

CLASSICAL FIELD THEORY - PROBLEM SHEET II

(1)

Problem 1.

Consider a theory of a 1-form field $A \in \Omega^1(X)$ on a metric manifold (spacetime) (X, g) , sign $g = (1, d)$, determined by the lagrangian density

$$\mathcal{L}(x, A, \partial A) := -\frac{1}{4} F(A) \wedge * F(A),$$

where $F(A) := dA$ & $*$ is the Hodge star on $\Omega^*(X)$ defined by g .

(i) Derive the E-L eq's

of the theory. What is this theory?

(ii) Derive the symplectic form of the theory.

(iii) Demonstrate that the transformation $(x, A) \mapsto (x, A + d\lambda)$, $\lambda \in C^\infty(X, \mathbb{R})$

are symmetries of the theory. ②

Derive the corresponding vector field X on P (please see)

Ex compute $X \rfloor \Omega = ?$

Problem 2.

Consider the 2-dimensional (bosonic) σ -model, that is the theory

of fields \rightarrow , consider this to be $(X^{\mu})_{\text{local coörd}}$

$$\left\{ \begin{array}{l} X \in C^\infty(\Sigma, M) \text{ (dynamical)} \\ \gamma \in \Gamma(T^*\Sigma \otimes_{\Sigma}^m T^*\Sigma) \text{ (non-dynamical)} \end{array} \right.$$

over 2-dim. spacetime (Σ, γ) termed: WORLD SHEET
with metric γ (sign $\gamma = (1, 1)$) , with $\Sigma \cong S^1$ (circle!)

with the field bundle given by

$$\begin{matrix} \Sigma \times M & \downarrow & \text{where } M \overset{\text{TARGET}}{\text{SPACE}} \\ \Sigma & & \text{is a manifold} \end{matrix}$$

endowed with a metric

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$$g \in \Gamma(T^*M \otimes_M^{\text{sym}} T^*M)$$

& a De Rham 3-coycle

$$H \in Z_{\text{dR}}^3(M) \quad (\Leftrightarrow \begin{cases} H \in \Omega^3(M) \\ dH = 0 \end{cases}),$$

the theory being determined locally,

i.e., for $X(\Sigma) \subset U \subset M$ such that

$$H|_U = dB \text{ for some } B \in \Omega^2(U),$$

by the Polyakov action functional
written out in local coords

$$\{\delta^\alpha\}_{\alpha \in \text{action}}$$
 on Σ

$$\{x^\mu\}_{\substack{\mu \in \{0, d\} \\ \alpha \in \text{dS}}} \text{ on } M, d \neq 1 = \dim M$$

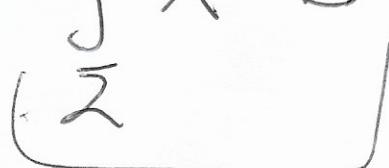
$$S_p[X, \gamma] := \int_{\Sigma} \text{vol}(\Sigma) \sqrt{|\det g|^{(0)}} \underbrace{(g^{-1})^{ab}_{(0)} g_{\mu\nu}(X) \partial_a X^\mu_b \partial_b X^\nu}_{=: S_{\text{meh}}[X]}$$



$$+ \int_{\Sigma} X^* B$$

$\sqsubseteq : S_{\text{meh}}[X]$

the metric term



$$\sqsubseteq : S_{WZ}[X]$$

the
Wess
-Zumino
term

Wute

$$\left\{ \begin{array}{l} S_{\text{met}}[X] = \int_{\Sigma} \text{Vol}(\Sigma) L_{\text{met}}(\sigma, X, \dot{x}, \partial X) \\ S_{\text{we}}[X] = \int_{\Sigma} \text{Vol}(\Sigma) L_{\text{we}}(\sigma, X, \partial X) \end{array} \right. \quad (4)$$

(i) Derive the S-L eq's of the Polyakov σ -model.

(ii) Express the pre-symplectic form on the phase space in terms of X, \dot{x} & the KINETIC momenta

$$P_\mu := \frac{\partial L_{\text{met}}}{\partial \dot{x}^\mu}.$$

(iii) Demonstrate that the following are symmetries of the σ -model:

$$\rightarrow (\sigma^a, X^\mu, \gamma_{ab}) \mapsto (f(\sigma), X^\mu, D_a^{(1)}{}^c D_b^{(2)}{}^d \gamma_{cd})$$

Difft-invariance

Consider small diffeomorphisms f on Σ ~ vector fields on Σ invertible for f - arbitrary diffeomorphisms $\in C^\infty(\Sigma, \Sigma)$

Weyl invariance $\rightarrow (\sigma^a, X^\mu, \gamma_{ab}) \mapsto (\sigma^a, X^\mu, e^{2w} \gamma_{ab})$
for arbitrary $w \in C^\infty(\Sigma, \mathbb{R})$

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$$\rightarrow (\sigma^a, X, \gamma^{ab}) \mapsto (\sigma^a, \bar{\Phi}(x), \gamma^{ab})$$

for $\bar{\Phi} \in C^\infty(M, M) \subset \text{Diff}(M)$ arbitrary such that

$$\bar{\Phi}^* g = g \quad (\text{isometries!})$$

$$\bar{\Phi}^* B = B + d\phi, \quad \phi \in \Omega^1(M).$$

Consider 'small' diffeomorphisms on M
(near Id_M) \sim vector fields V on M

When lifting V to the presece space

$$T^*LM, LM = C^\infty(S^1, M) \text{ for } \Sigma \cong S^1$$

impose the condition $\tilde{L}_V \Theta = 0$

on the lift \tilde{V} , for $\Theta \in \Omega^1(T^*LM)$

The part of the action form (presymplectic potential) coming from Lueh.

(ii) Find the corresponding Noether charges & conserved currents.