositions of diffeomorphisms

$$\tau_{ij} := \tau_i \circ \tau_j^{-1} \restriction_{\mathcal{O}_{ij} \times F} : \mathcal{O}_{ij} \times F \circlearrowleft : (x,f) \mapsto (x, g_{ij}(x)(f))$$

& termed the TRANSITION MAPS of the bundle, give rise to C^∞-smooth maps

$$\mathcal{O}_{ij} \times F \xrightarrow{C^\infty} F : (x,f) \longmapsto g_{ij}(x)(f).$$

$\diamond$

Remarks :

(1) $E_x := \pi_E^{-1}(\{x\})$, $x \in B$ is termed the FIBRE of E over x.

(2) $\mathrm{Aut}(F)$ is , in this context, termed the STRUCTURE GROUP of the bundle. Oftentimes, we can choose the local ~~sections~~ ~~&~~ trivialisations τ_i such that all transition maps take values in some proper subgroup $G \subset \mathrm{Aut}(F)$. We then speak of a REDUCTION of the structure group. ~~&~~

(3) It is customary to use the notation

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi_E"] \\ & B \end{tikzcd}$$

Beware of the different ontological status of the two arrows in the diagram, though ($\downarrow$ ~~is~~ represents a map, $\rightarrow$ does not represent any concrete map).

(4) Connection (1-form) ~~the~~ requires a somewhat more involved discussion which we omit for the time being...

In differential geometry, just like in algebra, it is natural & useful to distinguish those maps between sets endowed with a given structure that ~~to~~ transport that structure (e.g., group homomorphisms between sets with group structure, linear maps between vector spaces, continuous maps between topological spaces, smooth maps between differentiable manifolds etc.).

In the present context, we have

Def^n: A ~~morphism~~ BUNDLE MORPHISM ~~of~~ between fibre bundles $(E_A, B_A, F_A, \pi_{E_A})$, $A \in \{1,2\}$ is a pair ~~of~~ (Φ, f) of C^∞-maps that render the following diagram commutative

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_{E_1}"'] \arrow[dr, phantom, "\circlearrowleft"] & E_2 \arrow[d, "\pi_{E_2}"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

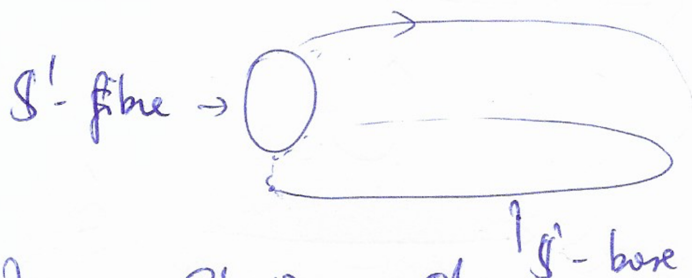
We say that Φ COVERS the base map f.

Bundles are ubiquitous in classical field theory (& in some approaches to quantisation) & sometimes actually indispensable, e.g., in the lagrangean modelling of Fermi fields (which calls for the notion of a spinor bundle). Accordingly, we shall encounter concrete examples of these structures in the course. Meanwhile, let us take a closer look at a bunch of structural examples...

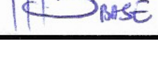
<u>Example 1</u>: The TRIVIAL BUNDLE

$$\begin{array}{ccc} F & \longrightarrow & B\times F \\ & & \big\downarrow \ pr_1 \ \left(\text{canonical projection}\right) \\ & & B \end{array} ,$$

e.g., - the (2-)torus : $\mathbb{T}^2 \equiv \mathbb{S}^1\times\mathbb{S}^1 \longrightarrow \mathbb{S}^1$



- the cylinder : $\mathbb{S}^1\times\mathbb{R} \longrightarrow \mathbb{S}^1$, or $\mathbb{R}\times\mathbb{S}^1 \longrightarrow \mathbb{R}$



Example 1 : The ∞-Möbius band as a NON-trivial bundle over S^1 with the typical fibre $\mathbb{R}$ (or some open segment, for a finite-size band).

Example 2: ~~Let~~ The PULLBACK BUNDLE.

Let (E, B, F, π_E) be a fibre bundle with local trivialisations $\tau_i : \pi_E^{-1}(O_i) \xrightarrow{\cong} O_i \times F$ associated with a trivialising cover $\{O_i\}_{i \in I} \equiv \mathcal{O}_B$ of the base B, & let $f : \tilde{B} \longrightarrow B$ be a C^∞-map (with ~~the~~ the domain $\tilde{B}$ - a C^∞-manifold). The quadruple

$$\left(f^*E \equiv \tilde{B} \times_B E,\ \tilde{B},\ F,\ pr_1 \right)$$ is a fibre bundle,

~~termed~~ the pullback bundle, ~~~~ ~~composed of~~ ~~can~~ has

→ the total space given by the fibred product

$$\begin{tikzcd} \tilde{B} \times_B E \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "\pi_E"] \\ \tilde{B} \arrow[r, "f"'] & B \end{tikzcd}$$

equipped with the differentiable structure
induced from the product — on $\widetilde{B} \times E$;

→ the projection ~~the~~ to the base

$$\pi_{f^*E} \equiv pr_1 : \widetilde{B} \times_B E \longrightarrow \widetilde{B}$$

given by the canonical projection onto the first cartesian factor;

→ the typical fibre F, & the fibre over $x \in \widetilde{B}$ given by $\{x\} \times \pi_E^{-1}(\{f(x)\}) \equiv E_{f(x)}$;

→ local trivialisations over the pullback cover $f^*\mathcal{O}_B \equiv \{ f^{-1}(\mathcal{O}_i) \}_{i \in I}$ (recall that f is continuous, therefore the $f^{-1}(\mathcal{O}_i)$ are open) given by

$$\tau_i^{f^*} := \left(\mathrm{id}_{f^{-1}(\mathcal{O}_i)} \times (pr_2 \circ \tau_i) \right)\Big|_{pr_1^{-1}(f^{-1}(\mathcal{O}_i))} : pr_1^{-1}(f^{-1}(\mathcal{O}_i)) \xrightarrow{\cong} pr_1^{-1}(f^{-1}(\mathcal{O}_i)) \times F$$

→ transition maps (!)

$$g_{ij}^{f^*} \equiv f^* g_{ij} : f^{-1}(\mathcal{O}_{ij}) \longrightarrow \mathrm{Aut}(F).$$

— × —

There is a canonical construction that associates a fibre bundle (with additional structure, but that we shall not discuss right now) with an arbitrary C^∞-manifold. ~~...~~ As the bundle is subsequently used in the definition of such physically ~~...~~ relevant objects as, e.g., vector fields, covector fields (e.g., the electromagnetic field), spinor fields, the metric tensor etc., we review it briefly hereunder...

Example 3. The TANGENT BUNDLE over a C^∞-manifold $(M, \mathcal{A})$ with an atlas composed of local charts

$$\kappa_i : \mathcal{O}_i \xrightarrow[\text{homeo}]{\cong} \mathcal{U}_i \ , \ i \in I \ , \quad \bigcup_{i \in I} \mathcal{O}_i = M$$

$$\mathcal{U}_i \subset \mathbb{R}^{\dim M} \text{ open}$$

is the fibre bundle with ~~...~~

$\to$ the base M (a given C^∞-manifold);

→ the total space

$$TM := \bigsqcup_{x\in M} P_x \,,$$

where $P_x = \{\gamma :]-\varepsilon,\varepsilon[\to M \mid \varepsilon>0 \wedge \gamma(0)=x\}/\sim_x$ is the set of equivalence classes of (local) paths γ through $x\in M$ with respect to the relation of co-tangency:

$$\gamma_1 \sim_x \gamma_2 \Longleftrightarrow \begin{cases} \gamma_2(0)=x=\gamma_1(0) \\ D(\kappa\circ\gamma_2)(0)=D(\kappa\circ\gamma_1)(0) \end{cases}$$

for some local chart $\kappa : \mathcal{O}_x \to \mathbb{R}^{\dim M}$ on a neighbourhood $\mathcal{O}_x$ of $x\in M$,

equivalence class:

$$\boxed{[\gamma_1]_{\sim_x} \ni \gamma_1,\gamma_2}$$

(tangent map the 'velocity')

where the induced structure of a differentiable manifold on TM thus defined is described below;

→ the typical fibre $\mathbb{R}^{\dim M}$ (with the natural structure of a differentiable manifold);

→ the projection to the base

$$\pi_{TM} : TM \to M : \left([\gamma]_{\sim_x}, x\right) \longmapsto x \,.$$

Above, the tangent mappings:

$$T\kappa_i : T\mathcal{O}_i := \pi_{TM}^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{U}_i \times \mathbb{R}^{\dim M}$$

$$: \left([\gamma]_{\sim_x}, x\right) \longmapsto \left(\kappa_i(\underset{x}{\underset{\shortparallel}{\gamma(0)}}), D(\kappa_i \circ \gamma)(0)\right)$$

induce on TM the so-called

STRONG PULLBACK TOPOLOGY

from the product topology on the $\mathcal{U}_i \times \mathbb{R}^{\dim M} \subset \mathbb{R}^{2\dim M}$, that is the topology in which $V \subset TM$ is open iff it satisfies the condition

$$\forall i \in I : T\kappa_i(V \cap T\mathcal{O}_i) \in \mathcal{T}(\mathcal{U}_i \times \mathbb{R}^{\dim M}).$$

In this topology, the mappings $T\kappa_i$ are – tautologically – homeomorphisms, & so can be employed as local charts – termed the NATURAL CHARTS – to define the desired differentiable manifold structure on TM. The corresponding transition mappings read:

$$T\kappa_{ij} \equiv T\kappa_i \circ \left(T\kappa_j\right)^{-1}\Big|_{T\mathcal{O}_{ij}} : \kappa_j(\mathcal{O}_{ij}) \times \mathbb{R}^{\dim M} \xrightarrow{\cong} \kappa_i(\mathcal{O}_{ij}) \times \mathbb{R}^{\dim M}$$

$$: \left(\kappa_j(\overset{\equiv\gamma(0)}{x}), D(\kappa_j \circ \gamma)(0)\right) \longmapsto \left(\kappa_i(x), D(\kappa_i \circ \gamma)(0)\right)$$

$$\left(\kappa_{ij}(\kappa_j(x)), D\kappa_{ij}(\kappa_j(x)) \circ D(\kappa_j \circ \gamma)(0)\right)$$

$\Leftarrow$ chain rule for derivatives!

for $\kappa_{ij} := \kappa_i \circ \left(\kappa_j \restriction \mathcal{O}_{ij}\right)^{-1}$ the transition mappings of M

Altogether, the dependence of the point in the image of $T\kappa_{ij}$ on the point in the domain is manifestly smooth, so we do get the structure (of a diff$^{\underline{ble}}$ manifold) sought-after. Clearly, we may use the $T\kappa_i$ also as local trivialisations of TM. Finally, the projection to the base is a $\overset{\text{smooth}}{\text{surjection}}$ because

$$\pi_{TM}\restriction_{T\mathcal{O}_i} \equiv \underbrace{\kappa_i^{-1}}_{\text{surjection}} \circ \overbrace{pr_1}^{\text{surjection}} \circ \overbrace{T\kappa_i}^{\text{surjection}} .$$

$\times$

A closer look at the last example shall guide us in the direction of vector bundles – later still – principal bundles that are instrumental in the description of the so-called gauging of global symmetries in a field theory. That, however, has to wait. Meanwhile, we return to a variation on our discussion of the geometrisation of local data of a de Rham 2-cocycle ... (29)

The Clutching Theorem

Adopt the hitherto notation.

The transitions maps g_{ij} of a fibre bundle (E, B, F, π_E) associated with a trivialising cover $\{O_i\}_{i \in I}$ of its base B satisfy the 1-cocycle condition:

$$\forall_{i,j,k \in I} : \forall_{x \in O_{ijk}} : g_{ij}(x) \circ g_{kj}(x)^{-1} \circ g_{ki}(x) = \mathrm{id}_F .$$

Conversely, let $\{O_i\}_{i\in I} = \mathcal{O}_B$ be an open cover of a manifold B & let F be a manifold with automorphism group $Aut(F)$. Every $\mathcal{O}_B$-indexed (auto-diffeomorphism) family of mappings

$$g_{ij} : O_{ij} \longrightarrow Aut(F) \quad . \; i,j \in I$$

inducing C^∞-smooth maps

$$O_{ij} \times F \longrightarrow F : (x,f) \longmapsto g_{ij}(x)(f)$$

& satisfying the above 1-cocycle condition defines a fibre bundle with transition maps over the O_{ij} given by the respective g_{ij}.
Whenever the g_{ij} are transition maps of a fibre bundle over B with the typical fibre F, the latter bundle is isomorphic with the one determined by the g_{ij}.

— x —

Proof: We leave the proof of the Thm to the Reader, confining ourselves to a number of guiding remarks (see: below, on the problem sheet).