
Gauging the Oïd
& Other Tales
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Part I
Geometric Field Theory
& Symmetry Models
MAKES BEN HUR
LOOK SO AN-EPIC!
G
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La Géométrie est une image,
mais il n'y a pas d'image de la Géométrie.

# The Gauge Principle

**Concepts:**

(*) Modelling dynamics on symmetry orbispaces & that of charged probes

(**) Configurational-symmetry reduction

(***) Emergence of projector fields

(****) Anomalies

(*****) Poly-phase dynamics with defects

(******) Descent of geometric/cohomological structures

(*******) Categorification of symmetry

**Def$^{n}$ 1.** GEOMETRIC FIELD THEORY is A SEXTUPLE

$$GFT := (\mathcal{F}, \Sigma, M, \pi_{\mathcal{F}}, \eta_{\Sigma}, \mathcal{A})$$ COMPOSED of

(*) METRIC MANIFOLD $(\Sigma, \eta_\Sigma)$ (of DIMENSION $D$) (SPACETIME)

(**) FIBRE BUNDLE $(\mathcal{F}, \Sigma, M, \pi_{\mathcal{F}})$ (CONFIGURATION BUNDLE)

(***) FUNCTIONAL $\mathcal{A} : \Gamma(\mathcal{F}) \longrightarrow U(1)$ (DIRAC-FEYNMAN AMPLITUDE)

with - TYPICAL FIBRE $M$ of $\mathcal{F}$ - SPACE of iDOFs $\equiv$ FIELD TYPE

e.g., $\mathbb{R}$, $\mathbb{R}^{\times D} \otimes \mathrm{Lie}\,G$, SPINOR MODULE, METRIC MANIFOLD w/ p-FORM FIELDS ...

- SECTIONS of $\mathcal{F}$ - FIELD CONFIGURATIONS, or FIELDS

- CRITICAL POINTS of $\mathcal{A}$ - CLASSICAL FIELD CONFIGURATIONS

$$\frac{d}{dt}\Big|_{t=0} \mathcal{A}[\Phi_V(t;\phi)] = 0 \quad \forall\, V \in \Gamma(\phi^* V\mathcal{F})$$

E.g.,

(*) STANDARD MODEL of FUNDAMENTAL INTERACTIONS

$$(\Sigma, \eta_\Sigma) = \mathbb{R}^{3,1}, \quad \mathcal{F} = \mathbb{R}^{3+1} \times (17), \quad \mathcal{A}_{SM} = \exp\Big(i \int_{\mathbb{R}^{3,1}} \mathcal{L}_{SM}(\cdot, T\cdot)\Big)$$

(**) 2d POLYAKOV-ALVAREZ-GAWĘDZKI $\sigma$-MODEL

$(\Sigma, \eta_\Sigma)$: $\dim \Sigma = 2$, ORIENTABLE, $\partial\Sigma = \emptyset$, COMPACT

EUCLIDEAN or PIECEWISE LORENTZIAN

$\mathcal{F} = \Sigma \times M$: $(M, g_M)$ METRIC, w/ $H \in Z^3_{dR}(M)$, $\mathrm{Per}(H) \subset 2\pi\mathbb{Z}$

$$\mathcal{A}_{PAG}[X] = \exp\Big(-\tfrac{i}{2} \|TX\|^2_{\eta_\Sigma, g_M}\Big) \cdot \mathrm{Hol}_{\mathcal{G}}(X(\Sigma))$$

$$\|TX\|^2_{\eta_\Sigma, g_M} \underset{loc.}{\longrightarrow} \int \sqrt{|\det \eta_\Sigma|}\; \eta_\Sigma^{-1\,ab}\, g_M(X)_{\mu\nu}\, \partial_a X^\mu \partial_b X^\nu$$

$$\mathrm{Hol}_{\mathcal{G}} \in \hat{H}^3(M) = \{\chi \in \mathrm{AbGrp}(Z_2(M), U(1)) \mid \exists \varpi \in Z^3_{dR}(M) : \chi \circ \partial_3(\cdot) = \exp(i \textstyle\int_\cdot \varpi)\}$$

## (***) POISSON $\sigma$-MODEL

$(\Sigma, \eta_\Sigma)$ : $\dim \Sigma = 2$ , ORIENTABLE , $\partial\Sigma = \emptyset$ , COMPACT

$\mathcal{F}$ : 2-COMPONENT : $\Sigma \times M$ , $(M, \Pi)$ POISSON

$$C^\infty(\Sigma, M) \ni X \quad : \quad \Gamma(T^*\Sigma \otimes X^* T^* M) \ni a$$

$$\begin{tikzcd}
T\Sigma \arrow[r, "a"] \arrow[d, "\pi_{T\Sigma}"'] & T^*M \arrow[d, "\pi_{T^*M}"] \\
\Sigma \arrow[r, "X"'] & M
\end{tikzcd}$$

$$\mathcal{A}_{PSM}[X, a] = \exp\left(i \int_\Sigma \mathcal{L}_{PSM}(X, TX; a)\right)$$

$$\mathcal{L}_{PSM}(X, TX; a) \underset{loc.}{=} \left(a_{\mu a} \partial_b X^\mu + \frac{1}{2} \Pi^{\mu\nu}(X)\, a_{\mu a}\, a_{\nu b}\right) d\sigma^a \wedge d\sigma^b \qquad \Pi \underset{loc.}{=} \Pi^{\mu\nu} \partial_\mu \wedge \partial_\nu$$

**Def$^{n}$ 2.** CONFIGURATIONAL SYMMETRY of GFT is

$$\Phi \in \mathrm{Aut}_{\mathrm{Bun}(\Sigma)/\Sigma}(\mathcal{F}) \quad \text{s.th.} \quad \forall \phi \in \Gamma(\mathcal{F}) : \mathcal{A}[\Phi \circ \phi] = \mathcal{A}[\phi]$$

w/ LOCAL MODEL

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\Phi"] \arrow[d, "\pi_{\mathcal{F}}"'] & \mathcal{F} \arrow[d, "\pi_{\mathcal{F}}"] \\ \Sigma \arrow[r, equal] & \Sigma \end{tikzcd}$$

(*) $\mathrm{Aut}(M) \subset \mathrm{Diff}(M)$ – RIGID

OR

(**) $\mathrm{Aut}(M)^{\vartheta}$, $\vartheta \in \mathcal{J}(\Sigma)$ – LOCAL or GAUGE

E.G., (*) $\mathbb{R}^{3,1} \longrightarrow SU(3)_c \times U(1) (\times SU(2)_Y) \wedge \ldots$ in SM

(**) $\{ g \in \mathrm{Iso}(M, g_M) \mid g^* \mathrm{Hol}_{\mathfrak{g}} = \mathrm{Hol}_{\mathfrak{g}} \}$

LINEARISATION: $\mathcal{K} \in \Gamma(TM)$ Killing : $\exists\, \kappa \in \Omega^1(M) : \mathcal{K} \lrcorner H = -d\kappa$

(***) TANGENTIAL : $\langle X^* Z^1_{dR}(M) \rangle_{C^\infty(\Sigma, \mathbb{R})} \ni \epsilon_\mu(\cdot)\, dX^\mu(\cdot)$

$$X^\mu \longmapsto X^\mu + \Pi^{\mu\nu}(X)\epsilon_\nu \;;\quad a_\mu \longmapsto a_\mu + d\epsilon_\mu + \partial_\mu \Pi^{\lambda\nu}(X)\epsilon_\lambda a_\nu$$

# SYMMETRY MODELS:

(*) CLASSIC – SMOOTH (LIE-)GROUP ACTION $\lambda : G \longrightarrow \mathrm{Diff}(M) \iff \tilde{\lambda} : G \times M \xrightarrow[C^\infty]{} M$

$\Longrightarrow$ LINEARISATION $\mathcal{K}^{(\lambda)} : \mathrm{Lie}\,G \times M \longrightarrow TM\,,\ (\xi, x) \longmapsto T_{(e,x)}\tilde{\lambda}(-\xi, 0_{TM}(x))$

FUNDAMENTAL VECTOR FIELD

**PROBLEM**: TRIVIAL FIBRATION of SYMMETRY MODEL $G$ OVER $M$:

$$G \times M \xrightarrow{\mathrm{pr}_2} M$$

FAILS to CAPTURE SIMPLE SYMMETRY SCENARIOS...

E.G., $U(1) \times \mathbb{D}^2 \longrightarrow \mathbb{D}^2 \searrow \{(u, z) \in U(1) \times \mathbb{D}^2_* \mid u \cdot z \in \mathbb{D}^2_*\} \longrightarrow \mathbb{D}^2_* := \mathbb{D}^2 \setminus \{(\tfrac{1}{2}, 0)\}$

NB: FIBRE OVER $|\cdot|^{-1}(\{\tfrac{1}{2}\})$ : $U(1) \setminus \{*\} \simeq \mathbb{R} \not\simeq U(1)$ : FIBRE OVER $|\cdot|^{-1}(\{z \neq \tfrac{1}{2}\})$

**UPSHOT**: NEED ABSTRACTION of (SMOOTH) SYMMETRY WHICH FIBRES OVER $M$

(WITHOUT TYPICAL FIBRE IN GENERAL)...

**ABSTRACTION**: (*) MANIFOLD of INVERTIBLE CORRESPONDENCES $\mathcal{G} \ni$
$$s(g) \in M \ni t(g), \quad s(g) \xrightarrow{g} t(g), \quad t(g) \xrightarrow{g^{-1}} s(g),$$

INCLUDING AUTO-CORRESPONDENCES $\mathrm{Id}_m$,
$$s(\mathrm{Id}_x) = x = t(\mathrm{Id}_x)$$

(**) ASSOCIATIVE COMPOSITION
$$m : \mathcal{G}\,{}_s\!\times_t \mathcal{G} \longrightarrow \mathcal{G}$$
$$\mathcal{G}\,{}_s\!\times_t \mathcal{G} \equiv \{(g_1,g_2) \in \mathcal{G}\times\mathcal{G} \mid s(g_1) = t(g_2)\}$$

(***) SMOOTH LOCAL CHARTING of $M$ by CORRESPONDENCES,
$$\forall x \in M\ \forall g \in s^{-1}(\{x\})\ \exists\, \mathcal{O} \ni x\ \exists\, \tilde{\mathcal{O}} \ni g : \left( s|_{\tilde{\mathcal{O}}} : \tilde{\mathcal{O}} \xrightarrow{\cong} \mathcal{O} \ \wedge\ t(\tilde{\mathcal{O}}) \simeq \mathcal{O} \right)$$

**Def$^n$ 3.** **LIE GROUPOID** is A CATEGORY $(M, \mathcal{G}, s, t, \mathrm{Id}, m)$ w/ ALL MORPHISMS INVERTIBLE, $\exists\ \mathrm{Inv} : \mathcal{G} \to \mathcal{G},\ g \mapsto g^{-1}$, WHICH is INTERNAL to $\mathbf{Man}$, & HAS $s$ & $t$ SUBMERSIVE.

**NOTATION**: $\mathcal{G} \Longrightarrow M$     **MORPHISM of GROUPOIDS** is A SMOOTH FUNCTOR.

E.g., (*) PAIR GROUPOID $\mathrm{Pair}(M) : M\times M \rightrightarrows M$

$$s(x_2,x_1)=x_1 \,,\quad t(x_2,x_1)=x_2 \,,\quad \mathrm{Id}_x=(x,x)$$

$$\mathrm{Inv}(x_2,x_1)=(x_1,x_2) \,,\quad (x_3,x_2).(x_2,x_1)=(x_3,x_1)$$

(**) SUBMERSION GROUPOID $\mathrm{Pair}(\pi_Y)$: GIVEN $\pi_Y : Y \twoheadrightarrow X$,

SURJECTIVE SUBMERSION

$$Y\,{}_{\pi_Y}\!\times_{\pi_Y} Y \rightrightarrows Y \quad \subset \quad \mathrm{Pair}(Y)$$

(***) DELOOPING GROUPOID $BG : G \rightrightarrows \bullet$ for $G\in \mathrm{LieGrp}$

(****) ACTION GROUPOID $G\ltimes_\lambda M : G\times M \rightrightarrows M$

$$s(g,x)=x \,,\quad t(g,x)=\lambda_g(x) \,,\quad \mathrm{Id}_x=(e,x)$$

$$\mathrm{Inv}(g,x)=(g^{-1},\lambda_g(x)) \,,\quad (h,\lambda_g(x)).(g,x)=(h\cdot g,x)$$

(*****) EHRESMANN–ATIYAH GROUPOID $\mathrm{At}(P)$: GIVEN $\pi_P : P\to\Sigma$

PRINCIPAL G-BUNDLE

$$\mathrm{Pair}(P)/G : (P\times P)/G \rightrightarrows P/G \equiv \Sigma$$

# GROUPOIDS AS MODELS of ORBISPACES

$$M/\!/\mathcal{G} = \{ [x] := t(s^{-1}(\{x\})) \mid x \in M \}$$

$\mathcal{G}$-ORBIT of $x$

CONSIDER **NERVE** of $\mathcal{G} \rightrightarrows M$ :

$$\mathcal{G}_\bullet := N_\bullet(\mathcal{G} \rightrightarrows M) : \quad \cdots \overset{d^{(4)}_\bullet}{\Rrightarrow} \mathcal{G} {}_s\!\times_t \mathcal{G} {}_s\!\times_t \mathcal{G} \overset{d^{(3)}_\bullet}{\Rrightarrow} \mathcal{G} {}_s\!\times_t \mathcal{G} \overset{d^{(2)}_\bullet}{\Rrightarrow} \mathcal{G} \overset{d^{(1)}_\bullet}{\rightrightarrows} M$$

w/ **FACE MAPS** $d^{(\cdot)}_\cdot$ :

$$d^{(1)}_0 = t \,,\; d^{(1)}_1 = s \;; \qquad d^{(n)}_i(g_n, g_{n-1}, \dots, g_1) = \begin{cases} (g_n, g_{n-1}, \dots, g_2) & \text{if } i = 0 \\ (g_n, g_{n-1}, \dots, g_{i+2}, g_{i+1} \cdot g_i, g_{i-1}, g_{i-2}, \dots, g_1) & \text{if } 0 < i < n \\ (g_{n-1}, g_{n-2}, \dots, g_1) & \text{if } i = n \end{cases}$$

FORM STRATIFIED MANIFOLD $\displaystyle\coprod_{n \in \mathbb{N}} (\mathcal{G}_n \times \Delta_n)$ w/ $\Delta_n = \{ (t_1, t_2, \dots, t_{n+1}) \in \mathbb{R}^{\times n+1}_{\geq 0} \mid \sum_{k=1}^{n+1} t_k = 1 \}$

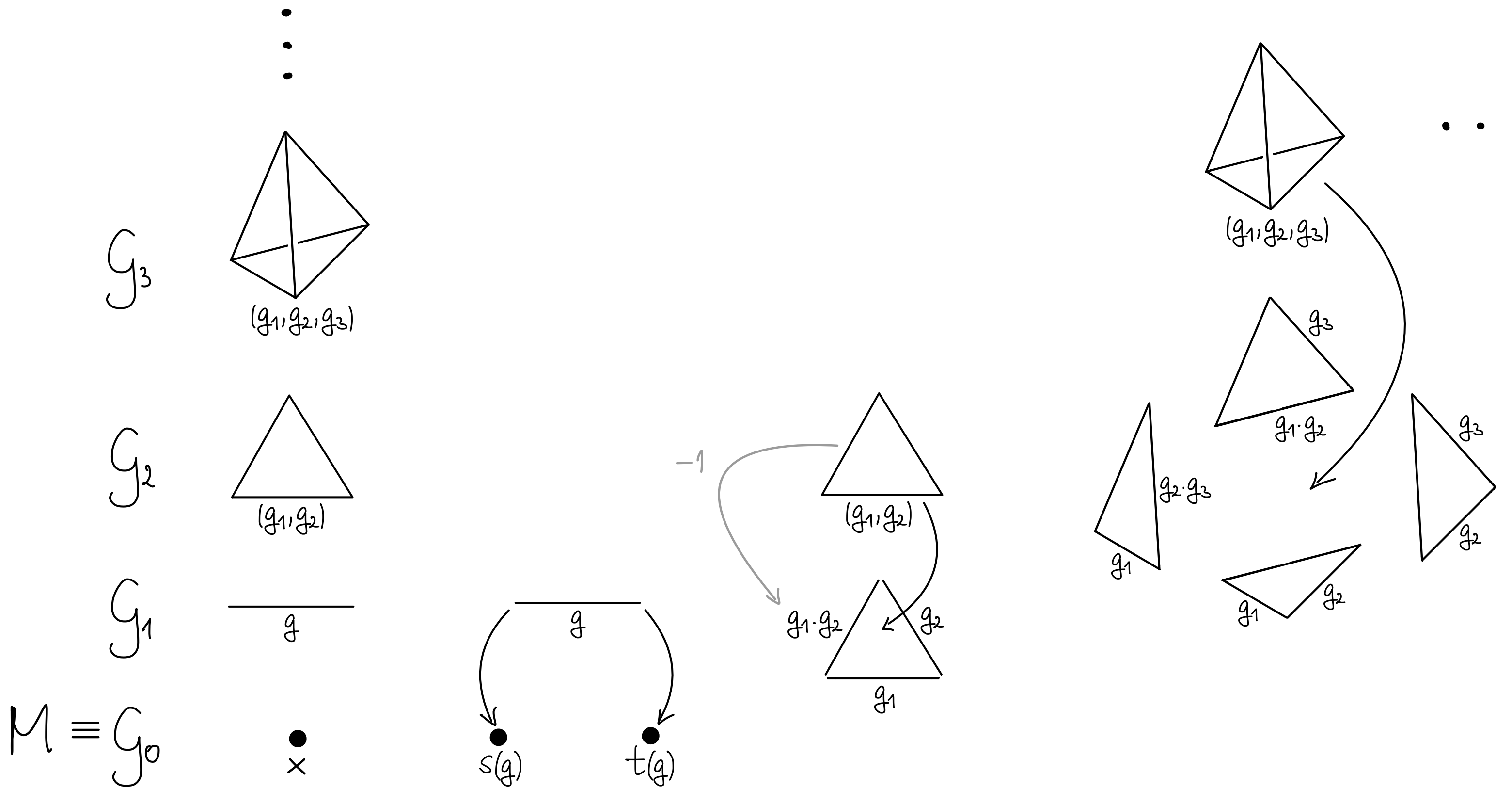
STANDARD $n$-SIMPLEX

QUOTIENT SPACE $\displaystyle \|\mathcal{G}_\bullet\| := \coprod_{n \in \mathbb{N}} (\mathcal{G}_n \times \Delta_n) /\sim$ w.r.t. $\left(d^{(n+1)}_i(\gamma), t\right) \sim \left(\gamma, \partial^{(n+1)}_i(t)\right)$

$$\partial^{(n+1)}_i : \Delta^n \to \Delta^{n+1},\; (t_1, t_2, \dots, t_{n+1}) \mapsto (t_1, t_2, \dots, t_i, 0, t_{i+1}, t_{i+2}, \dots, t_{n+1})$$

IS A HOMOTOPY MODEL of $M/\!/\mathcal{G}$ : **HOMOTOPY QUOTIENT**

# ENGINEERING THE HOMOTOPY MODEL ...



# IMPLEMENTATIONS of SYMMETRY by GROUPOIDS:

**Def$^n$ 4.** RIGHT $\mathcal{G}$-MODULE is A TRIPLE $(X, \mu, \rho)$ COMPOSED of

(*) MANIFOLD $X$;

(**) MAPS: $\mu : X \to M$ (MOMENT) & $\rho : X\,{}_{\mu}\!\times_{t} \mathcal{G} \to X$, $(x,g) \mapsto x \triangleleft g$ (ACTION)

s.th.

(RM1) $\mu(x \triangleleft g) = s(g)$; (RM2) $x \triangleleft \mathrm{Id}_{\mu(x)} = x$; (RM3) $(x \triangleleft g) \triangleleft h = x \triangleleft g.h$.

LEFT $\mathcal{G}$-MODULES $(X, \mu, \lambda)$ ARE DEFINED ANALOGOUSLY.

NOTATION:
$$\begin{tikzcd} X \arrow[dr, "\mu"'] & & \mathcal{G} \arrow[ll, bend right, "\rho"'] \arrow[dl, Rightarrow] \\ & M & \end{tikzcd}$$
RESP.
$$\begin{tikzcd} \mathcal{G} \arrow[rr, bend left, "\lambda"] \arrow[dr, Rightarrow] & & X \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

MORPHISM of RIGHT $\mathcal{G}$-MODULES is A MAP $\Phi : X_1 \xrightarrow[C^\infty]{} X_2$ s.th.
$$\begin{tikzcd} X_1\,{}_{\mu_1}\!\times_t \mathcal{G} \arrow[r, "\Phi \times id_{\mathcal{G}}"] \arrow[d, "\rho_1"'] & X_2\,{}_{\mu_2}\!\times_t \mathcal{G} \arrow[d, "\rho_2"] \\ X_1 \arrow[r, "\Phi"'] & X_2 \end{tikzcd}$$
is WELL-DEFINED & COMMUTATIVE.

E.g., (*) TARGET ACTION:

$$\begin{tikzcd} \mathcal{G} \arrow[rr, bend left, "t\circ pr_1"] \arrow[dr, Rightarrow] & & M \arrow[dl, equal] \\ & M & \end{tikzcd}$$

$$(t\circ pr_1)(g,x) = t(g)$$
$$x = s(g)$$

(**) RIGHT-FIBRED ACTION:

$$\begin{tikzcd} \mathcal{G} \arrow[dr, "s"'] & & \mathcal{G} \arrow[ll, bend right, "r=m"'] \arrow[dl, Rightarrow] \\ & M & \end{tikzcd}$$

$$r(g,h) = g\cdot h$$
$$s(g) = t(h)$$

(***) LEFT-FIBRED ACTION:

$$\begin{tikzcd} \mathcal{G} \arrow[rr, bend left, "\ell=m"] \arrow[dr, Rightarrow] & & \mathcal{G} \arrow[dl, "t"] \\ & M & \end{tikzcd}$$

$$\ell(h,g) = h\cdot g$$
$$t(g) = s(h)$$

PROBLEM: $g\in\mathcal{G}$ ACTS ONLY on $\mu^{-1}(\{t(g)\}) \Longrightarrow$ NOT by DIFFEOs IN GENERAL

WHEREAS GFT SYMMETRIES ARE MODELLED on $Aut(M)\subset Diff(M)$...

NEED: IMPLEMENTATION of $\mathcal{G}$-ACTION by $Diff(X)$

**Def$^{n}$ 5.** GLOBAL BISECTION of $\mathcal{G} \rightrightarrows M$ is A MAP $\beta : M \xrightarrow[C^\infty]{} \mathcal{G}$

s.th. **(B1)** $s \circ \beta = \mathrm{id}_M$ ; **(B2)** $t \circ \beta \in \mathrm{Diff}(M)$.

$\Longleftrightarrow$ IT IS A SUBMANIFOLD $\beta(M) \subset \mathcal{G}$ : $s|_{\beta(M)}, t|_{\beta(M)} \in \mathrm{Diff}(\beta(M), M)$.

**NOTATION:** $\mathbb{B}(\mathcal{G})$ SET of BISECTIONS of $\mathcal{G} \rightrightarrows M$

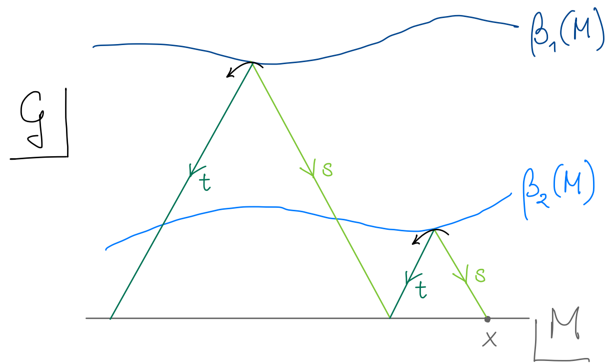
**Prop$^{n}$ 1.** $\mathbb{B}(\mathcal{G})$ is A GROUP.

**P:** BINARY OPERATION : $\beta_1 \cdot \beta_2(\bullet) := \beta_1 \circ t \circ \beta_2(\bullet) \cdot \beta_2(\bullet)$

w/ UNIT $\mathrm{Id}$

& INVERSE

$\beta^{-1} = \mathrm{Inv} \circ \beta \circ (t \circ \beta)^{-1}$



E.g., (*) $\mathbb{B}(\mathrm{Pair}(M)) \simeq \mathrm{Diff}(M)$ $\qquad \beta = (f, \mathrm{id}_M)$

(**) $\mathbb{B}(BG) \simeq G$ $\qquad \beta(\bullet) \in G$

(***) $\mathbb{B}(G \ltimes_\lambda M) \supset \{ (g, \cdot) : M \to G \times M \mid g \in G \} \xleftarrow{\sim} G$

(****) $\mathbb{B}(\mathrm{At}(P)) \simeq \mathrm{Aut}_{\mathrm{Bun}_G(\Sigma)}(P)$ **!**

**Prop$^{\underline{n}}$ 2.** $\forall\ (X, \mu, \lambda)$ **(LEFT)** $\mathcal{G}$-**MODULE** $\exists\ B\lambda : \mathbb{B}(\mathcal{G}) \xrightarrow[(\mathrm{Grp})]{} \mathrm{Diff}(X)$

P:

$$\begin{tikzcd}
\mathbb{B}(\mathcal{G}) \times M \times X \arrow[r, "ev \times \mathrm{id}_X"] & \mathcal{G}\, {}_s\!\times_\mu X \arrow[d, "\lambda"] \\
\mathbb{B}(\mathcal{G}) \times X \arrow[u, "\mathrm{id}_{\mathbb{B}(\mathcal{G})} \times (\mu, \mathrm{id}_X)"] \arrow[r, "\widetilde{B\lambda}"'] \arrow[ur, phantom, "!!"] & X
\end{tikzcd}$$

$\blacksquare$

E.g., (*) TARGET ACTION $t \circ pr_1$ INDUCES SHADOW ACTION

$$t_* : B(\mathcal{G}) \longrightarrow \mathrm{Diff}(M), \quad \beta \longmapsto t \circ \beta(\cdot) \equiv \beta^{\trianglerighteq}$$

(**) RIGHT-FIBRED ACTION $r$ INDUCES RIGHT-MULTIPLICATION

$$R : B(\mathcal{G}) \longrightarrow \mathrm{Diff}(\mathcal{G}), \quad \beta \longmapsto m \circ \left(\cdot, \beta \circ (t \circ \beta)^{-1} \circ s(\cdot)\right)$$

(***) LEFT-FIBRED ACTION $\ell$ INDUCES LEFT-MULTIPLICATION

$$L : B(\mathcal{G}) \longrightarrow \mathrm{Diff}(\mathcal{G}), \quad \beta \longmapsto m \circ \left(\beta \circ t(\cdot), \cdot\right)$$

& SO ALSO

CONJUGATION $\quad C : B(\mathcal{G}) \longrightarrow \mathrm{Diff}(\mathcal{G}), \quad \beta \longmapsto L_\beta \circ R_{\beta^{-1}}$

## STRUCTURAL PROPERTIES RELEVANT to PHYSICAL APPLICATIONS:

**Prop$^{\underline{n}}$ 3.** $\mathrm{Diff}_{r(\mathcal{G})}(\mathcal{G}) \left(\equiv \mathrm{Mod}_{\mathcal{G}^{op}}(\mathcal{G},\mathcal{G})\right) \simeq L(\mathbb{B}(\mathcal{G}))$

**?:** EXERCISE!

**COMPARE:** $\mathrm{Diff}_{r(G)}(G) \simeq \ell(G) \qquad \left(G \simeq \mathbb{B}(BG)\right)$

**Prop$^{\underline{n}}$ 4.** $C(\mathbb{B}(\mathcal{G}))$ PRESERVES IDENTITY BISECTION $\mathrm{Id}(M)$:

$$C_\beta(\mathrm{Id}_x) = \mathrm{Id}_{t_*\beta(x)}\,.$$

**?:** EXERCISE!

**COMPARE:** $\mathrm{Ad}_g(e) = e$

# TANGENT STRUCTURE, & "INFINITESIMAL" SYMMETRIES

**Def$^{\underline{n}}$ 6.** TANGENT LIE ALGEBROID of $\mathcal{G} \Longrightarrow M$ IS THE LIE ALGEBROID (SEE: KG's COURSE)

$$\Big( \underset{\mathrm{Lie}(\mathcal{G})}{\overset{}{\mathcal{E}}} \equiv \mathrm{Id}^*\mathrm{Ker}\,Ts \xrightarrow{\pi_{\mathcal{E}}} M,\ [\cdot,\cdot]_{\mathcal{E}} \equiv \Theta_* \circ [\cdot,\cdot]_{\Gamma(T\mathcal{G})} \circ \left(\Theta_*^{-1} \times \Theta_*^{-1}\right),\ \alpha_{\mathcal{E}} \equiv Tt \Big)$$

$\mathcal{E} \equiv \mathrm{Lie}(\mathcal{G})$

VECTOR BUNDLE — LIE BRACKET — ANCHOR

WHERE $\mathrm{Id}^*\mathrm{Ker}\,Ts \equiv M\ {}_{\mathrm{Id}}\!\times_{\pi_{T\mathcal{G}}|}\ \mathrm{Ker}\,Ts \xrightarrow{pr_2} \mathrm{Ker}\,Ts \equiv T\Big(\bigsqcup_{x\in M} s^{-1}(\{x\})\Big) \subset T\mathcal{G}$

$$\begin{tikzcd}
\mathrm{Id}^*\mathrm{Ker}\,Ts \arrow[r, "pr_2"] \arrow[d, "\pi_{\mathcal{E}} = pr_1"'] & \mathrm{Ker}\,Ts \arrow[d, "\pi_{T\mathcal{G}}|"] \\
M \arrow[r, "\mathrm{Id}"'] & \mathcal{G}
\end{tikzcd}$$

& $\Theta_* : \underset{\text{RIGHT-INVARIANT VECTOR FIELDS}}{\Gamma(T\mathcal{G})_{r(\mathcal{G})}} \xrightarrow{\cong} \Gamma(\mathrm{Id}^*\mathrm{Ker}\,Ts)$

$\quad : \mathcal{R} \longmapsto \Theta \circ \mathcal{R}$

for

$$\begin{tikzcd}
\mathrm{Ker}\,Ts \arrow[r, "\Theta"] \arrow[d, "\pi_{T\mathcal{G}}|"'] & \mathcal{E} \arrow[d, "\pi_{\mathcal{E}}"] \\
\mathcal{G} \arrow[r, "t"'] & M
\end{tikzcd}$$

$\pi_{T\mathcal{G}}(v) = g$

$\Theta(v) = \left(t(g), T_g r_{g^{-1}}(v)\right)$

$\mathrm{Ker}\,Ts$-FOLIATED $\mathcal{E}$-VALUED **MAURER-CARTAN FORM**

**OBSERVATION**: SPLITTING of $\mathrm{T}\mathcal{G}\restriction_{\mathrm{Id}(M)}$ IS GENERIC:

$$\forall \beta \in \mathrm{B}_{(\mathrm{loc})}(\mathcal{G}),\ x \in \mathrm{dom}(\beta):\ \underline{\mathrm{T}_{\beta(x)}\mathcal{G} \simeq \mathrm{T}_x\beta(\mathrm{T}_x M) \oplus \mathrm{Ker}\,\mathrm{T}_{\beta(x)} s \simeq \mathrm{T}_x M \oplus \mathcal{E}_{t_*\beta(x)}}$$

& SHALL PLAY A RÔLE IN THE GAUGING...

**E.G.**, **(*)** $\mathrm{Lie}(\mathrm{Pair}(M)) \simeq \mathrm{T}M$, $[\cdot,\cdot]_{\mathrm{T}M} \equiv [\cdot,\cdot]_{\Gamma(\mathrm{T}M)}$, $\alpha_{\mathrm{T}M} = \mathrm{id}_{\mathrm{T}M}$

**(**)** $\mathrm{Lie}(\mathrm{B}G) \simeq \mathrm{Lie}\,G$

**(***)** $\mathrm{Lie}(G \ltimes_\lambda M) \simeq M \times \mathrm{Lie}\,G$, $[\cdot,\cdot]_{M\times\mathrm{Lie}G}$ DETERMINED by $[\cdot,\cdot]_{\mathrm{Lie}G}$ & LEIBNIZ PROPERTY

$\alpha_{M\times\mathrm{Lie}G} \equiv \mathcal{K}^{(\lambda)}$ FUNDAMENTAL VECTOR FIELD

THE LATTER DEFINES AN ACTION of $\mathrm{Lie}\,G$ on $M$ $\equiv$ "INFINITESIMAL" SYMMETRY

**GENERALISATION**: $\alpha_{\mathcal{E}*}: \Gamma(\mathcal{E}) \to \Gamma(\mathrm{T}M)$, OR EVEN $\alpha_{\mathrm{Lie}(\mathcal{G}\ltimes_\lambda X)*}$ ...

# BACK TO GFT, & ITS $\mathcal{G}$-MODELLED SYMMETRIES...

IDEA: REINTERPRET $\mathcal{G} \curvearrowright M$ (resp. $X$) PASSIVELY, & CONSIDER DYNAMICS INDEPENDENT of (LOCALLY) SMOOTHLY $\Sigma$-INDEXED CHANGES of REFERENCE FRAME/OBSERVER/GAUGE...

CLASSIC REALISATION [EHRESMANN-CARTAN]: GIVEN $G \curvearrowright M$, CONSTRUCT

BUNDLE of GAUGES

PRINCIPAL $G$-BUNDLE

$$\begin{tikzcd} P \arrow[d, "\pi_P"'] \arrow[dr, dashed] & & G \arrow[ll, bend right, "e"'] \arrow[dl, Rightarrow, dashed] \\ P/G \simeq \Sigma & \bullet & \end{tikzcd}$$

GLOBAL CHANGES of GAUGE

LOCALLY MODELLED ON $\Sigma \times (G, \cdot, r)$,

w/ $G$-INVARIANT EHRESMANN CONNECTION

$$\begin{tikzcd} TP \arrow[r, "H"] \arrow[d, "\pi_{TP}"'] \arrow[rr, bend left, "\mathcal{A}"] & VP \equiv \operatorname{Ker} T\pi_P \arrow[r, "\kappa^{(e)-1}", "\simeq"'] \arrow[d, "\pi_{TP}|"] & P \times \operatorname{Lie} G \arrow[d, "pr_1"] \\ P \arrow[r, equal] & P \arrow[r, equal] & P \end{tikzcd}$$

$$\operatorname{Ker} H \equiv \operatorname{Ker} \mathcal{A} = HP$$

$$\forall p \in P, g \in G: \; T_p e_g(H_p P) = H_{e_g(p)} P$$

$$\begin{tikzcd}
& G\times M \arrow[d, hook] & \\
& P\times M \arrow[dl, "\pi_P\circ pr_1"'] \arrow[d, "pr_2"] & G\times M \arrow[l, bend right, "e"'] \arrow[ul, bend right, "r"'] \arrow[dl, Rightarrow] \\
\Sigma & M &
\end{tikzcd}$$

$r$ (LOCAL MODEL)

NEXT, EXTEND $P \nearrow P\times M$ & EQUIP with $\widetilde{H} = pr_1^* H + id_{pr_2^* TM}$

NB: $r$ IS PRINCIPAL RELATIVE to $M$ along $t \equiv \lambda$, i.e.,

$$(pr_{1,2}, r)^{-1} : (G\times M)\,{}_\lambda\!\times_\lambda (G\times M) \xrightarrow{\ \cong\ } (G\times M)\,{}_{pr_2}\!\times_\lambda (G\times M)$$

$$\equiv (pr_{1,2}, \phi)$$ DIVISION MAP $$\phi\big((g_1,x),(g_2,\lambda_{g_2^{-1}g_1}(x))\big) = (g_1,x)^{-1}\cdot(g_2,\lambda_{g_2^{-1}g_1}(x)) = (g_1^{-1}g_2, \lambda_{g_2^{-1}g_1}(x))$$

$\Longrightarrow$ $e$ IS FIBREWISE PRINCIPAL RELATIVE to

along $\pi_\sim : P\times M \twoheadrightarrow P\times_\lambda M, \quad (p,x) \longmapsto [(p,x)]$

LOCALLY MODELLED on $\lambda$

& $\widetilde{H}$ INDUCES $H_C \circ T\pi_\sim := T\pi_\sim \circ \widetilde{H}$ CRITTENDEN CONNECTION

$$\begin{tikzcd}
M \arrow[d, hook] \\
P\times_\lambda M \equiv (P\times M)/G \arrow[d, "{[\pi_P]}"'] \\
\Sigma
\end{tikzcd}$$

ASSOCIATED BUNDLE

**LOCAL PICTURE**: G-EQUIVARIANT TRIVIALISATIONS $\tau_i : \pi_P^{-1}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times G$ YIELD

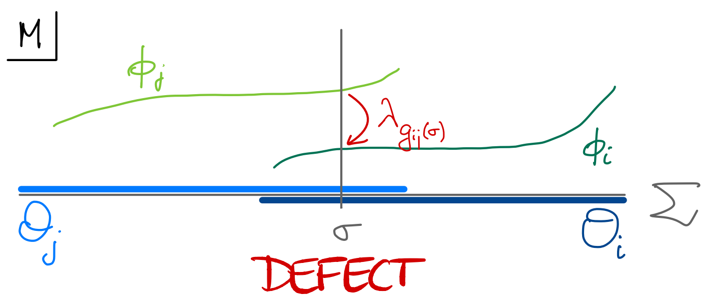
**TRANSITION 1-COCYCLE** $g_{ij} : \mathcal{O}_{ij} \longrightarrow G$ AS $\tau_i \circ \tau_j^{-1}(\sigma, g) = (\sigma, \ell_{g_{ij}(\sigma)}(g))$, $\sigma \in \mathcal{O}_{ij}$

$$\Rightarrow \quad P \simeq \bigsqcup_{i \in I} \mathcal{O}_i \times G / \sim_{g_{\cdot\cdot}} \, , \quad (\sigma, g, j) \sim (\sigma, \ell_{g_{ij}(\sigma)}(g), i) \qquad \text{LOCAL MODEL}$$

THESE INDUCE $[\tau_i] : \pi_{P \times_\lambda M}(\mathcal{O}_i) \xrightarrow{\cong} \mathcal{O}_i \times M$, $[(\tau_i^{-1}(\sigma, g), x)] \longmapsto (\sigma, \lambda_g(x))$,

& THEN $\phi \in \Gamma(P \times_\lambda M)$ IS LOCALLY PRESENTED AS

$$[\tau_i] \circ \phi = (\mathrm{id}_{\mathcal{O}_i}, \phi_i) \quad \text{for} \quad \phi_i : \mathcal{O}_i \longrightarrow M \quad \text{s.th.} \quad \phi_i(\sigma) = \lambda_{g_{ij}(\sigma)}(\phi_j(\sigma)), \, \sigma \in \mathcal{O}_{ij}$$




DISCONTINUOUS
**G-TWISTED SECTOR**
of GAUGE FIELD THEORY
(-TO-BE)

# TEASER for LECTURE II :

WHEN GAUGING SYMMETRY MODELLED on $\mathcal{G} \rightrightarrows M$ & IMPLEMENTED by $\mathbb{B}(\mathcal{G})$,

START from BUNDLE of GAUGES

$$
\begin{tikzcd}
\mathcal{G} \arrow[d, hook] & & \\
\mathcal{P} \arrow[d, "\pi_{\mathcal{P}}"'] \arrow[dr, "\mu"] & & \mathcal{G} \arrow[ll, bend right, "e"'] \arrow[ull, bend right, "r"'] \arrow[dl, shift left] \arrow[dl, shift right] \\
\Sigma & M &
\end{tikzcd}
$$

**PRINCIPALOID**

$\mathcal{G}$-**BUNDLE**

[arXiv: 2503.09886 [math.DG]]

## PHYSICS :

(*) CONFIGURATIONAL BUNDLE AS SMOOTH QUOTIENT $\mathcal{P}/\mathcal{G}$

(**) GAUGE FIELD from $\mathcal{G}$-INVARIANT EHRESMANN CONNECTION on $\mathcal{P}$

(***) COVARIANT DERIVATIVE from INDUCED EHRESMANN CONNECTION on $\mathcal{P}/\mathcal{G}$

## INTRICACIES :

(*) $\mathcal{E}$-VALUED GAUGE FIELD

(**) STRUCTURE-GROUP REDUCTIONS in PRESENCE of STRUCTURE on $\mathcal{G}$

(***) STACKY DESCENT of TOPOLOGICAL COUPLINGS...