## Problem 1.

Let $(P_L, \Omega_L)$ be a symplectic manifold, i.e., $\Omega_L \in Z^2_{dR}(P_L)$ is non-degenerate, & let $\{\mathcal{O}_i\}_{i\in I}$ be an open cover of $P_L$ ~~for~~ which trivialises $\Omega_L$, i.e., there exist local symplectic potentials

$$\Theta_i \in \Omega^1(\mathcal{O}_i) \quad : \quad \delta\Theta_i = \Omega_L\restriction_{\mathcal{O}_i}.$$

Consider ~~locally~~ smooth maps

$$\psi_i : \mathcal{O}_i \longrightarrow \mathbb{C}$$

& define on the space of such maps linear operators $\hat{h}_i$, $i \in I$ that 'quantise' the classical hamiltonians $h$ (the latter satisfying $-\delta h = V_h \lrcorner\, \Omega_L$ for the corresponding hamiltonian vector field $V_h \in \Gamma(TP_L)$) as per

$$\hat{h}_i := -i\,\mathcal{L}_{V_h} - V_h \lrcorner\, \Theta_i + h\restriction_{\mathcal{O}_i}$$

Demonstrate that $[\hat{h}_{1i}, \hat{h}_{2i}] = i\,\widehat{\{h_1, h_2\}}_i$.

## Problem 2.

Prove the identity

$$g^{f^*}_{ij} = f^* g_{ij}$$

for the transition maps of the pullback fibre bundle of Example 2. from pages 23-24 of Lecture Notes I.

---

## Problem 3a

Consider the lagrangean model of a massive (m = 1) point-like particle of charge $q = 1$ moving in external fields: gravitational: $g \xrightarrow[loc.]{} g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu$ & electromagnetic: $F \xrightarrow[loc.]{} F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu$, $dF = 0$, on a manifold $M$.

Parametrise classical states of the theory with the classical action functional (for worldline $S^1$)

$$S[x] = S_{metr}[x] + S_{top}[x],$$

with the Nambu-Goto metric component

$$S_{mech}[x] := \int_{S^1} \sqrt{\det x^* g}$$

& the topological term (Wess-Zumino) locally given by

$$S_{top}[x] \xrightarrow[loc.]{} \int_{S^1} x^* A \,,$$

(i.e., for $x : S^1 \to M$, $x: S^1 \to \mathcal{U} \to M$, $F|_{\mathcal{U}} = dA$)

by the position $x$ & the KINETIC momentum

$$p_\mu = \frac{\partial L_{mech}}{\partial \dot{x}^\mu} \,,$$

where $L_{mech} = \sqrt{\det x^* g}$.

Write the presymplectic form of the theory in these coordinates & derive the Noether currents/hamiltonians associated with global symmetries of $S$ induced from ISOMETRIES of $(M, g)$.

(hint: write out the conditions satisfied by the corresponding vector fields on $M$). In the derivation, fix the lifts $\tilde{\mathcal{V}} \in \Gamma(T^*M)$ of the vector fields $\mathcal{V} \in \Gamma(TM)$ engendered

by the isometries to $T^*M$ by imposing (4)
the condition of invariance of the
KINETIC-ACTION 1-FORM:

$$\vartheta[x,p] = p_\mu \,\delta x^\mu$$

under the flows of the $\tilde{\mathcal{V}}$,

$$\mathcal{L}_{\tilde{\mathcal{V}}}\,\vartheta = 0 .$$

Compute the Poisson-Lie algebra of these charges.

Problem 3b*

Consider the generalised tangent bundle ~~$TM\times_M(M\times\mathbb{R})$~~

$$\begin{tikzcd} \mathcal{E}^{1,0}M \arrow[r, phantom, "="] & TM\times_M(M\times\mathbb{R}) \arrow[d] \\ & M \end{tikzcd}$$

of type $(1,0)$

$\Omega^{(0)}(M)$

endowed with the skew Vinogradov bracket (twisted by $F$ à la Ševera-Weinstein)

$$[\cdot,\cdot]^F_V : \Gamma(\mathcal{E}^{1,0}M)\times\Gamma(\mathcal{E}^{1,0}M) \longrightarrow \Gamma(\mathcal{E}^{1,0}M)$$

$$: \big((\mathcal{W}_1,f_1),(\mathcal{W}_2,f_2)\big) \longmapsto \big([\mathcal{W}_1,\mathcal{W}_2],\ \mathcal{L}_{\mathcal{W}_1} f_2 - \mathcal{L}_{\mathcal{W}_2} f_1 + \mathcal{W}_1\lrcorner\mathcal{W}_2\lrcorner F\big)$$

on the space of its sections.

Show that the bracket closes on sections associated with infinitesimal ~~gl.~~ symmetries of $S$ introduced in 3a.

Fix a basis $\{V_A\}_{A\in\overline{1,N}}$ in the Lie algebra of vector fields ~~that~~ whose flows give the global symmetries of $S$, & consider the corresponding sections $\overline{V}_A \in \Gamma(\mathcal{E}^{1,0}M)$.

Derive conditions that have to be satisfied so that the functional span

$$\mathcal{S} := \langle \overline{V}_A \mid A \in \overline{1,N} \rangle_{C^\infty(M,\mathbb{R})}$$

of the $\overline{V}_A$ has properties:

(1) $[\cdot,\cdot]^F_V$ closes on $\mathcal{S}$;

(2) When viewed as a derivation on the $C^\infty(M,\mathbb{R})$-module $\mathcal{S}$, $[\cdot,\cdot]^F$ satisfies the Leibniz rule,

$$[\overline{V}_A, f\overline{V}_B]^F_V = f[\overline{V}_A,\overline{V}_B]^F_V + (V_A \lrcorner df)\overline{V}_B$$

(3) $[\cdot,\cdot]^F_V$ on $\mathcal{S}$ satisfies the Jacobi identity.

Problem 3c** Consider the global-symmetry group $G$, acting on $M$ as $\lambda : G\times M \to M$. (6)

Interpret the conditions derived in 3b* as conditions ensuring the identity

$$\lambda^* F = pr_2^* F + d\varrho$$

satisfied by $F$ over $G\times M$ for some (specific — to be derived!) 1-form $\varrho \in \Omega^1(G\times M)$

## Problem 4



Complete the proof of the Clutching Theorem from pp. 29–30 of the Lecture Notes on the basis of the following hints for the second part of the proof (the first being straightforward)!

(1) Consider $\bigsqcup_{i \in I} \mathcal{O}_i \times F$ with the rel$^{n}$:

$$\underset{\mathcal{O}_i}{(x, i, f)} \sim_{g..} \underset{\mathcal{O}_j}{(y, j, g)} \iff \begin{cases} x = y \in \mathcal{O}_{ij} \\ g = g_{ji}(x)(f) \end{cases},$$

What kind of rel$^{n}$ is it?

Can we divide out by it (i.e., does $\bigsqcup_{i \in I} \mathcal{O}_i \times F$ decompose into classes of abstraction?)?

(2) How many representatives with a fixed cover index $i \in I$ does a class $[(x, i, f)]_{\sim_{g..}}$ contain?

Use the result to define local trivialisations

$[\tau_i] : \pi^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times F$ over the $\mathcal{O}_i$

for $\pi : \overbrace{\left(\bigsqcup_{i \in I} O_i \times F\right)/\sim_{g.}}^{=: L} \longrightarrow B$

$: [(x, i, f)]_{\sim_{g.}} \longmapsto x$

Induce on $L$ the quotient topology ~~along~~ from the disjoint-sum topology on $\bigsqcup_{i \in I} O_i \times F$ along $\pi$. Show that this topology is Hausdorff.

(3) Use ~~the~~ local charts on the $O_i$ (induced by those on $B$) & those on $F$ to define an atlas on $L$. Check the smoothness of the ensuing coordinate-transition maps. (use the assumed smoothness of $(x, f) \mapsto g_{ij}(x)(f)$).

(4) Convince yourself that the $[\tau_i]$ are (tautologically) diffeomorphisms with respect to the differentiable structure from point (3). Check the smoothness of $\pi$.

— x —