CLASSICAL FIELD THEORY (interaction at a distance)

VI ELEMENTS of LIE-GROUP THEORY - ctd.

+ GLOBAL ASPECTS

We begin with

Prop$^{\underline{n}}$ 1.: Left- & right-invariant vector fields on a Lie group are complete.

Proof: Consider a smooth integral ~~line~~ curve $\gamma :]a,b[\to G$ (with $\gamma(t_0) = g_0 \in G$ for $t_0 \in]a,b[$) of a left-invariant (LI) $V \in \mathfrak{X}_L(G)$, i.e., the solution to the initial-value problem (IVP)

$$D\gamma(t) = V(\gamma(t)) \quad , \quad \gamma(t_0) = g_0$$

Choose arbitrarily intermediate times t_1, t_2 : $a < t_1 < t_2 < b$ & denote $\Delta t := t_2 - t_1 > 0$.

We shall extend γ using transitivity of the left regular action of G on itself,

$$\ell : \underset{\substack{\uparrow \\ \text{acts}}}{G} \times \underset{\substack{\uparrow \\ \text{is acted upon}}}{G} \longrightarrow G : (g, h) \longmapsto \underset{\substack{\| \\ \ell_g(h)}}{gh}$$ (2)

We define a path

$$\gamma_{\Delta t} : \,]a+\Delta t, b+\Delta t[\, \longrightarrow G : t \longmapsto \ell_{g_{21}} \circ \gamma(t-\Delta t)$$

for $g_{21} := \gamma(t_2) \cdot \gamma(t_1)^{-1}$.

It solves the IVP

$$D\gamma_{\Delta t}(t) = T_{\gamma(t-\Delta t)} \ell_{g_{21}} \circ D\gamma(t-\Delta t) \quad , \quad \gamma_{\Delta t}(t_2) = \gamma(t_2)$$

but $t - \Delta t \in \,]a, b[$ for all $t \in \,]a+\Delta t, b+\Delta t[$, & so – by left-invariance of V –

$$D\gamma_{\Delta t}(t) = T_{\gamma(t-\Delta t)} \ell_{g_{21}} \circ V(\gamma(t-\Delta t)) \overset{\downarrow}{\equiv} V(\ell_{g_{21}} \circ \gamma(t-\Delta t))$$

$$\equiv V(\gamma_{\Delta t}(t)).$$

Therefore, $\gamma_{\Delta t}$ is ALSO an integral ~~line~~ curve of V, whence – in virtue of the Picard-Lindelöf Thm (Cauchy Thm) – (uniqueness of sol$^{\text{n}}$ to the Cauchy problem) we have – on the non-empty interval

$$\Delta I := \,]a+\Delta t, b[\, \ni t_2 \ –$$

$$\gamma_{\Delta t} \restriction_{\Delta I} \equiv \gamma \restriction_{\Delta I}$$

But then, we obtain a smooth

extension of γ to $]a, b+\Delta t[$ given by

$$\tilde{\gamma} :]a, b+\Delta t[\to G : t \mapsto \begin{cases} \gamma(t) & \text{for } t \in]a,b[\\ \gamma_{\Delta t}(t) & \text{for } t \in]a+\Delta t, b+\Delta t[\end{cases}$$

Upon iteration, this procedure gives us unbounded extensions ~~'upwards'~~ forward in time, i.e., to $]a, \infty[$. An analogous argument shows that γ extends to $]-\infty, b[$, &, so – altogether – to $\mathbb{R}$. But g_0 was chosen arbitrarily, (V is defined over the entire G) hence – the thesis of the Prop.$^{\text{n}}$.

The proof develops along similar lines for RI vector fields.

$\square$

The structure of the integral lines is deciphered in

Prop$^{\underline{n}}$ 2: In the hitherto notation, for $X \in \mathfrak{g}$ ($\equiv$ Lie G), ~~the~~ the 1-parameter groups of diffeomorphisms of G ~~associated with~~ engendered by ~~the~~ the LI vector field L_X,

$$\mathcal{L}^X_{\cdot_1}(\cdot_2) \equiv \Phi_{L_X}(\cdot_1,\cdot_2) : \mathbb{R}\times G \to G :$$
$$: (t, g) \longmapsto \Phi_{L_X}(t,g) \equiv \mathcal{L}^X_t(g),$$

'time' initial cond$^{\underline{n}}$

& ~~the~~ the RI vector field R_X,

$$\mathcal{R}^X_{\cdot_1}(\cdot_2) \equiv \Phi_{R_X}(\cdot_1,\cdot_2) : \mathbb{R}\times G \longrightarrow G$$
$$: (t,g) \mapsto \Phi_{R_X}(t,g) \equiv \mathcal{R}^X_t(g),$$

are of the ~~form~~ respective forms:

$$(1) \quad \mathcal{L}^X_t \equiv P_{\mathcal{L}^X_t(e)} \quad \& \quad \mathcal{R}^X_t = l_{\mathcal{R}^X_t(e)}$$

right multiplication by $\mathcal{L}^X_t(e)$ — left multiplication by $\mathcal{R}^X_t(e)$

Proof: (Induction of 1-parameter ⑤
local groups of local diffeomorphisms follow from the ~~linear~~ course on Diff. Geom I, their globality is implied by Prop^{n} 1.).
We merely check Eq.(1). We obtain

$$\frac{d}{dt}\Big|_{t=0} \mathcal{L}_t^X(g) = L_X(g) \equiv T_e l_g(L_X(e))$$

$$= T_e l_g\left(\frac{d}{dt}\Big|_{t=0} \mathcal{L}_t^X(e)\right) \underset{\substack{\uparrow \\ \text{chain rule} \\ \text{for derivatives}}}{\equiv} \frac{d}{dt}\Big|_{t=0} (l_g \circ \mathcal{L}_t^X(e))$$

Thus, the paths $\mathcal{L}_\cdot^X(g)$ & $l_g \circ \mathcal{L}_\cdot^X(e)$ cross at $t=0$,

$$l_g \circ \mathcal{L}_0^X(e) = l_g(e) \equiv g \cdot e = g \equiv \mathcal{L}_0^X(g),$$

$\Phi_X^X(0,g) \equiv g$

& so they are co-tangent at $t=0$, with the 'velocity' vector $= L_X(g)$.
But they are both integral ~~lines~~ curves of L_X as $\frac{d}{dt} \mathcal{L}_t^X(g) = L_X \circ \mathcal{L}_t^X(g)$ for any t!

&

$$\frac{d}{dt} l_g \circ \mathcal{L}_t^X(e) = T_{\mathcal{L}_t^X(e)} l_g \left(L_X \circ \mathcal{L}_t^X(e) \right) \quad ⑥$$

$$\underset{LI}{=} L_X \circ l_g \circ \mathcal{L}_t^X(e).$$

In virtue of ~~the~~ the Picard-Lindelöf Th$\underline{m}$, (PLTh$\underline{m}$)
they <u>coincide</u>!

The statement for R_X is proven analogously. $\square$

Having reduced our considerations to the e-based paths $\mathcal{L}^X_\cdot(e)$ & $\mathcal{R}^X_\cdot(e)$, we shall next take a closer look at these...

<u>Prop$\underline{n}$</u> 3. The smooth paths

$$\lambda_X \equiv \mathcal{L}^X_\cdot(e) : \mathbb{R} \to G$$

& $\rho_X \equiv \mathcal{R}^X_\cdot(e) : \mathbb{R} \to G$ satisfy the identities:

$$\forall_{t \in \mathbb{R}} : \left(\lambda_{t \triangleright X}(1) = \lambda_X(t) \ \wedge \ \rho_{t \triangleright X}(1) = \rho_X(t) \right.$$

action of scalar t on vector X

Proof: In virtue of the PLThm, the integral ~~line~~ curve λ_X is uniquely determined by the conditions:

$$\begin{cases} \frac{d}{dt}\lambda_X(t) = L_X \circ \lambda_X(t) \underset{\substack{\uparrow \\ LI}}{\equiv} T_e l_{\lambda_X(t)}(X) \\ \\ \lambda_X(0) = e \end{cases}$$

The former yields

$$\forall s,t \in \mathbb{R} : T_e l_{\lambda_X(st)}(X) = \frac{d}{d(st)}\lambda_X(st) = \frac{1}{s}\frac{d}{dt}\lambda_X(st)$$

$$\Leftrightarrow \quad \frac{d}{dt}\lambda_X(st) = T_e l_{\lambda_X(st)}(s \triangleright X).$$

Define a 1-parameter ~~family~~ family of paths:

$$\gamma_s : \mathbb{R} \to G : t \longmapsto \lambda_X(st), \quad s \in \mathbb{R}$$

to rewrite the above as

$$\begin{cases} \frac{d}{dt}\gamma_s(t) = T_e l_{\gamma_s(t)}(s \triangleright X) \\ \\ \gamma_s(0) = \lambda_X(0) = e \end{cases},$$

whence the equality $\forall s \in \mathbb{R} : \gamma_s \equiv \lambda_{s \triangleright X}$, & so - ultimately - $\lambda_{s \triangleright X}(t) = \gamma_s(t) = \lambda_X(st)$.

Consequently,

$$\lambda_{s\cdot X}(1) = \lambda_X(s).$$

The proof for e_X develops along similar lines. $\square$

We may, next, relate the two families of paths ...

Prop$^{\underline{n}}$ 4. In the hitherto notation, the smooth paths λ_X & e_X are homomorphisms of the additive Lie group $\mathbb{R}$ into G, satisfying the initial cond$^{\underline{n}}$: $\frac{d}{dt}\big|_{t=0} \lambda_X(t) = X = \frac{d}{dt}\big|_{t=0} e_X(t)$.
Conversely, <u>any</u> homomorphism of Lie groups $\mathcal{X} : \mathbb{R} \to G$
satisfying the initial cond$^{\underline{n}}$

$$\frac{d}{dt}\Big|_{t=0} \mathcal{X}(t) = X \in \mathfrak{g} \quad (2)$$

is of the form $\lambda_X = \mathcal{X} = e_X$.
In particular, λ_X & e_X ~~are~~ coincide!

Proof: The only thing that we need (9) to verify ~~is~~ for the ~~λ_X~~ $\gamma \in \{\lambda_X, \varrho_X\}$ is the group-homomorphism property,

$$\forall_{s,t \in \mathbb{R}} : \gamma(s+t) = \gamma(s) \cdot \gamma(t). \quad (3)$$

We do that for λ_X. ~~The~~ Our task boils down to checking the identity of λ_X & $\gamma_s(\cdot) := \ell_{\lambda_X(s)^{-1}} \circ \lambda_X(s+\cdot)$ at a given (arbitrary) $s \in \mathbb{R}$. For that, we calculate:

$$\frac{d}{dt}\gamma_s(t) = T_{\lambda_X(s+t)}\ell_{\lambda_X(s)^{-1}}\left(\frac{d}{dt}\lambda_X(s+t)\right)$$

$$\equiv T_{\lambda_X(s+t)}\ell_{\lambda_X(s)^{-1}}\left(\frac{d}{d(s+t)}\lambda_X(s+t)\right)$$

$$= T_{\lambda_X(s+t)}\ell_{\lambda_X(s)^{-1}}\left(L_X \circ \lambda_X(s+t)\right)$$

$$\overset{LI}{\equiv} L_X \circ \ell_{\lambda_X(s)^{-1}} \circ \lambda_X(s+t) \equiv L_X \circ \gamma_s(t),$$

& so γ_s is an integral ~~line~~ curve of L_X just as λ_X, but also

$$\frac{d}{dt}\Big\lceil_{t=0} \gamma_s(t) = L_X \circ \gamma_s(0) = L_X(e) = \frac{d}{dt}\Big\lceil_{t=0} \gamma(t)$$

&

$$\gamma_s(0) = e = \gamma(0),$$

whence their identity (by the PLTh^{m}).

As for $\gamma = \varrho_X$, we consider $\tilde{\gamma}_s \overset{(\cdot)}{:=} p_{\varrho_X(s)^{-1}} \circ \varrho_X(s+\cdot)$ to the same effect.

Let, next, X be a ~~homomorphism~~ group homomorphism subject to the ~~constraints~~ constraint (2).

Upon differentiation of the functional identity (3), we obtain

$$\frac{d}{ds}X(s) \equiv \frac{d}{d\xi}\Big\lceil_{\xi=s} X(\xi) = \frac{d}{dt}\Big\lceil_{t=0} X(s+t)$$

$$= \frac{d}{dt}\Big\lceil_{t=0} l_{X(s)} \circ X(t) = T_{X(0)} l_{X(s)}\left(\frac{d}{dt}\Big\lceil_{t=0} X(t)\right)$$

$$= T_e l_{X(s)}\left(\frac{d}{dt}\Big\lceil_{t=0} X(t)\right) \equiv L_{\frac{d}{dt}\lceil_{t=0} X(t)} \circ X(s),$$

& so we see that X is the integral curve of, through $X(0) = e$ ($\Leftarrow$ (3)), of the LI vector field associated with $\frac{d}{dt}\restriction_{t=0} X(t)$

by the isomorphism L. from the previous lecture. Repeating the same reasoning for

$$\cancel{\frac{d\xi}{dt}}\ \frac{d}{dt} X(t) = \frac{d}{d\xi}\restriction_{\xi=t} X(\xi) = \frac{d}{ds}\restriction_{s=0} X(s+t)$$

$$= \frac{d}{dt}\restriction_{s=0} P_{X(t)} \circ X(s) = T_{X(0)} P_{X(t)} \left(\frac{d}{ds}\restriction_{s=0} X(s)\right)$$

$$= T_e P_{X(t)} \left(\frac{d}{ds}\restriction_{s=0} X(s)\right) \equiv R_{\frac{d}{ds}\restriction_{s=0} X(s)} \circ X(t),$$

we recover the last part of the statement of the Prop$^{\underline{n}}$. □

Our general knowledge of the dependence of flows of vector fields on their arguments leads to

<u>Prop$^{\underline{n}}$ 5.</u> : The mapping

$$\lambda_{\cdot} : \mathfrak{g} \times \mathbb{R} \to G : (X, t) \mapsto \lambda_X(t)$$

is <u>smooth</u>.

Proof:
Smoothness of the dependence of $\lambda_{\cdot}$ on the 2nd argument follows directly from the PLThm as λ_X is a flow of L_X. Consequently, we only need to show smoothness of the dependence of $\lambda_{\cdot}$ on X. To this end, we consider λ_X as an integral curve of a smooth vector field on $\mathfrak{g}\times G$ with initial data (X,e). In so doing, we identify $\mathrm{Der}_e C^1(G,\mathbb{R})$ (derivations of $C^1(G,\mathbb{R})$ at e) with T_eG.

Define

$$V : \mathfrak{g}\times G \longrightarrow T(\mathfrak{g}\times G)$$

$$: (X,g) \longmapsto (0_{\mathfrak{g}}, T_e l_g(X)) \equiv (0_{\mathfrak{g}}, L_X(g))$$

zero vector in vector space $\mathfrak{g}$

$\mathfrak{g}\oplus T_e l_g(\mathfrak{g})$

$\shortparallel$

$\mathfrak{g}\oplus T_gG \equiv T_{(X,g)}(\mathfrak{g}\times G)$

Its flow Φ_V satisfies ~~the~~

$$\begin{cases} \frac{d}{dt}\Phi_V(t;\underset{\text{initial time}}{t_0},(X_0,g_0)) = V(\Phi_V(t;t_0,(X_0,g_0))) \\ \\ \Phi_V(t_0;t_0,(X_0,g_0)) = (X_0,g_0) \quad \text{initial condition at } t=t_0 \end{cases}$$

Let $(X_0,g_0)=(X,e)$ at $t_0=0$ to obtain

$$\frac{d}{dt}\underset{\text{canonical projection}}{pr_2}\circ\Phi_V(t;0,(X,e)) = pr_2\circ V(\Phi_V(t;0,(X,e)))$$

$$= L_{pr_1\circ\Phi_V(t;0,(X,e))}(pr_2\circ\Phi_V(t;0,(X,e)))$$

$$= L_X(pr_2\circ\Phi_V(t;0,(X,e)))$$

by construction (of V)

Therefore, $pr_2\circ\Phi_V(\cdot\,;0,(X,e))$ is the unique integral curve of L_X through $g_0=e$ (at $t_0=0$), i.e.,

$$pr_2\circ\Phi_V(\cdot\,;0,(X,e)) = \lambda_X(\cdot)\,, \qquad \square$$

which concludes the proof by virtue of the PLThm.

We shall now introduce the universally employed notation :

Def^{n}. 1 : The mapping

$$\exp \equiv \exp^{G} := \lambda_{\cdot}(1) : \mathfrak{g} \to G$$

is termed the EXPONENTIAL MAPPING on G.

Problem: Justify the notation in the case of matrix Lie groups, e.g., for $G = SU(2)$

$$\equiv \{ m \in \mathbb{C}(2) \mid m^{\dagger} \cdot m = \mathbb{1}_2 = m \cdot m^{\dagger} \wedge \det m = 1 \}$$

We have

Prop^{n} 6. There exist open neighbourhoods $\mathcal{O}_{\mathfrak{g}}$ of $\emptyset_{\mathfrak{g}} \in \mathfrak{g}$ & $\mathcal{O}_e$ of $e \in G$ such that $\exp \restriction \mathcal{O}_{\mathfrak{g}}$ is a diffeomorphism (of class C^{∞}) onto $\mathcal{O}_e$.

<u>Proof</u>: We compute the tangent of exp at $\emptyset_{\mathfrak{g}}$ (zero vector) on an arbitrary vector $X \in \mathfrak{g} \equiv T_{0_{\mathfrak{g}}}\mathfrak{g}$ (we have $T_v V \equiv V$ for V - top. vector space), using Prop$^{\underline{n}}$ 3, to obtain

$$T_{\emptyset_{\mathfrak{g}}} \exp (X) \equiv \frac{d}{dt}\Big|_{t=0} \exp(\emptyset_{\mathfrak{g}} + t\cdot X)$$

$$\equiv \frac{d}{dt}\Big|_{t=0} \lambda_{t\cdot X}(1) \equiv \frac{d}{dt}\Big|_{t=0} \lambda_X(t) = X,$$

or $\quad T_{\emptyset_{\mathfrak{g}}} \exp = \mathrm{id}_{\mathfrak{g}}$,

& so the statement of the Prop$^{\underline{n}}$ follows from the Inverse-~~Fu~~Mapping Th$^{\underline{m}}$. $\square$

We have the structural

Prop^n 7. [Naturality of exp]

Let $\chi : G_1 \to G$ be a Lie-group homomorphism. The following diagram is commutative:

tangent Lie algebras

$$\begin{tikzcd}
\mathrm{Lie}\, G_1 \arrow[r, "\mathrm{Lie}\,\chi"] \arrow[d, "\exp^{G_1}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Lie}\, G_2 \arrow[d, "\exp^{G_2}"] \\
G_1 \arrow[r, "\chi"'] & G_2
\end{tikzcd}$$

Proof: Consider the smooth path

$$\gamma := \chi \circ \lambda_X : \mathbb{R} \longrightarrow G_2$$

through $\gamma(0) \equiv \chi \circ \lambda_X(0) = \chi(e_1) \overset{\text{homomorphism property!}}{=} e_2$

We calculate directly:

$$\frac{d}{dt}\gamma(t) = T_{\lambda_X(t)}\chi\left(\frac{d}{dt}\lambda_X(t)\right) = T_{\lambda_X(t)}\chi\left(L_X \circ \lambda_X(t)\right)$$

$$= T_{\lambda_X(t)} \circ T_{e_1} \ell^{(G_1)}_{\lambda_X(t)}(X) = T_{e_1}\left(\chi \circ \ell^{(G_1)}_{\lambda_X(t)}\right)(X)$$

$$= T_{e_1}\left(\ell^{(G_2)}_{\chi \circ \lambda_X(t)} \circ \chi\right)(X) = T_{\chi(e_1)} \ell^{(G_2)}_{\chi \circ \lambda_X(t)}\left(T_{e_1}\chi(X)\right)$$

$$= T_{e_2} \ell^{(G_2)}_{\chi \circ \lambda_X(t)}\left(T_{e_1}\chi(X)\right) \equiv T_{e_2} \ell^{(G_2)}_{\gamma(t)}\left(\mathrm{Lie}\,\chi(X)\right)$$

$$\equiv L_{\mathrm{Lie}\,\chi(X)} \circ \gamma(t)$$

and

$$\gamma(0) = e_2 \equiv \lambda_{\mathrm{Lie}\,\chi(X)}(0),$$

so that – in virtue of the PLThm – we have the identity

$$\chi \circ \lambda_X \equiv \gamma = \lambda_{\mathrm{Lie}\,\chi(X)},$$

which yields (at $t=1$)

$$\chi \circ \exp^{G_1}(X) \equiv \chi \circ \lambda_X(1) = \lambda_{\mathrm{Lie}\,\chi(X)}(1)$$

$$\equiv \exp^{G_2} \circ \mathrm{Lie}\,\chi(X). \qquad \square$$

We are finally ready to write out some (directly) practical statements: (18)

Prop.$^{\underline{n}}$ 8. Let $\left((T_e\mathrm{Ad}_\cdot)_A{}^B\right)_{A,B\in\overline{1,\dim G}}$ be the matrix valued in $C^\infty(G,\mathbb{R})$ determined by the equations:

$$T_e\mathrm{Ad}_\cdot(t_A) =: (T_e\mathrm{Ad}_\cdot)_A{}^B\, t_B \qquad A\in\overline{1,\dim G}$$

(cp the previous lecture)

The LI & RI vector fields L_A, R_A, $A\in\overline{1,\dim G}$ satisfy the rel$^{\underline{n}}$s:

structure constants of $\mathfrak{g}\equiv\mathrm{Lie}\,G$

$$[L_A, L_B] = f_{AB}{}^C\, L_C\,, \qquad [R_A, R_B] = -f_{AB}{}^C\, R_C$$

$$[L_A, R_B] = 0\,.$$

Furthermore,

$$L_A(\cdot) = (T_e\mathrm{Ad}_\cdot)_A{}^B\, R_B(\cdot)$$

&

$$\mathrm{Inv}_* L_A = -R_A\,, \qquad \mathrm{Inv}_* R_A = -L_A\,,$$

as well as – for any $g\in G$ –
$$\begin{cases} p_{g*}L_A = (T_e\mathrm{Ad}_{g^{-1}})_A{}^B\, L_B \\ l_{g*}R_A = (T_e\mathrm{Ad}_g)_A{}^B\, R_B\end{cases}$$

Proof: The first part follows directly from the def^n of the commutator in Lie G as – in consequence of left-invariance of ~~that~~ the commutator ~~of~~ of LI vector fields – we find

$$f_{AB}{}^{C} \triangleright L_C(\cdot) = f_{AB}{}^{C} \triangleright T_e l_\cdot \left(L_C(e)\right)$$

$$= f_{AB}{}^{C} \triangleright T_e l_\cdot \left(t_C\right) = T_e l_\cdot \left([t_A, t_B]\right)$$

$$= T_e l_\cdot \left([L_A, L_B](e)\right) = [L_A, L_B](\cdot)$$

The next commutator computes on an arbitrary $f \in C^1(G, \mathbb{R})$ at $g \in G$ as (cp Prop^n 2.):

$$[L_A, R_B](f)(g) \equiv L_A\left(R_B(f)\right)(g) - R_B\left(L_A(f)\right)(g)$$

$$= L_A\left(\frac{d}{dt}\Big|_{t=0} f \circ R_t^{t_B}\right)(g) - R_B\left(\frac{d}{ds}\Big|_{s=0} f \circ \mathcal{L}_s^{t_A}\right)(g)$$

$$= \frac{d}{ds}\Big|_{s=0} \left(\frac{d}{dt}\Big|_{t=0} f \circ R_t^{t_B}\right) \circ \mathcal{L}_s^{t_A}(g) - \frac{d}{dt}\Big|_{t=0} \left(\frac{d}{ds}\Big|_{s=0} f \circ \mathcal{L}_s^{t_A}\right) \circ R_t^{t_B}(g)$$

$$= \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \Big[f\big(\mathcal{R}_t^{t_B}(e) \cdot (g \cdot \mathcal{L}_s^{t_A}(e))\big) - f\big((\mathcal{R}_t^{t_B}(e) \cdot g) \cdot \mathcal{L}_s^{t_A}(e)\big)\Big]$$

$\equiv 0$ (the two actions of G on G commute!)

Finally, we derive

$$L_A(g) = T_e l_g (L_A(e)) \equiv T_e l_g (t_A)$$

$$\equiv T_e p_g \circ T_g p_{g^{-1}} \circ T_e l_g (t_A)$$

$$\equiv T_e p_g \circ T_e \mathrm{Ad}_g (t_A) = (T_e \mathrm{Ad}_g)_A{}^B \triangleright T_e p_g (t_B)$$

$$\equiv (T_e \mathrm{Ad}_g)_A{}^B \triangleright T_e p_g (R_B(e)) = (T_e \mathrm{Ad}_g)_A{}^B \triangleright R_B(g),$$

&, – by our previous results –

$$\mathrm{Inv}_* L_A(g) = T_{g^{-1}} \mathrm{Inv}(L_A(g^{-1})) = T_{g^{-1}} \mathrm{Inv} \circ T_e l_{g^{-1}} (t_A)$$

$$= -T_e p_g \circ T_{g^{-1}} l_g \circ T_e l_{g^{-1}} (t_A) = -T_e p_g (t_A) \equiv -R_A(g),$$

whence also

$$\mathrm{Inv}_* R_A = -\mathrm{Inv}_* \circ \mathrm{Inv}_* L_A = -(\mathrm{Inv} \circ \mathrm{Inv})_* L_A = -L_A.$$

~~Next, we consider~~ From the above, it follows that

$$-f_{AB}{}^{C}{}_{\triangle} R_C = f_{AB}{}^{C}{}_{\triangle}\, \mathrm{Inv}_* L_C \equiv \mathrm{Inv}_*([L_A, L_B])$$

$$= [\mathrm{Inv}_* L_A, \mathrm{Inv}_* L_B] \equiv [R_A, R_B],$$

& — for any $g, h \in G$ —

$$(p_{g*} L_A)(h) \equiv \cancel{T_{p_{g^{-1}}(h)}}\, T_{p_{g^{-1}}(h)}\, p_g (L_A \circ p_{g^{-1}}(h))$$

$$= T_{hg^{-1}}\, p_g \circ T_e l_{hg^{-1}}(t_A) = T_e l_h \circ T_e \mathrm{Ad}_{g^{-1}}(t_A)$$

$$= (T_e \mathrm{Ad}_{g^{-1}})_A{}^{B}{}_{\triangle}\, T_e l_h(t_B) = (T_e \mathrm{Ad}_{g^{-1}})_A{}^{B}{}_{\triangle}\, L_B(h)$$

as well as

$$(l_* R_A)(h) \equiv T_{l_{g^{-1}}(h)}\, l_g (R_A \circ l_{g^{-1}}(h))$$

$$= T_{g^{-1}h}\, l_g \circ T_e p_{g^{-1}h}(t_A) = T_e(l_g \circ p_h \circ p_{g^{-1}})(t_A)$$

$$= T_e(p_h \circ \mathrm{Ad}_g)(t_A) = (T_e \mathrm{Ad}_g)_A{}^{B}{}_{\triangle}\, T_e p_h(t_B)$$

$$\equiv (T_e \mathrm{Ad}_g)_A{}^{B}{}_{\triangle}\, R_B(h)$$

□

The above means that the Lie algebras of LI & RI vector fields on G are mutually anti-isomorphic ~~the~~ * subalgebras in $(\Gamma(TG), [\cdot,\cdot])$, & $\left((T_e \mathrm{Ad}_g)_A{}^B \right)_{A,B \in \overline{1,\dim G}}$ is – at a given $g \in G$ – the transition matrix between the two bases of $T_g G$ given by values of the LI resp. RI vectors fields at g.

We may, next, ~~[illegible]~~ dualise the objects considered, which we do in

Def.^{n} 2. Let $\{\theta_L^A\}^{A \in \overline{1,\dim G}}$ resp. $\{\theta_R^A\}^{A \in \overline{1,\dim G}}$ be bases of the $C^\infty(G,\mathbb{R})$-module of $\Omega^1(G) \equiv \Gamma(\Lambda^1 T^* G)$ dual to the bases $\{L_A\}_{A \in \overline{1,\dim G}}$ resp. $\{R_A\}_{A \in \overline{1,\dim G}}$, i.e., such that

$$L_A \lrcorner \theta_L^B = \delta_A{}^B = R_A \lrcorner \theta_R^B, \quad A,B \in \overline{1,\dim G}$$

The $\mathfrak{g}$-valued 1-form $\theta_L := \theta_L^A \otimes_{\mathbb{R}} t_A$

is called the canonical LI 1-form field, or the LI MAURER-CARTAN (MC) FORM.

Similarly, $\theta_R := \theta_R^A \otimes_{\mathbb{R}} t_A$

is called the canonical RI 1-form field, or the RI MAURER-CARTAN FORM

We have

Prop.ⁿ 9. The MC forms are LI resp. RI, that is

$$\forall g \in G : \begin{cases} (l_g^* \otimes \mathrm{id}_{\mathfrak{g}})\,\theta_L = \theta_L \\ (p_g^* \otimes \mathrm{id}_{\mathfrak{g}})\,\theta_R = \theta_R \end{cases}$$

Moreover, they satisfy the identities:

$$\theta_R = (\mathrm{id}_{T^*G} \otimes T_e \mathrm{Ad}_\cdot) \circ \theta_L ,$$

$$(\mathrm{Inv}_*^* \otimes \mathrm{id}_{\mathfrak{g}})\,\theta_L = -\theta_R \iff (\mathrm{Inv}^* \otimes \mathrm{id}_{\mathfrak{g}})\,\theta_R = -\theta_L$$

So $(p_g^* \otimes \mathrm{id}_{\mathfrak{g}})\,\theta_L = (\mathrm{id}_{T^*G} \otimes T_e \mathrm{Ad}_{g^{-1}}) \circ \theta_L$ & $(l_g^* \otimes \mathrm{id}_{\mathfrak{g}})\,\theta_R = (\mathrm{id}_{T^*G} \otimes T_e \mathrm{Ad}_g) \circ \theta_R$

Proof: We leave the proof as an exercise for the Reader... □

~~Next, we~~ The last entity that we introduce is

Def.ⁿ 3. Let $(M, \mathcal{A})$ be a smooth manifold & let G be a Lie group.

ATLAS

The left logarithmic derivative on $C^\infty(M,G)$ is the mapping

$$\delta_L \log : C^\infty(M,G) \longrightarrow \Omega^1(M) \otimes_{\mathbb{R}} \mathfrak{g}$$

given by the formula

$$(\delta_L \log f)(x)(v_x) := T_{f(x)} l_{f(x)^{-1}} \circ T_x f(v_x)$$

written for arbitrary $x \in M$, $v_x \in T_x M$ & $f \in C^\infty(M,G)$.

Similarly, the right logarithmic derivative on $C^\infty(M,G)$ is given by

$$\delta_R \log : C^\infty(M,G) \to \Omega^1(M) \otimes_{\mathbb{R}} \mathfrak{g}$$

$$(\delta_R \log f)(x)(v_x) := T_{f(x)} P_{f(x)^{-1}} \circ T_x f(v_x).$$

We have

Prop$\underline{\text{n}}$ 10.

$\forall f_1, f_2 \in C^\infty(M, G)$: $\delta_L \log (\underbrace{f_1 \cdot f_2}_{\text{point-wise product}})(x) = \delta_L \log f_2(x)$
$x \in M$

$+ \left(\mathrm{id}_{T^*G} \otimes T_e Ad_{f_2(x)^{-1}}\right) \circ \delta_L \log f_1(x)$

$$\delta_R (f_1 \cdot f_2)(x) = \delta_R \log f_1(x) + \left(\mathrm{id}_{T^*G} \otimes T_e Ad_{f_1(x)}\right) \circ \delta_R \log f_2(x)$$

Proof: Left to the Reader. $\square$

Prop$\underline{\text{n}}$ 11. $\forall f_1, f_2 \in C^\infty(M, G)$:

$$\begin{cases} \delta_L \log f_1 = \delta_L \log f_2 \iff \exists g \in G : f_2 = l_g \circ f_1 \\ \delta_R \log f_1 = \delta_R \log f_2 \iff \exists g \in G : f_2 = r_g \circ f_1 \end{cases}$$

Proof: $\Leftarrow$ is obvious (left to the Reader).

$\boxed{\Rightarrow}$ ~~Not~~ Consider the identity (= Prop$\underline{\text{n}}$ 10.)

$\delta_L \log (f_2 \cdot \mathrm{Inv} \circ f_1) = \delta_L \log (\mathrm{Inv} \circ f_1) + (\mathrm{id}_{T^*G} \otimes T_e Ad_{f_1}) \cdot \delta_L \log f_2$ -

given $0 \equiv \delta_L \log (f_1 \cdot \mathrm{Inv} \circ f_1) = \delta_L \log (\mathrm{Inv} \circ f_1)$
$+ (\mathrm{id}_{T^*G} \otimes T_e Ad_{f_1}) \delta_L \log f_1$

we ~~[illegible]~~ may rewrite the former as (26)

$$\delta_L \log(f_2 \cdot \mathrm{Inv} \circ f_1) = -(\mathrm{id}_{T^*G} \otimes T_e \mathrm{Ad}_{f_1})\,\delta_L \log f_1 + (\mathrm{id}_{T^*G} \otimes T_e \mathrm{Ad}_{f_1})\,\delta_L \log f_2\,,$$

so that $\delta_L \log f_1 = \delta_L \log f_2$

$$\Updownarrow$$

$$\delta_L \log(f_2 \cdot \mathrm{Inv} \circ f_1) = 0$$

But

$$\| $$

$$T_{f_2(\cdot)\cdot f_1(\cdot)^{-1}} \ell_{f_1(\cdot)\cdot f_2(\cdot)^{-1}} \circ T_\cdot\left(f_2(\cdot)\cdot f_1(\cdot)^{-1}\right),$$

& so – in view of the invertibility of – we have

$$\delta_L \log f_1 = \delta_L \log f_2 \Leftrightarrow T_\cdot\left(f_2(\cdot)\cdot f_1(\cdot)^{-1}\right) = 0,$$

i.e. $\Longrightarrow \forall x \in M\ \forall v \in T_x M : v \lrcorner\, d\left(f_2(\cdot)\cdot f_1(\cdot)^{-1}\right)_{(x)} = 0$

$\Leftrightarrow \forall x \in M : f_2(x) \cdot f_1(x)^{-1} = \mathrm{const} \in G \Rightarrow$ const is the desired constant $g \in G$. $\square$

The other part is proven similarly.

The two constructions introduced above (27)
are related in

Prop.^{n} 12. $\begin{cases} \theta_L = \delta_L \log \mathrm{id}_G \\ \theta_R = \delta_R \log \mathrm{id}_G \end{cases}$ (cp the formula encountered in physics literature: $\theta_L(g) = g^{-1}dg$, $\theta_R(g) = dg \cdot g^{-1}$ meaningful ONLY for matrix groups!!!)

Proof: We check

$$(L_A \lrcorner \delta_L \log \mathrm{id}_G)(g) \equiv T_g \ell_{g^{-1}} \circ T_g \mathrm{id}_G (L_A(g))$$

$$= T_g \ell_{g^{-1}} \circ \mathrm{id}_{T_gG} (L_A(g)) = T_g \ell_{g^{-1}} \circ T_e \ell_g (L_A(e))$$

$$= L_A(e) \equiv t_A ,$$

& similarly for the other equality. □

We may employ the last Prop.^{n} in the proof of

Prop.^{n} 13. Let $m : G \times G \to G$ be the binary operation (multiplication) on G.

Then, for any $g, h \in G$,

$$\begin{cases} m^* \theta_L(g,h) = \theta_L(h) + (\mathrm{id}_{T^*G} \otimes T_e \mathrm{Ad}_{h^{-1}}) \circ \theta_L(g) \\ m^* \theta_R(g,h) = \theta_R(g) + (\mathrm{id}_{T^*G} \otimes T_e \mathrm{Ad}_g) \circ \theta_R(h) \end{cases}$$

Proof: Left to the Reader. □

The lecture leaves us with a nontrivial arsenal of computational tools used amply in theoretical modelling. Prior to moving in the direction of its physically motivated applications, we state one last Theorem, this time without proof (the latter is not difficult but lengthy – we add it in a separate note (also, in Polish)).

Th$^{\underline{m}}$ 1. [the Cartan closed-subgroup theorem] Every closed subgroups of a Lie group is a submanifold & hence also a Lie subgroups of the latter. Conversely, every Lie subgroup of a Lie group is closed.