Problem 1.

Consider a theory of a 1-form field $A \in \Omega^1(X)$ on a metric manifold (spacetime) (X, g), sign $g = (1, d)$, determined by the lagrangian density

$$\mathcal{L}(x, A, \partial A) := -\frac{1}{4} F(A) \wedge * F(A),$$

where $F(A) := dA$ & $*$ is the Hodge op on $\Omega^\bullet(X)$ defined by g.

(i) Derive the E-L eq$^{\underline{n}}$s of the theory. What is this theory?

(ii) Derive the symplectic form of the theory

(iii) Demonstrate that the transformation

$$(x, A) \longmapsto (x, A + d\lambda), \quad \lambda \in C^\infty(X, \mathbb{R})$$

are symmetries of the theory.

Derive the corresponding vector field X on P (phase space)

& compute $X \lrcorner \Omega = ?$

Problem 2.

Consider the 2-dimensional (bosonic) σ-model, that is the theory of fields

consider this to be $(X^\mu)^{\mu \in \overline{0, \dim M - 1}}$ ← local coörds

$$\begin{cases} X \in C^\infty(\Sigma, M) & \text{(dynamical)} \\ \gamma \in \Gamma(T^*\Sigma \overset{\text{sym.}}{\underset{\Sigma}{\otimes}} T^*\Sigma) & \text{(non-dynamical)} \end{cases}$$

over 2-dim. spacetime (Σ, γ) ← termed: WORLDSHEET with $\Sigma_t \simeq \mathbb{S}^1$ (circle!)

with metric γ ($\operatorname{sign} \gamma = (1,1)$),

with the field bundle given by

$$\begin{array}{c} \Sigma \times M \\ \downarrow \\ \Sigma \end{array}$$

, where M ← termed: TARGET SPACE is a manifold endowed with a metric ③

$$g \in \Gamma(T^*M \otimes_M^{\text{sym.}} T^*M)$$

& a de Rham 3-cocycle

$$H \in Z^3_{dR}(M) \quad \left(\Leftrightarrow \begin{cases} H \in \Omega^3(M) \\ dH = 0 \end{cases}\right),$$

the theory being determined locally,

i.e., for $X(\Sigma) \subset \mathcal{U} \subset M$ such that

$$H\restriction_{\mathcal{U}} = dB \text{ for some } B \in \Omega^2(M),$$

by the Polyakov action functional

(written out in local coörds

$\{\sigma^a\}^{a \in \{0,1\}}$ on Σ

$\{x^\mu\}^{\mu \in \overline{0,d}}$ on M , $d+1 \equiv \dim M$)

$$S_P[X,\gamma] := \underbrace{\int_\Sigma \underset{\text{"} d\sigma^0 d\sigma^1}{\mathrm{vol}(\Sigma)} \sqrt{|\det \gamma|(\sigma)}\,(\gamma^{-1})^{ab}_{(\sigma)}\, g_{\mu\nu}(X_{(\sigma)})\, \partial_a X^\mu_{(\sigma)}\, \partial_b X^\nu_{(\sigma)}}_{=: S_{\text{met}}[X] \text{ the metric term}}$$

$$+ \underbrace{\int_\Sigma X^* B}_{=: S_{WZ}[X] \text{ the Wess-Zumino term}}$$

Write
$$\begin{cases} S_{\text{metr}}[X, \gamma] = \int_\Sigma \mathrm{Vol}(\Sigma)\, \mathcal{L}_{\text{metr}}(\sigma, X, \gamma, \partial X) \\ S_{\text{WZ}}[X] \equiv \int_\Sigma \mathrm{Vol}(\Sigma)\, \mathcal{L}_{\text{WZ}}(\sigma, X, \partial X) \end{cases}$$

(i) Derive the E-L eq$^{\text{n}}$s of the Polyakov σ-model.

(ii) Express the pre-symplectic form on its phase space in terms of X, γ & the KINETIC momentum
$$p_\mu := \frac{\partial \mathcal{L}_{\text{metr}}}{\partial \partial_0 X^\mu}.$$

(iii) Demonstrate that the following are symmetries of the σ-model:

- $(\sigma^a, X^\mu, \gamma_{ab}) \longmapsto (f^a(\sigma), X^\mu, D^{-1\,c}_{\ \ a}(\sigma) D^{-1\,d}_{\ \ b}(\sigma)\gamma_{cd})$

$\boxed{\mathrm{Diff}_\Sigma\text{-invariance}}$! Consider small ! diffeomorphisms on Σ $\sim$ vector fields on Σ

$\frac{\partial f^a}{\partial \sigma^b} =: D^a_{\ b}$ assumed invertible

for f - arbitrary diffeomorphisms $\in C^\infty(\Sigma, \Sigma)$

- $(\sigma^a, X^\mu, \gamma_{ab}) \longmapsto (\sigma^a, X^\mu, e^{2\omega}\gamma_{ab})$

$\boxed{\text{Weyl invariance}}$ for arbitrary $\omega \in C^\infty(\Sigma, \mathbb{R})$

$\longrightarrow (\sigma^a, X, \gamma_{ab}) \longmapsto (\sigma^a, \Phi(X), \gamma_{ab})$

for $\Phi \in C^\infty(M,M) \equiv \mathrm{Diff}(M)$ arbitrary such that

$\Phi^* g = g$ (isometries!)

$\Phi^* B = B + d\phi$, $\phi \in \Omega^1(M)$.

Consider 'small' diffeomorphisms on M (near id_M) $\sim$ vector fields V on M

When lifting V to the phase space T^*LM , $LM \equiv C^\infty(S^1, M)$ for $\Sigma_t \simeq S^1$ impose the condition $\mathcal{L}_{\tilde{V}} \Theta \overset{!}{=} 0$ on the lift $\tilde{V}$, for $\Theta \in \Omega^1(T^*LM)$ the part of the action 1-form (presymplectic potential) coming from $\mathcal{L}_{\mathrm{met}}$.

(iv) Find the corresponding Noether charges & conserved currents.