

Gauging the Oïd
& Other Tales

PART III
Homeopathic
2-Gauge Theory

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PALES BEN MAR,
2008. SO AN-ERIC P.

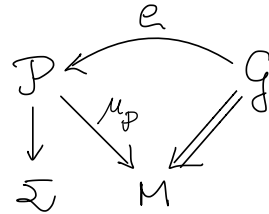
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*La Géométrie est une image,
mais il n'y a pas d'image de la Géométrie*

IN THE LAST EPISODE : CONSTRUCTION of

(*) BUNDLE of FRAMES=GAUGES MODELLED on G à la Ehresmann

— THE PRINCIPALOID BUNDLE



with STRUCTURE GROUP

$$B(G) \leftarrow \mathcal{O}_i: \beta_{ij}$$

REALISED by $L: B(G) \rightarrow \text{Diff}(G)$

(**) CONFIGURATION BUNDLE — THE SHADOW $\mathcal{F} \simeq P/G \xleftarrow{D} P$

with β_{ij} REALISED by $t_*: B(G) \rightarrow \text{Diff}(M)$

$$\downarrow \Sigma$$

(***) LOCAL GAUGE FIELD with VALUES in $\mathcal{E} = \text{Id}^* \text{Ker } T\Sigma$,

from G -EQUIVARIANT PROJECTOR FIELDS $\odot: TP \rightarrow VP$

(****) SHADOW GAUGE SYMMETRY GROUP $\text{Gau}(\mathcal{F}) = \mathcal{F}_*(\text{Gau}(P))$

Ad (iv) WHEN LIFTED TO THE TANGENT, $\mathcal{G}$ -EQUIVARIANCE YIELDS

Propⁿ 9. COMPATIBLE CONNECTION $(\mathcal{H})$ CANONICALLY INDUCES ONE ON $\mathcal{F}$
 $\mathcal{J}$ -th. $(\mathcal{H})_{\mathcal{F}} \circ T\mathcal{D} := T\mathcal{D} \circ (\mathcal{H})$.

9: INDUCE à la KOBAYASHI-NOMIZU: $\begin{cases} H_{\mathcal{D}(p)} \mathcal{F} := T_p \mathcal{D}(H_p \mathcal{P}) \\ V_{\mathcal{D}(p)} \mathcal{F} := T_p \mathcal{D}(V_p \mathcal{P}) \end{cases}$. $\square$

THIS GIVES THE DESIRED COUPLING

Defⁿ 12. COVARIANT DERIVATIVE on $\Gamma(\mathcal{F})$ IS

$$\nabla^{(\mathcal{H})} : \Gamma(\mathcal{F}) \ni \phi \longmapsto (\mathcal{H})_{\mathcal{F}} \circ T\phi \in \Gamma(T^*\Sigma \otimes \phi^* V\mathcal{F})$$

EXTENDING $\text{Gau}(\mathcal{P})$ AS $(P, (\mathcal{H})) \xrightarrow{\bar{\mathcal{F}}} (P, \underbrace{T\bar{\mathcal{F}} \circ (\mathcal{H}) \circ T\bar{\mathcal{F}}^{-1}}_{\text{GAUGE TRANSFORM}} \equiv (\mathcal{H})^{\bar{\mathcal{F}}})$, WE GET

Propⁿ 10. $\forall (\phi, \bar{\mathcal{F}}) \in \Gamma(\mathcal{F}) \times \text{Gau}(\mathcal{P}) : \nabla^{(\mathcal{H})^{\bar{\mathcal{F}}}} \phi^{\bar{\mathcal{F}}} = T\bar{\mathcal{F}}_*(\bar{\mathcal{F}}) \circ \nabla^{(\mathcal{H})} \phi$

Proof: Follows from $(\mathcal{H})^{\bar{\mathcal{F}}} = T\bar{\mathcal{F}}_*(\bar{\mathcal{F}}) \circ (\mathcal{H})_{\mathcal{F}} \circ T\bar{\mathcal{F}}^{-1}$. $\square$

IN THE LOCAL PICTURE, THERE EMERGES THE BASIC GAUGE-TO-MATTER COUPLING:

$$\mathcal{D}^{A_i} \phi_i := p_{\mathbb{R}} \circ T\mathcal{F}_i \circ \nabla^{\oplus} \phi = T\phi_i - \underbrace{\alpha_{\varepsilon}(\phi_i(\cdot)) \circ \underline{A}_i(\cdot, \phi_i(\cdot))}_{\text{the Higgsoid at its HIGGSDEST!}}$$

the Higgsoid at its HIGGSDEST!

PROBLEM: THE GLUONOID A_i SEEMS GLUED TO THE HIGGSOID...

SOLUTION: ASSOCIATION $\mathcal{P} \rightsquigarrow (\mathcal{P}_{\mu_{\mathbb{P}}} \times_{\mu_{X_{\alpha}}} X_{\alpha}) / G, \alpha \in \text{Matter}$

[arXiv: 2603.20914 [hep-th]]

WE FIND THE DESIRED GAUGE-TRANSFORMATION LAWS:

$$\phi_i(\sigma) \longmapsto t_*(\gamma_i(\sigma))(\phi_i(\sigma)) \quad \text{GLUING of LOCAL MATTER by } g\text{-ACTION}$$

$$\underline{A}_i(\sigma, x) \longmapsto \underline{A}_i^{\bar{\Phi}}(\sigma, t_*(\gamma_i(\sigma))(x)) = T_{\text{Id}_x} C_{\gamma_i(\sigma)} \circ \underline{A}_i(\sigma, x) + \Theta \circ T_{\sigma} \tilde{\gamma}_i(\cdot, x)$$

AFFINE g -GAUGE TRAO of GLUONOID

for $\tilde{\gamma}_i : \mathcal{O}_i \times M \rightarrow G, (\sigma, x) \longmapsto \gamma_i(\sigma)(x)$

$$\text{So } \mathcal{D}^{A_i} \phi_i(\sigma) \longmapsto \mathcal{D}^{A_i^{\bar{\Phi}}} \phi_i^{\bar{\Phi}}(\sigma) = T_{\phi_i(\sigma)} t_*(\gamma_i(\sigma)) \circ \mathcal{D}^{A_i} \phi_i(\sigma)$$

SHADOWY COVARIANCE
of LOCAL GAUGE-TO-MATTER COUPLINGS

AD (v) THE GAUGING OF $B_*(g)$ -INVARIANT TENSORIAL COUPLINGS

GIVEN $T \in \Gamma(T^*M^{\otimes n})$, DEFINE $B_T(g) := \{ \beta \in B(g) \mid (t_*\beta)^* T = T \}$
T-BISECTION GROUP

* REDUCE THE STRUCTURE GROUP OF $P \rightarrow F$: $B(g) \rightarrow B_T(g)$

$$\Rightarrow \text{Gau}_T(P) \xrightarrow{F^*} \text{Gau}_T(F)$$

** MINIMALLY COUPLE A_i to T : $T \leadsto T[A_i] := T \circ \bigoplus_{F_i}^{\otimes n}$
 with $\bigoplus_{F_i}(\sigma_i x) = \text{id}_{T_x M} - \alpha_E(x) \circ A_i(\sigma_i x)$

** DESCEND $\{ \tilde{T}_i := F\tau_i^* T[A_i] \}_{i \in I}$ along $\check{Y}_F := \bigsqcup_{i \in I} \pi_F^{-1}(\mathcal{O}_i) \rightarrow F$

THE SHEAFY WAY $\tilde{T}_i|_{\pi_F^{-1}(\mathcal{O}_{ij})} = \tilde{T}_j|_{\pi_F^{-1}(\mathcal{O}_{ij})} \Rightarrow \tilde{T} \in \Gamma(T^*F^{\otimes n}) : \tilde{T}|_{\pi_F^{-1}(\mathcal{O}_i)} = \tilde{T}_i$ (34)

UPSHOT: $\text{Gau}_T(\mathcal{F})$ - INVARIANT COUPLING

$$\phi^* \widetilde{T}$$

E.g., RECALL THE METRIC COUPLING in the PAC σ -MODEL (9.2):

$$\mathcal{A}_{\text{PAC}}^{\text{mch}}[X] = \exp\left(-\frac{i}{2} \|TX\|_{\eta_\Sigma, g_M}^2\right)$$

$$\|TX\|_{\eta_\Sigma, g_M}^2 \xrightarrow{\text{loc.}} \int \sqrt{|\det \eta_\Sigma|} \cdot \eta_\Sigma^{-1 ab} g_M(x)_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$$

Upon METRIC REDUCTION $B(\mathcal{G}) \xrightarrow{\text{red}} t_*^{-1}(\text{Iso}(M, g_M)) = B_{g_M}(\mathcal{G})$.

WE OBTAIN $\text{Gau}_{g_M}(\mathcal{F})$ - INVARIANT COUPLING

$$\Gamma(\mathcal{F}) \ni \phi \longmapsto \widetilde{\mathcal{A}}_{\text{PAC}}^{\text{mch}}[\phi] = \exp\left(-\frac{i}{2} \|T\phi\|_{\eta_\Sigma, g_\mathcal{F}}^2\right)$$

for $g_\mathcal{F} \in \Gamma(S^2 T^* \mathcal{F})$, $g_\mathcal{F}|_{\pi_i^{-1}(\theta_i)} = \pi_i^*(g_M \circ \otimes_{\mathcal{F}_i}^{\otimes 2})$.

THE STORY of the TOPOLOGICAL COUPLING is QUITE A BIT MORE INVOLVED... (35)

BACK to PAC

...WHERE TWO (CATEGORIFIED) GAUGE SYSTEMS MEET...

IN HYPERCOHOMOLOGY

THE GAUGING OF Hol_g PREREQUIRES THAT THE HIGHER-COMOLOGICAL OBJECT g BE UNDERSTOOD...

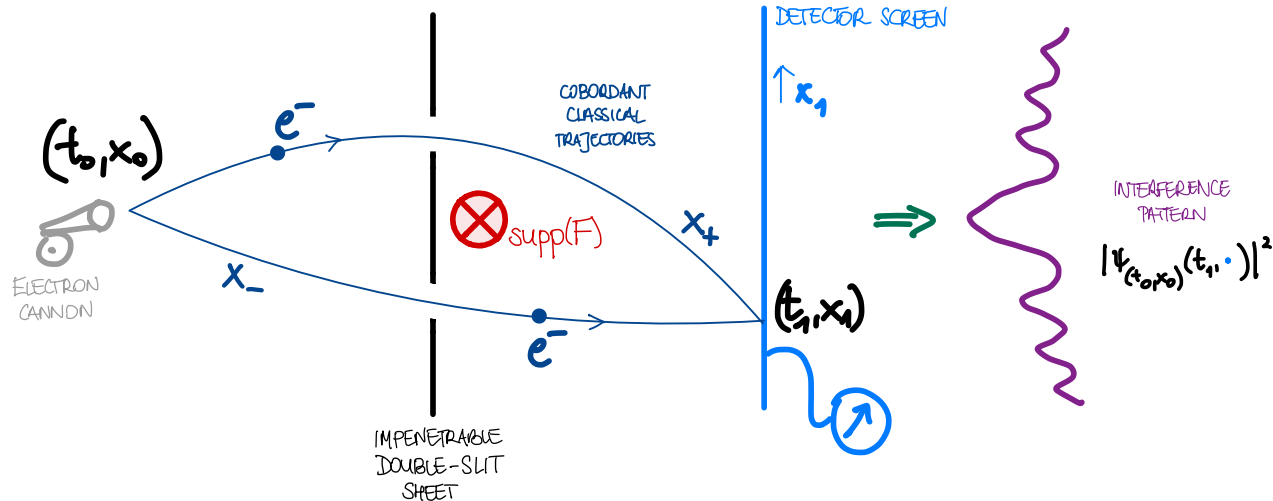
'TOY MODEL': CONSIDER, INSTEAD, 1d DIRAC σ -MODEL:

$$X : \Sigma \xrightarrow{C^\infty} M \quad \text{for } \Sigma \simeq S^1 \quad (\partial\Sigma = \emptyset)$$

$$\mathcal{E}_C(M, g_M, F), \quad F \in \mathbb{Z}_{\text{dr}}^2(M),$$

FARADAY TENSOR

MODELLING THE INTERFERENCE PHASE in THE AHARONOV-BOHM EXPERIMENT...



LORENTZ FORCE on $e \equiv 1$ MODELLED by $\int_{[t_0, t_n]} x_{\pm}^* A_{\pm}$, BUT THERE MAY BE

NO GLOBAL POTENTIAL over NON-CONTRACTIBLE LOOP $x_{+}(\overrightarrow{[t_0, t_n]}) \cup x_{-}(\overleftarrow{[t_0, t_n]}) \dots$

SOLUTION: USE LOCAL TRIVIALISATION of F :

$$F|_{\mathcal{O}_i} = d\alpha_i, \quad \alpha_i \in \Omega^1(\mathcal{O}_i)$$

$$(\alpha_j - \alpha_i)|_{\mathcal{O}_{ij}} = df_{ij}, \quad f_{ij} \in C^\infty(\mathcal{O}_{ij}, \mathbb{R})$$

$$(f_{jk} - f_{ik} + f_{ij})|_{\mathcal{O}_{ijk}} = N_{ijk} \in \mathbb{R}$$

over COVER of M

$$\mathcal{I} = \bigsqcup_{i \in I} \mathcal{O}_i$$

(EQ., GOOD [WEIL-DE RHAM])

& INVOKE [DIRAC '33] TO GET WELL-DEFINED (TOPOLOGICAL) DF AMPLITUDE

$$\mathcal{A}_D^{\text{top}}[x] = \exp \left[i \left(\sum_{e \in E_{\Delta_S^1}, x(e)} \int \alpha_{\lambda(e)} + \sum_{v \in V_{\Delta_S^1}} f_{\lambda_+(v)\lambda_-(v)}(x(v)) \right) \right] \quad \text{for TESSELLATION } \Delta_S^1 = E_{\Delta_S^1} \cup V_{\Delta_S^1}$$

SUBORDINATE to $\mathcal{I}$ along x

& for $N_{ijk} \in 2\pi\mathbb{Z}$ ($\Leftrightarrow \text{Per}(F) \subset 2\pi\mathbb{Z}$)

DIRAC'S QUANTISATION CONDITION

$$\exists \iota: \Delta_S^1 \rightarrow \mathbb{I} \quad \forall \tau \in \Delta_S^1 : x(\tau) \subset \mathcal{O}_{\iota(\tau)}$$

WELL-DEFINEDNESS of $\mathcal{A}_D^{\text{top}} \iff$ INDEPENDENCE of ARBITRARY CHOICES MADE,

INCLUDING CHOICE of REPRESENTATIVES : $\alpha_i \sim \alpha_i + dh_i$, $h_i \in C^\infty(\mathcal{O}_i, \mathbb{R})$

AS LONG AS $d_{ij} \in 2\pi\mathbb{Z}$

$$f_{ij} \sim f_{ij} + (h_j - h_i)|_{\mathcal{O}_{ij}} + d_{ij} \text{ , } d_{ij} \in \mathbb{R}$$

$$N_{ijk} \sim N_{ijk} + d_{jk} - d_{ik} + d_{ij}$$

UPSHOT : $F \in \text{Per}^{-1}(2\pi\mathbb{Z})$ REPRESENTED by $(\alpha_i, g_{ij} := e^{-if_{ij}}) \in \Omega^1(\mathcal{O}_i) \oplus C^\infty(\mathcal{O}_{ij}, U(1))$

$$\text{s.t. } (-i \log g_{ij} + (\alpha_j - \alpha_i)|_{\mathcal{O}_{ij}}, (g_{jk}^{-1} \cdot g_{ik} \cdot g_{ij}^{-1})|_{\mathcal{O}_{ik}}) = (0, 1) \equiv 0$$

$$\text{UP TO } (\alpha_i, g_{ij}) \mapsto (\alpha_i - i \log \chi_i, g_{ij} \cdot (\chi_j^{-1} \cdot \chi_i)|_{\mathcal{O}_{ij}}) \equiv (\alpha_i, g_{ij}) + (-i \log \chi_i, (\chi_j^{-1} \cdot \chi_i)|_{\mathcal{O}_{ij}})$$

$$\text{WHERE } (\chi_i := e^{ih_i}) \in C^\infty(\mathcal{O}_i, U(1))$$

(HIGHER) PRINCIPAL $\bigsqcup_{i \in I} \mathcal{O}_i \times U(1) / \sim_g$
GEOMETRY : $U(1)$ -BUNDLE
 $\downarrow$
 M

with CONNECTION of CURVATURE F
 PRESENTED LOCALLY by $\mathcal{A}_i(x, z) = \frac{dz}{z} + \alpha_i(x)$

...

UP TO CONNECTION-PRESERVING AUTOMORPHISM with LOCAL DATA χ_i (38)

... & (HOMOLOGICAL) ALGEBRA:

CONSIDER

ČECH GROUPOID $\equiv$ SUBMERSION GROUPOID of

$$\mathcal{O} = \bigsqcup_{i \in I} \mathcal{O}_i \xrightarrow{\tilde{\pi}} M \\ (x, i) \mapsto x$$

Defⁿ B.

$\check{C}(\mathcal{O})$ with ARROW MANIFOLD

$$\mathcal{O} \times_M \mathcal{O} \equiv \bigsqcup_{i, j \in I} \mathcal{O}_{ij} \text{ etc.} \\ ((x, i), (x, j)) \equiv (x, i, j)$$

BUILD ITS NERVE - THE ČECH NERVE

$$N_n \check{C}(\mathcal{O}) \equiv \bigsqcup_{i_0, i_1, \dots, i_n \in I} \mathcal{O}_{i_0 i_1 \dots i_n}$$

with FACE MAPS

$$d_k^{(n)}(x, i_0, i_1, \dots, i_n) = (x, i_0, i_1, \dots, \hat{i}_k, \dots, i_n)$$

SIMPLICIAL IDENTITIES SATISFIED by $d_k^{(n)}$ & $d_k^{(n+1)}$ ENSURE $\check{\Delta}_{(n+1)} \circ \check{\Delta}_{(n)} = 0$

for DUPONT OPERATORS $\check{\Delta}_{(n)} = \sum_{k=0}^{n+1} (-1)^k d_k^{(n+1)*} : \Omega^\bullet(N_n \check{C}(\mathcal{O})) \rightarrow \Omega^\bullet(N_{n+1} \check{C}(\mathcal{O}))$

& SO WE ARRIVE @ COCHAIN BICOMPLEX... (39)

Defⁿ 14.

ČECH-DE RHAM
BICOMPLEX

$$\begin{array}{c}
 \vdots \\
 \uparrow d_{(q)} \\
 \check{C}^p(\underline{\Omega}^q) \\
 \uparrow d_{(q-1)} \\
 \vdots
 \end{array}
 \xrightarrow{\check{\Delta}_{(p-1)}} \check{C}^p(\underline{\Omega}^q) \xrightarrow{\check{\Delta}_{(p)}} \dots$$

COMMUTING

with $\check{C}^p(\underline{\Omega}^q) = \prod_{i_0, i_1, \dots, i_p \in \mathbb{I}} \Omega^q(\mathcal{O}_{i_0 i_1 \dots i_p}) \ni (\omega_{i_0 i_1 \dots i_p})$

where $\underline{\Omega}^0 \equiv C^\infty(\cdot, U(1))$ & $d_{(0)} = -i d \log$

WHICH WE TRUNCATE @ FORM DEGREE $n \in \mathbb{N} \Rightarrow$ ČECH-DELIQUE
BICOMPLEX

WE THEN FORM THE DIAGONAL COMPLEX:

$$BD_{(r)} := \bigoplus_{\substack{p+q=r \\ q \leq n}} \check{C}^p(\underline{\Omega}^q) \text{ with } \underline{DELIQUE} \text{ OPERATORS } \mathcal{D}_{(r)}|_{\check{C}^p(\underline{\Omega}^q)} = d_{(q)} + (-1)^{q+1} \check{\Delta}_{(p)}$$

& CONSIDER ITS COHOMOLOGY $H^\bullet(\mathcal{O}, \mathcal{D}(n)^\bullet) \xrightarrow{\text{REFINEMENT}} H^\bullet(M, \mathcal{D}(n)^\bullet)$

BEILINSON-DELIQUE
COHOMOLOGY

THIS IS WHERE OUR CHARGE FIELDS LIVE!

INDEED, WE HAVE

$$\begin{array}{ccccccc}
 \alpha_i & & & & & & \\
 \check{C}^0(\underline{\mathcal{Q}}^1) & \xrightarrow{\check{\Delta}_{(0)}} & \check{C}^1(\underline{\mathcal{Q}}^1) & \xrightarrow{\check{\Delta}_{(1)}} & \check{C}^2(\underline{\mathcal{Q}}^1) & \xrightarrow{\check{\Delta}_{(2)}} & \dots \\
 \uparrow d_{(0)} & \oplus & \uparrow d_{(0)} & & \uparrow d_{(0)} & & \\
 \check{C}^0(\underline{\mathcal{Q}}^0) & \xrightarrow{\check{\Delta}_{(0)}} & \check{C}^1(\underline{\mathcal{Q}}^0) & \xrightarrow{\check{\Delta}_{(1)}} & \check{C}^2(\underline{\mathcal{Q}}^0) & \xrightarrow{\check{\Delta}_{(2)}} & \dots \\
 \chi_i & & g_{ij} & & & &
 \end{array}$$

$$D_{(1)}(\alpha_i, g_{ij}) = 0$$

$$(\alpha_i, g_{ij}) \sim (\alpha_i, g_{ij}) + D_{(0)}(\chi_i)$$

MOREOVER, THERE ARISES A STRUCTURE OF MONOIDAL CATEGORY:

OBJECTS : $(\alpha_i, g_{ij}) \in \text{Ker } D_{(1)}$

IDENTITIES : (1)

(ISO) MORPHISMS : $(\alpha_i, g_{ij}) \xrightarrow{(\chi_i)} (\alpha_i - \text{id} \log \chi_i, g_{ij} \cdot (\chi_j^{-1} \cdot \chi_i))|_{\alpha_j}$

'POOR MAN'S' VERSION
of GROUPOID $\text{Bun}_{U(1)}(M)$
with BAYLINSKI'S CONTRACTION

COMPOSITION : $(\chi_i) \circ (\psi_i) = (\chi_i \cdot \psi_i)$

TENSOR UNITS : $(0, 1), (1)$

TENSOR FUNCTOR : $(\alpha_i, g_{ij}) \otimes (\beta_i, h_{ij}) = (\alpha_i + \beta_i, g_{ij} \cdot h_{ij}), (\chi_i) \otimes (\psi_i) = (\chi_i \cdot \psi_i)$

IN THE PAC σ -MODEL, THE KALB-RAMOND FIELD $H \in \mathcal{L}_{dr}^3(M)$, $\text{Per}(H) \subset 2\pi\mathbb{Z}$

TRIVIALISES AS $(B_i, A_{ij}, g_{ijk}) \in \text{Ker } \mathcal{D}_{(2)}$ UP TO GAUGE EQUIVALENCE

with $dB_i = H|_{\partial_i}$ $(B_i, A_{ij}, g_{ijk}) \sim (B_i, A_{ij}, g_{ijk}) + \mathcal{D}_{(1)}(\mathcal{P}_i, k_{ij}),$

BUT NOW... WE HAVE SECONDARY GAUGE EQUIVALENCES

$$\begin{array}{ccccccc}
 \overset{B_i}{\check{C}^0(\underline{\mathcal{Q}^2})} & \xrightarrow{\check{\Delta}_{(0)}} & \check{C}^1(\underline{\mathcal{Q}^2}) & \xrightarrow{\check{\Delta}_{(1)}} & \check{C}^2(\underline{\mathcal{Q}^2}) & \xrightarrow{\check{\Delta}_{(2)}} & \dots \\
 \uparrow d_{(1)} & \oplus & \uparrow d_{(1)} & & \uparrow d_{(1)} & & \\
 \overset{\mathcal{P}_i}{\check{C}^0(\underline{\mathcal{Q}^1})} & \xrightarrow{\check{\Delta}_{(0)}} & \check{C}^1(\underline{\mathcal{Q}^1}) & \xrightarrow{\check{\Delta}_{(1)}} & \check{C}^2(\underline{\mathcal{Q}^1}) & \xrightarrow{\check{\Delta}_{(2)}} & \dots \\
 \uparrow d_{(0)} & \oplus & \uparrow d_{(0)} & \oplus & \uparrow d_{(0)} & & \\
 \check{C}^0(\underline{\mathcal{Q}^0}) & \xrightarrow{\check{\Delta}_{(0)}} & \check{C}^1(\underline{\mathcal{Q}^0}) & \xrightarrow{\check{\Delta}_{(1)}} & \check{C}^2(\underline{\mathcal{Q}^0}) & \xrightarrow{\check{\Delta}_{(2)}} & \dots \\
 h_i & & k_{ij} & & g_{ijk} & &
 \end{array}$$

$(\mathcal{P}_i, k_{ij}) \sim (\mathcal{P}_i, k_{ij}) + \mathcal{D}_{(0)}(h_i)$
 ENCODED NEATLY
 BY
 $H^2(M, \mathcal{D}(2)^{\bullet})$

CLUTCHING
MODEL

$$\Delta^{(3)} \mu_L = \text{id}$$

$$\mu_L: d_0^{(\omega)*} L \otimes d_2^{(\omega)*} L \simeq d_1^{(\omega)*} L$$

$$L = N_1 \check{C}(\sigma) \times U(1), \mathcal{A}$$

1-GERBE
 $\mathcal{G}$

$$N_3 \check{C}(\sigma) \rightrightarrows N_2 \check{C}(\sigma) \rightrightarrows N_1 \check{C}(\sigma) \rightrightarrows N_0 \check{C}(\sigma), B \in \Omega^2(N_0 \check{C}(\sigma))$$

$$\downarrow \check{\pi}$$

$$B|_{\partial_i} = B_i$$

ANALOGOUSLY, $(P_i, k_{ij}) \sim$ PRINCIPAL $U(1)$ -BUNDLE
with CONNECTION +

$(h_i) \sim$ CONNECTION-PRESERVING
ISOMORPHISM
of PRINCIPAL $U(1)$ -BUNDLES +

$$\begin{array}{ccc} f^*P = X_f \times_{\pi_P} P & \xrightarrow{p_2} & P \\ p_1 \downarrow & & \downarrow \pi_P \\ X & \xrightarrow{f} & B \end{array}$$

CHOICE of HONEYCOMB TESSELLATION $\Delta_\Sigma = P_{\Delta_\Sigma} \cup E_{\Delta_\Sigma} \cup V_{\Delta_\Sigma}$ SUBORDINATE to σ along X ,
(3-VALENT) $\exists \nu: \Delta_\Sigma \rightarrow I \quad \forall e \in \Delta_\Sigma: x(e) < \sigma_{\nu(e)}$

YIELDS TOPOLOGICAL DF AMPLITUDE $\equiv$ SURFACE HOLONOMY (CONSTANT on $[[B_i, A_{ij}, g_{jk}]]$)

$$\text{Hol}_g(X(\Sigma)) = \exp \left[i \left(\sum_{p \in P_{\Delta_\Sigma}} \left\{ B_{\nu(p)} \right\}_{X(p)} + \sum_{e \in E_{\Delta_\Sigma}} \left\{ A_{\nu(p(e)) \nu(p(e))} \right\}_{X(e)} \right) \right] \prod_{v \in V_{\Delta_\Sigma}} g_{\nu(p(v)) \nu(p(v)) \nu(p(v))}(x(\sigma))$$

THIS TIME, WE OBTAIN A MONOIDAL 2-CATEGORY...

OBJECTS: $\textcircled{b} = (B_i, A_{ij}, g_{ijk}) \in \text{Ker } D_\omega$
 0-CELLS

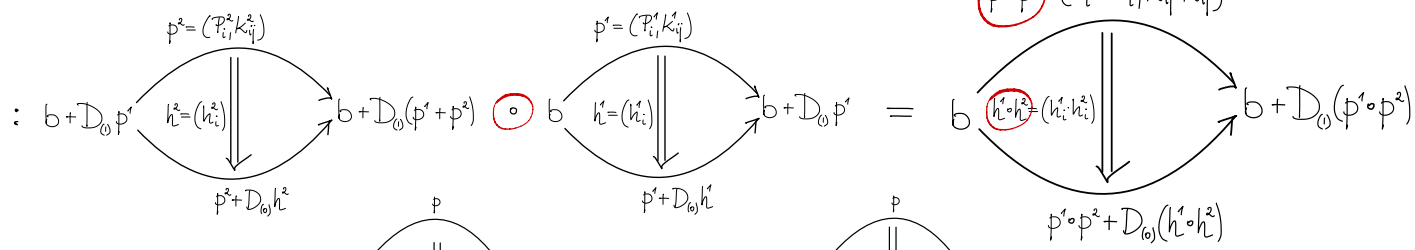
OBJECT IDENTITIES: $(0, 1)$

1-(ISO) MORPHISMS: $(B_i, A_{ij}, g_{ijk}) \xrightarrow{(P_i, k_{ij})} (B_i + dP_i, A_{ij} + (P_j - P_i)|_{O_j} - \text{id} \log k_{ij}, g_{ijk} \cdot (k_{jk}^{-1} \cdot k_{ik} \cdot k_{ij}^{-1})|_{O_{jk}})$
 1-CELLS

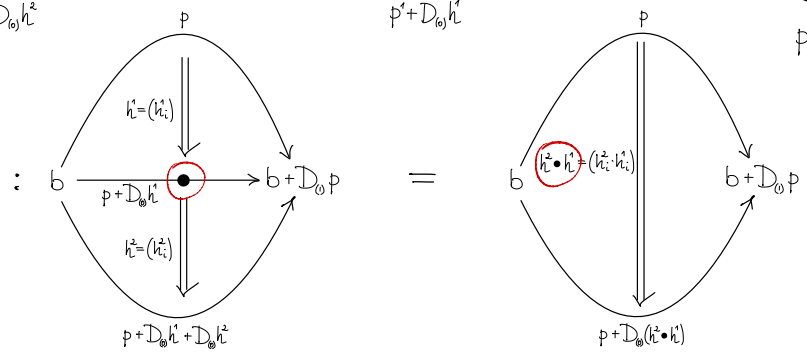
2-(ISO) MORPHISMS: $b \xrightarrow{(P_i, k_{ij})} b + D_\omega p$
 2-CELLS
 $(P_i - \text{id} \log h_i, k_{ij} \cdot (h_j^{-1} \cdot h_i)|_{O_j})$

1-MORPHISM IDENTITIES: (1)

HORIZONTAL COMPOSITION FUNCTOR



VERTICAL COMPOSITION



TENSOR 2-FUNCTOR: $(B_i^1, A_{ij}^1, g_{ijk}^1) \otimes (B_i^2, A_{ij}^2, g_{ijk}^2) = (B_i^1 + B_i^2, A_{ij}^1 + A_{ij}^2, g_{ijk}^1 \cdot g_{ijk}^2)$

$$(P_i^1, k_{ij}^1) \otimes (P_i^2, k_{ij}^2) = (P_i^1 + P_i^2, k_{ij}^1 \cdot k_{ij}^2)$$

TENSOR UNITS: $(0, 0, 1), (0, 1), (1)$ $(h_i^1) \otimes (h_i^2) = (h_i^1 \cdot h_i^2)$

NB: $\text{curv}(g_1 \otimes g_2) = \text{curv}(g_1) + \text{curv}(g_2)$
 $\text{curv}^{-1}(H)$ IS $\text{curv}^{-1}(0) \simeq H^2(M, U(1))$ -TORSOR

...THE 2-CATEGORY of (BD) GERBES $G_{\nabla}^{\text{BD}}(M)$

GIVEN $f: M_1 \xrightarrow{C^\infty} M_2$ & f -SUBORDINATION of COVERS $\bigcup_{i \in I_2} \mathcal{O}_i^{(k)} = M_2$,
 i.e., $\varphi: I_1 \rightarrow I_2: f(\mathcal{O}_i^{(n)}) \subset \mathcal{O}_{\varphi(i)}^{(k)}$ $k \in \{1, 2\}$

WE OBTAIN MONOIDAL PULLBACK 2-FUNCTOR $f^*: G_{\nabla}^{\text{BD}}(M_2) \rightarrow G_{\nabla}^{\text{BD}}(M_1)$

with COMPONENTS $(B_i, A_{ij}, g_{ijk}) \mapsto (f^*B_{\varphi(i)}, f^*A_{\varphi(i)\varphi(j)}, f^*g_{\varphi(i)\varphi(j)\varphi(k)}) =: (B_i^f, A_{ij}^f, g_{ijk}^f)$
 $(P_i, k_{ij}) \mapsto (f^*P_{\varphi(i)}, f^*k_{\varphi(i)\varphi(j)}) =: (P_i^f, k_{ij}^f)$

CHARACTERS BEHAVE NATURALLY: $\text{Hol}_{g_j}(f_*(C_2)) = \text{Hol}_{f^*g_j}(C_2)$

NOTE THE DISTINGUISHED TRIVIAL GERBES $(B, 0, 1) =: \mathbb{I}_B$,

SATISFYING THE SIMPLE ALGEBRA $\mathbb{I}_{B_1} \otimes \mathbb{I}_{B_2} = \mathbb{I}_{B_1 + B_2}$ & $f^*\mathbb{I}_B = \mathbb{I}_{f^*B}$.

(45)

STARTERS
before
THE MAIN DISH

GOAL: CONNECTION-AIDED DESCENT of GEOMETRISABLE/PREQUANTISABLE
 COHOMOLOGICAL STRUCTURES on M to $\mathcal{F} \sim \mathcal{L}_0 M//G$

SOURCE of INTUITION: STRUCTURAL HOMOLOGICAL INTERPRETATION of MINIMAL COUPLING of GLOBAL $\underline{A}_i \equiv \underline{A}$
 to INVARIANT TRIVIAL OBJECT $\mathcal{I}_B, B \in \Omega^2(M)^{\mathbb{B}(G)}$, OR -SIMPLY STILL- $B \in \Omega^2(M)^G$

WE HAVE $\widetilde{B} = B[\underline{A}] = p_2^* B + p_2^*(\kappa_a \lrcorner B) \wedge p_1^* \underline{A}_a - \frac{1}{2} p_2^*(\kappa_a \lrcorner (\kappa_b \lrcorner B)) p_1^*(\underline{A}_a \wedge \underline{A}_b)$,

with $\kappa := -\kappa_a \lrcorner B: \text{Lie } G \times M \rightarrow T^*M$

$$\begin{array}{ccc} & & \\ p_2 \downarrow & & \downarrow \pi_{T^*M} \\ M & \xlongequal{\quad} & M \end{array}$$

WHICH SATISFIES $\kappa_a \lrcorner H = d\kappa$.

by G -INVARIANCE of B , with $dB = H$

NB: $M/G \in \text{Man} \Rightarrow$ INTEGRATING OUT $\underline{A}$ PRODUCES B' (or g'_M) G -BASIC $\Rightarrow$ PAG DESCENDS to M/G

LEMMA 1. $\lambda^* \omega(g, x) = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \widetilde{\nu}_{\kappa^{(A)} \circ \theta_L(g)}^n \lambda_g^* \omega \right)(x)$ for $\widetilde{\nu}_{\kappa^{(A)} \circ \theta_L(g)} \eta(x) = \theta_L^a(g) \wedge \kappa_a^{(A)} \lrcorner \eta(x)$
 $\theta_L \rightsquigarrow \underline{A}$

1: EXERCISE. $\square$

$\Rightarrow \underline{\lambda^* H = p_2^* H + d\omega_{\text{GSW}}}$, $\omega_{\text{GSW}} = -p_2^* \kappa_a \wedge p_1^* \theta_L^a + \frac{1}{2} p_2^*(\kappa_a \lrcorner \kappa_b) p_1^*(\theta_L^a \wedge \theta_L^b) \equiv \omega_{\text{GSW}}[\theta_L]$ (46)

OBSERVATION: $(\lambda, p_2) \equiv (t, s) = (d_0^{(i)}, d_1^{(i)})$ for $N.(G \ltimes M)$, & SO THE CURVATURE
 IDENTITY REWRITES AS $d\omega_{\text{GSW}} - \Delta_{(0)}H = 0$ in TERMS of $\Delta_{(0)} = d_0^{(1)*} - d_1^{(1)*}$.
 HOWEVER, $\Delta_{(0)}: \mathcal{Q}^*(N_0(G \ltimes M)) \rightarrow \mathcal{Q}^*(N_1(G \ltimes M))$ BELONGS TO FAMILY $\{\Delta_{(n)}\}_{n \in \mathbb{N}}$
 of DUPONT COBOUNDARY OPERATORS $\Delta_{(n)} = \sum_{k=0}^{n+1} (-1)^k d_k^{(n+1)*}$ (SEE: p.39),
 EACH of WHICH COMMUTES with $d \dots$

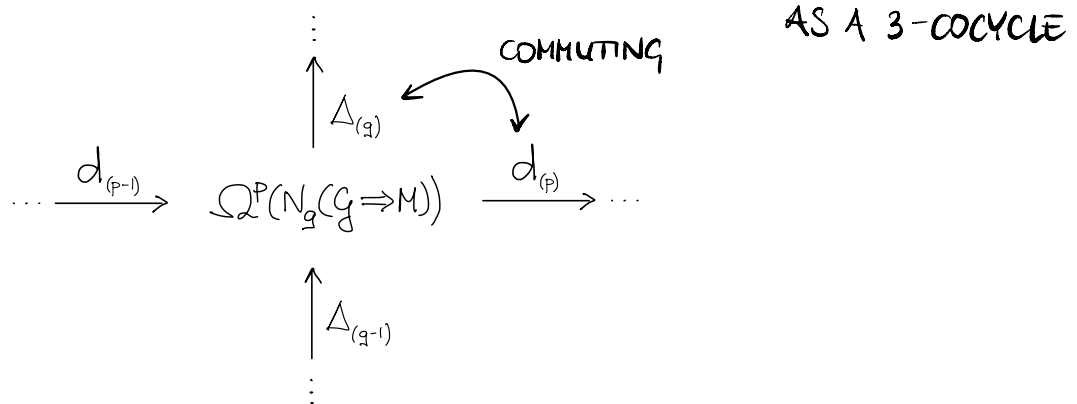
ONE CHECKS $\Delta_{(1)}\omega_{\text{GSW}} = 0$, & THIS ALTOGETHER INSCRIBES $(H_1, \omega_{\text{GSW}})$ in...

Defⁿ 15.

BOTT-SCHULMAN-STASHEFF

BICOMPLEX

for $\mathcal{G} \equiv G \times M$



OUR OBSERVATION PROMPTS... (47)

QUESTION: IS THE BSS COHOMOLOGY

$$H_{\text{BSS}}^*(\mathcal{G} \Rightarrow M) = H^*(B^\bullet, D^\bullet), \quad B^p = \bigoplus_{r+s=p} \Omega^r(N_s(\mathcal{G} \Rightarrow M)), \quad D^{(p)}: B^p \rightarrow B^{p+1}$$

$$D^{(p)}|_{\Omega^r(N_s(\mathcal{G} \Rightarrow M))} = d_{(r)} + (-1)^r \Delta_{(s)}$$

in ANY WAY NATURAL for THE PROBLEM in HAND?

TEASER for LECTURE IV :

NOW THAT WE HAVE UNDERSTOOD THE TOPOLOGICAL COUPLING in the PAG σ -MODEL in COHOMOLOGICAL TERMS, WE MAY INVOKES HOMOTOPY INVARIANCE of COHOMOLOGY & LOOK for A MODEL of the COHOMOLOGY of the EFFECTIVE CONFIGURATION FIBRE $M//G$ of the GAUGED PAG σ -MODEL (-to-be) IN WHICH to PUT A SUITABLY A_i -COUPLED (AS SUGGESTED by the MINIMAL-COUPLING RECIPE for $G = \mathbb{I}_B$ with $B \in \Omega^2(M)^{B(G)})$ GERBE of the MODEL. THIS LEADS US TO G -EQUIVARIANT (HYPER)COHOMOLOGY & THE STUDY of MULTIPLICATIVE FORMS. ONCE A_i -COUPLED LOCALLY over $\mathcal{F}$, THE GERBE STILL NEEDS to DESCEND to $\mathcal{F}$, WHICH IS WHERE THE 2-STACK STRUCTURE COMES INTO PLAY...