

A surreal landscape featuring a waterfall, a crane, a woman, and ducks. The scene is set in a dark, rocky environment with a large waterfall in the center. A crane is positioned on the right, lifting a large fish. A woman in a classical dress sits on the right, holding a bundle of wheat. On the left, four ducks are stacked on a stone pillar. The sky is dark and cloudy, with a bat flying in the upper left. The overall mood is mysterious and fantastical.

Gauging the Oïd & Other Tales

PART IV

Multiplicative Equivariance,
Holonomic Reduction,
& 2-Stacky Descent

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*La Géométrie est une image,
mais il n'y a pas d'image de la Géométrie*

IN THE LAST EPISODE :

(*) THE 'UNIVERSAL' GEOMETRIC STRUCTURE for GROUPOIDAL GAUGE QFT: THE SITTING DUCK & ITS g -EQUIVARIANTLY DESCENDABLE AUTOMORPHISMS

$$\text{Gau}(\mathcal{P}) \xrightarrow{\mathcal{F}_*} \text{Gau}(\mathcal{F})$$
$$\begin{array}{ccc} (\mathcal{P} \circledast \mathcal{V}) & \longrightarrow & (\mathcal{F} \circledast \mathcal{F}) \\ \downarrow & & \downarrow \\ \Sigma & \xlongequal{\quad} & \Sigma \end{array}$$

(**) SHEAFY DESCENT of INVARIANT TENSORIAL COUPLINGS through GAUGE-COVARIANT MINIMAL COUPLING

(***) THE BD HYPERCOHOMOLOGY & 2-CATEGORY behind the 2-GAUGE FIELD DRIVING THE PAG DYNAMICS of CHARGED LOOPS, & ITS EQUIVARIANTISATION on the HORIZON...

GOAL: CONNECTION-AIDED DESCENT of GEOMETRISABLE/PREQUANTISABLE
 COHOMOLOGICAL STRUCTURES on M to $\mathcal{F} \sim \mathcal{E} M/G$

SOURCE of INTUITION: STRUCTURAL HOMOLOGICAL INTERPRETATION of MINIMAL COUPLING of GLOBAL $\underline{A}_i \equiv \underline{A}$
 to INVARIANT TRIVIAL OBJECT $\mathcal{I}_B, B \in \Omega^2(M)^{\mathbb{B}(G)}$, OR -SIMPLY STILL- $B \in \Omega^2(M)^G$

WE HAVE $\widetilde{B} = B[\underline{A}] = p_2^* B + p_2^*(\kappa_a \lrcorner B) \wedge p_1^* \underline{A}_a - \frac{1}{2} p_2^*(\kappa_a \lrcorner (\kappa_b \lrcorner B)) p_1^*(\underline{A}_a \wedge \underline{A}_b)$,

with $\kappa := -\kappa_a \lrcorner B: \text{Lie } G \times M \rightarrow T^*M$

$$\begin{array}{ccc} & & \\ p_2 \downarrow & & \downarrow \pi_{T^*M} \\ M & \xlongequal{\quad} & M \end{array}$$

WHICH SATISFIES $\kappa_a \lrcorner H = d\kappa$.

by G -INVARIANCE of B , with $dB = H$

NB: $M/G \in \text{Man} \Rightarrow$ INTEGRATING OUT $\underline{A}$ PRODUCES B' (or g'_M) G -BASIC $\Rightarrow$ PAG DESCENDS to M/G

LEMMA 1. $\lambda^* \omega(g, x) = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \tilde{v}_{\kappa^{(A)} \circ \theta_L(g)}^n \lambda_g^* \omega \right)(x)$ for $\tilde{v}_{\kappa^{(A)} \circ \theta_L(g)} \eta(x) = \theta_L^a(g) \wedge \kappa_a^{(A)} \lrcorner \eta(x)$
 $\theta_L \rightsquigarrow \underline{A}$

1: EXERCISE. $\square$

$\Rightarrow \underline{\lambda^* H = p_2^* H + d\omega_{\text{GSW}}}$, $\omega_{\text{GSW}} = -p_2^* \kappa_a \wedge p_1^* \theta_L^a + \frac{1}{2} p_2^*(\kappa_a \lrcorner \kappa_b) p_1^*(\theta_L^a \wedge \theta_L^b) \equiv \omega_{\text{GSW}}[\theta_L]$ (46)

OBSERVATION: $(\lambda, p_2) \equiv (t, s) = (d_0^{(i)}, d_1^{(i)})$ for $N.(G \times M)$, & SO THE CURVATURE
 IDENTITY REWRITES AS $d\omega_{\text{GSW}} - \Delta_{(0)}H = 0$ in TERMS of $\Delta_{(0)} = d_0^{(1)*} - d_1^{(1)*}$.

HOWEVER, $\Delta_{(0)}: \mathcal{Q}^*(N_0(G \times M)) \rightarrow \mathcal{Q}^*(N_1(G \times M))$ BELONGS TO FAMILY $\{\Delta_{(n)}\}_{n \in \mathbb{N}}$
 of DUPONT COBOUNDARY OPERATORS $\Delta_{(n)} = \sum_{k=0}^{n+1} (-1)^k d_k^{(n+1)*}$ (SEE: p.39),

EACH of WHICH COMMUTES with $d \dots$

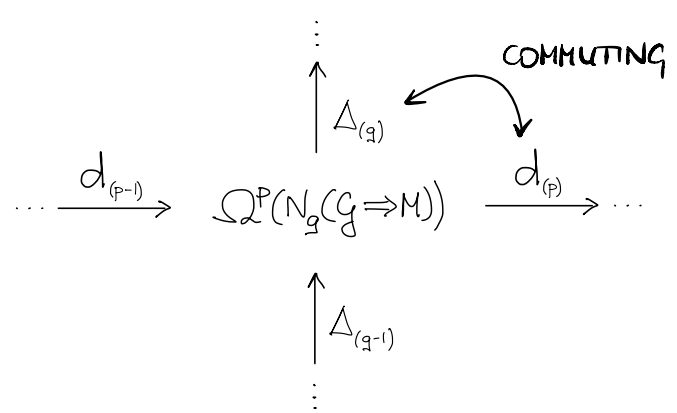
ONE CHECKS $\Delta_{(1)}\omega_{\text{GSW}} = 0$, & THIS ALTOGETHER INSCRIBES (H, ω_{GSW}) in...

Defⁿ 15.

BOTT-SCHULMAN-STASHEFF

BICOMPLEX

for $\mathcal{G} \equiv G \times M$



AS A 3-COCYCLE

OUR OBSERVATION PROMPTS... (47)

QUESTION: IS THE BSS COHOMOLOGY

$$H_{BSS}^*(\mathcal{G} \Rightarrow M) = H^*(B^\bullet, D^\bullet), \quad B^p = \bigoplus_{r+s=p} \Omega^r(N_S(\mathcal{G} \Rightarrow M)), \quad D^{(p)}: B^p \rightarrow B^{p+1}$$
$$D^{(p)} \upharpoonright \Omega^r(N_S(\mathcal{G} \Rightarrow M)) = d_{(r)} + (-1)^r \Delta_{(s)}$$

in ANY WAY NATURAL for THE PROBLEM in HAND?

ANSWER: CALLS for IDENTIFICATION of RELATION between (B)COMPLEXES
WHICH INDUCES ISOMORPHISM in COHOMOLOGY — THAT RELATION IS...

Defⁿ 16. HOMOTOPY EQUIVALENCE between COCHAIN COMPLEXES
 $(C_\alpha^\bullet, d_\alpha^\bullet)$, $\alpha \in \{1, 2\}$ IS A QUADRUPE $(f_1^{(\cdot)}, f_2^{(\cdot)}, h_1^{(\cdot)}, h_2^{(\cdot)})$ of FAMILIES

of MAPS: (*) COCHAIN MAPS: $(C_1^\bullet, d_1^\bullet) \xrightarrow{f_1^{(\cdot)}} (C_2^\bullet, d_2^\bullet) \xrightarrow{f_2^{(\cdot)}} (C_1^\bullet, d_1^\bullet)$

(**) HOMOTOPIES: $f_2^{(\cdot)} \circ f_1^{(\cdot)} \underset{h_1^{(\cdot)}}{\sim} id_{C_1} \quad \& \quad f_1^{(\cdot)} \circ f_2^{(\cdot)} \underset{h_2^{(\cdot)}}{\sim} id_{C_2}$ (48)

i.e., $\forall n \in \mathbb{N} : f_1^{(n)} : C_1^n \rightarrow C_2^n$ s.th. $d_2^{(n)} \circ f_1^{(n)} = f_1^{(n+1)} \circ d_1^{(n)}$

$f_2^{(n)} : C_2^n \rightarrow C_1^n$ s.th. $d_1^{(n)} \circ f_2^{(n)} = f_2^{(n+1)} \circ d_2^{(n)}$

& $h_\alpha^{(n)} : C_\alpha^n \rightarrow C_\alpha^{n-1}, \alpha \in \{1, 2\}$

s.th. $f_2^{(n)} \circ f_1^{(n)} - \text{id}_{C_1^n} = d_1^{(n-1)} \circ h_1^{(n)} + h_1^{(n+1)} \circ d_1^{(n)}$

$f_1^{(n)} \circ f_2^{(n)} - \text{id}_{C_2^n} = d_2^{(n-1)} \circ h_2^{(n)} + h_2^{(n+1)} \circ d_2^{(n)}$

Th^m 4. HOMOTOPY EQUIVALENT COMPLEXES HAVE ISOMORPHIC COHOMOLOGY GROUPS.

Q: NOTE THAT $f^{(i)} \underset{h^{(i)}}{\sim} g^{(i)}$ IMPLIES $H(f^{(i)}) = H(g^{(i)})$
for $H(f^{(i)})[c_n] := [f^{(n)}(c_n)]$.



AN ELUCIDATION OF THE 'MEANING' OF $H_{BSS}^*(G \Rightarrow M)$ NOW FOLLOWS FROM

Thm 5. THE BSS BICOMPLEX IS HOMOTOPY EQUIVALENT TO THE SUBCOMPLEX

OF THE DE RHAM BICOMPLEX OF THE STRATIFIED MANIFOLD $\bigsqcup_{n \in \mathbb{N}} (G_n \times \Delta_n)$

COMPOSED OF $\|G_\bullet\|$ -DESCENDABLE FORMS $\omega_{(\rho)}^* = (\omega_{(\rho)}^{(n)})^{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \Omega^p(G_n \times \Delta_n)$,

I.E., THOSE $\mathbb{N}$ -INDEXED FAMILIES OF RESTRICTIONS $\omega_{(\rho)}^{(n)} = \tilde{\omega}_{(\rho)}^{(n)} \upharpoonright_{G_n \times \Delta_n}$ OF SMOOTH

FORMS $\tilde{\omega}_{(\rho)}^{(n)} \in \Omega^p(G_n \times \tilde{\Delta}_n)$ ON HYPERPLANES

$$\tilde{\Delta}_n = \{ (t_1, t_2, \dots, t_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{k=1}^{n+1} t_k = 1 \}$$

$$\begin{aligned} & (id_{G_{n+1}} \times \tilde{\mathcal{J}}_i^{(n+1)})(r, t) \\ & \parallel \\ & (d_i^{(n+1)}(r), t) \sim (\tilde{\mathcal{J}}_i^{(n+1)}(t)) \\ & \parallel \\ & (d_i^{(n+1)} \times id_{\Delta_n})(r, t) \end{aligned}$$

WHICH SATISFY DESCENT IDENTITIES : $(d_i^{(n+1)} \times id_{\Delta_n})^* \omega_{(\rho)}^{(n)} = (id_{G_{n+1}} \times \tilde{\mathcal{J}}_i^{(n+1)})^* \tilde{\omega}_{(\rho)}^{(n+1)} \upharpoonright_{G_{n+1} \times \Delta_n}$

WRITTEN IN TERMS OF $\tilde{\mathcal{J}}_i^{(n+1)}: \tilde{\Delta}_n \longrightarrow \tilde{\Delta}_{n+1}, (t_1, t_2, \dots, t_{n+1}) \longmapsto (t_1, t_2, \dots, t_i, 0, t_{i+1}, t_{i+2}, \dots, t_{n+1})$
(EMBEDDINGS)

T. Dupont: DE RHAM OPERATOR on $\bigsqcup_{n \in \mathbb{N}} (G_n \times \Delta_n)$ IS PRESENTED AS TOTAL DIFFERENTIAL ON BICOMPLEX $(\prod_{n \in \mathbb{N}} \Omega^*(\Delta_n) \otimes_{\infty} \Omega^*(G_n), d_{(c_n)} \otimes \text{id}, \text{id} \otimes d_{(c_n)})$, & COCHAIN MAPS BETWEEN BICOMPLEXES ARE INTEGRATION OVER HOMOLOGY FIBRES Δ . & AUGMENTATION BY WHITNEY'S ELEMENTARY FORMS, RESPECTIVELY. $\square$

UPSHOT: BSS COHOMOLOGY IS A MODEL OF DE RHAM COHOMOLOGY OF M/G ,

$$\underline{H_{BSS}^*(G \Rightarrow M) \simeq H^*(\|G\|) =: H_{dR}^*(M/G)}$$

THE FORMER IS OFTEN REFERRED TO AS G -EQUIVARIANT COHOMOLOGY.

IN THE σ -MODEL CONTEXT, WE SHALL CONSIDER Π GEOMETRISED/PREQUANTISED:

$$(H, \omega, 0, 0) \in \Omega^3(M) \oplus \Omega^2(G) \oplus \Omega^1(G_2) \oplus \Omega^0(G_3)$$

$$dH = 0$$

$$t^*H = s^*H + d\omega$$

$$m^*\omega = pr_1^*\omega + pr_2^*\omega$$

Def^y 17. ω -TWISTED G -EQUIVARIANT STRUCTURE on G_j

$$\mathcal{I}: t^*G_j \xrightarrow{\cong} s^*G_j \otimes I_\omega, \quad \gamma: (pr_2^*\mathcal{I} \otimes \text{id}_{I_{pr_1^*\omega}}) \circ pr_1^*\mathcal{I} \xrightarrow{\cong} m^*\mathcal{I}$$

$$d_2^{(3)*} \gamma \cdot (\text{id} \circ d_0^{(3)*} \gamma) = d_1^{(3)*} \gamma \cdot ((d_3^{(3)*} \gamma \otimes \text{id}) \circ \text{id}) \quad (51)$$

NEED: RECONCILE ABSTRACT PREQUANTISATION of NATURAL COHOMOLOGY
EXTENSION $(H, \omega, 0, 0)$ of 2-GAUGE FIELD H with IDEA of IMPLEMENTING
(RIGID) GROUPOIDAL SYMMETRY $G \Rightarrow M$ by BISECTIONS...

LET $\beta \in \mathcal{B}(G)$ BE A SYMMETRY of TOPOLOGICAL DF AMPLITUDE,

$$\text{Hol}_{(t_*\beta)^*G_j}(X(\Sigma)) = \text{Hol}_{G_j}((t_*\beta \cdot X)(\Sigma)) \stackrel{!}{=} \text{Hol}_{G_j}(X(\Sigma))$$

BUT

$$\beta^*r \stackrel{!}{=} \text{Hol}_{(s \circ \beta)^*G_j}(X(\Sigma)) \cdot \text{Hol}_{I_{\beta^*\omega}}(X(\Sigma)) = \text{Hol}_{G_j}(X(\Sigma)) \cdot \exp\left(i \int_{X(\Sigma)} \beta^*\omega\right)$$

$\Updownarrow \forall \Sigma, X$

SIMPLE SOLUTION: $\beta^*\omega \stackrel{!}{=} 0 \Rightarrow \text{Per}(\beta^*\omega) \subset 2\pi\mathbb{Z}$

Prop 11. $\omega \in \text{Ker } \Delta_{(2)}$ $\Rightarrow \{\beta \in \mathcal{B}(G) \mid \beta^*\omega = 0\} =: \mathcal{B}_\omega(G)$ IS A SUBGROUP!

MULTIPLICATIVE 2-FORM ω -HOLONOMIC GROUP

9: EXERCISE. $\square$

BACK TO THE SOURCE OF INTUITION (p.47):

(*) $B_{\omega_{\text{GSW}}}(G \ltimes_\lambda M) \supseteq \iota(G)$, i.e., STRUCTURE GROUP IS REDUCED from THE START!

(**) INSPECTION of MECHANISM of $\iota(G)$ -INVARIANCE of $\text{Hol}_{G[A]}$

for AUGMENTED GERBE $G[A] = p_2^* G \otimes \mathcal{I}_{\omega_{\text{GSW}}[A]}$ (AS SUGGESTED by MINIMAL COUPLING)

with COUPLING to A MEDIATED by $k: \text{Lie } G \times M \rightarrow T^*M$ ($\kappa^{(A)} \lrcorner H = -dk$):

LET $\Phi: P \times_\lambda M \equiv \Sigma \times M \rightarrow \Sigma \times M$, $(\sigma, x) \mapsto (\sigma, \lambda_{\gamma(\sigma)}(x)) \equiv (p_{\Sigma}, \lambda^0(\gamma \times \text{id}_M))(\sigma, x)$

BE A GAUGE TRANSFORMATION. WE OBTAIN

$$\begin{aligned} \Phi^* G[A^\Phi] &\equiv \Phi^* (p_2^* G \otimes \mathcal{I}_{\omega_{\text{GSW}}[A^\Phi]}) \equiv (\lambda^0(\gamma \times \text{id}_M))^* G \otimes \mathcal{I}_{\Phi^* \omega_{\text{GSW}}[A^\Phi]} \simeq (\gamma \times \text{id}_M)^* p_2^* G \otimes \mathcal{I}_{(\gamma \times \text{id}_M)^* \omega_{\text{GSW}}[\Theta_L] + \Phi^* \omega_{\text{GSW}}[A^\Phi]} \\ &= p_2^* G \otimes \mathcal{I}_{(\gamma \times \text{id}_M)^* \omega_{\text{GSW}}[\Theta_L] + \Phi^* \omega_{\text{GSW}}[A^\Phi]} \equiv p_2^* G \otimes \mathcal{I}_{\omega_{\text{GSW}}[A]} \otimes \mathcal{I}_{(\gamma \times \text{id}_M)^* \omega_{\text{GSW}}[\Theta_L] + \Phi^* \omega_{\text{GSW}}[A^\Phi] - \omega_{\text{GSW}}[A]} \equiv G[A] \otimes \mathcal{I}_{\Delta(\Phi)} \end{aligned}$$

$$\boxed{\Delta(\Phi) = (\gamma \times \text{id}_M)^* \omega_{\text{GSW}}[\Theta_L] + \Phi^* \omega_{\text{GSW}}[A^\Phi] - \omega_{\text{GSW}}[A]}$$

USING

BUILT INTO/
IMPLIED by
MULTIPLICATIVITY
of ω_{GSW}

$\left\{ \begin{array}{l} \rightarrow G\text{-EQUIVARIANCE of } \underline{\kappa}: \text{Lie}G \rightarrow \Omega^1(M), \kappa \circ T_e \text{Ad}_g = \lambda_g^* \kappa ! \\ \rightarrow \forall X, Y \in \text{Lie}G : \kappa_X^{(A)} \lrcorner \kappa_Y = -\kappa_Y^{(A)} \lrcorner \kappa_X ! \end{array} \right.$

WE READILY VERIFY

$$\underline{\Delta(\Xi) = 0},$$

WHICH GIVES US $\mathfrak{r}(G)^{\circledast}$ -INVARIANCE OF $\text{Hol}_{g[A]}$.

QUESTION: CAN WE REPRODUCE IT FOR ARBITRARY $g \Rightarrow M$

AFTER ω -HOLONOMIC REDUCTION $B(g) \rightarrow B_{\omega}(g) ?$

ANSWER: $\omega \in \text{Ker } \Delta_{(2)} \cap \Omega^2(\mathfrak{g})$ DETERMINES

$$\begin{array}{ccc} \text{Def}^n 18. & \mathcal{K} : \mathcal{E} & \longrightarrow T^*M, \quad \epsilon \longmapsto \underline{\text{Id}^*(\epsilon \lrcorner \omega)} \\ & \downarrow & \downarrow \\ & M & = M \end{array}$$

1-COMOMENTUM

THE DESIRED 'ALGEBRAIC' PROPERTY OF $\mathcal{K}$ FOLLOWS FROM MULTIPLICATIVITY OF ω .

Propⁿ 12. $\forall \epsilon_1, \epsilon_2 \in \Gamma(\mathcal{E}) : \underline{\alpha_{\mathcal{E}}(\epsilon_1) \lrcorner \mathcal{K}(\epsilon_2) + \alpha_{\mathcal{E}}(\epsilon_2) \lrcorner \mathcal{K}(\epsilon_1) = 0}$

9: EXERCISE. $\square$

THE REMAINING ONES ARISE THROUGH ITS INTERPLAY WITH HOLONOMICITY...

Propⁿ 13. $\forall \beta \in B_{\omega}(\mathfrak{g}) \quad \forall x \in M \quad \forall X_{\alpha} = T_x \beta(\xi_{\alpha}) + T_{\text{Id}_{t_{\beta}(x)}} \tau_{\beta(x)}(\epsilon_{\alpha}) \in T_{\beta(x)} \beta(M) \oplus \text{Ker } T_{\beta(x)} S, \alpha \in \{1, 2\} :$ (SEE: p. 16.)

$$\omega(X_1, X_2) = T_x(t_{\beta}) \lrcorner \mathcal{K}(\epsilon_1) - T_x(t_{\beta}) \lrcorner \mathcal{K}(\epsilon_2) + \alpha_{\mathcal{E}}(\epsilon_2) \lrcorner \mathcal{K}(\epsilon_1)$$

See $\underline{\mathcal{K}(T_{\text{Id}_x} C_{\beta}(\epsilon)) \circ T_x(t_{\beta}) = \mathcal{K}(\epsilon)}$. ($B_{\omega}(\mathfrak{g})$ -EQUIVARIANCE) (ω FULLY DETERMINED by $\mathcal{K}$)

9: EXERCISE. (NB: TC_{β} RESTRICTS TO $\mathcal{E}$.) $\square$

NB: $\uparrow$ NOT TRUE ELSEWHERE in $\mathfrak{g}$ in GENERAL. (54)

Ex. 1. For $G \ltimes M$, WE FIND $\forall (g, x) \in G \ltimes M \exists \beta \in B_{\omega_{\text{GSW}}}(G \ltimes M) : (g, x) = \beta(x)$ —
 INDEED $\beta := (g, \cdot) \equiv \nu(g)$ DOES THE JOB. THIS EXPLAINS WHY ω_{GSW} IS EVERYWHERE
 DETERMINED by $\kappa(\xi, \cdot) = (e, \cdot)^* \xi + \omega_{\text{GSW}}$, $\xi \in \text{Lie } G$.

WE MAY PULL BACK $\mathcal{G}$ TO $P \times M$ ALONG MOMENT MAP $\mu_{P \times M} \equiv \text{pr}_2$
 & AUGMENT WITH... $A \in \Omega^1(P) \otimes \text{Lie } G$ AS SUGGESTED BY MINIMAL COUPLING

$$\mathcal{G} \rightsquigarrow \text{pr}_2^* \mathcal{G} \otimes I_{\omega_{\text{GSW}}[A]} =: \mathcal{G}[A],$$

WHEREUPON WE INVOKE

Thm 6. $X/G \in \text{Man} \Rightarrow \text{Grb}_\nabla(X/G) \simeq \text{Grb}_\nabla(X)^{(G, \omega_{\text{GSW}}=0)}$ UN-TWISTED
G-EQUIVARIANT
1-GERBES

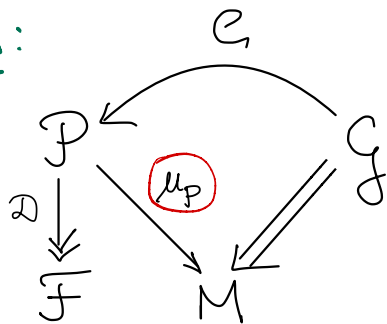
9: USES 2-STACK STRUCTURE. $\square$

for $X = P \times M$

LUCKILY, WE DO NOT NEED TO WRITE OUT & PROVE THE THEOREM BECAUSE...

THIS IS WHERE THE IMITATION BREAKS DOWN.

PROBLEM:



WHILE μ_P COULD BE USED TO PULL BACK G to P for DESCENT,

WE ONLY HAVE $(4) : TP \rightarrow VP$ THERE,

with $VP \underset{\text{loc.}}{\cong} TG \neq \epsilon$ for $\dim M > 0$

IDEA: GO DOWN DIRECTLY to F , & GO LOCAL

[cz '26]

$$F \xleftarrow[\substack{\text{f. 34.}}]{\tilde{\pi}} \check{Y}_F \equiv \bigsqcup_{i \in I} \pi_F^{-1}(\partial_i) \xrightarrow[\simeq]{\tilde{F}\tilde{\tau}_i} \bigsqcup_{i \in I} \partial_i \times M$$

THIS IS WHERE BOTH: G & A_i LIVE

WE MAY DEFINE

LOCAL A_i -AUGMENTED CERBES

$$G_j[A_i] := \tilde{F}\tilde{\tau}_i^*(p_2^* G_j \otimes I_{\omega[A_i]})$$

$$\omega[A_i] \equiv (\text{id}_2 \times \text{id})^*(\omega \circ ((4)_i \otimes (4)_i))$$

(EXERCISE!)

A TACTICAL PAUSE

- for CONTEMPLATION of THE QUESTION

WHEN DOES A FAMILY of LOCAL GERBES COME from A GLOBAL ONE?

IN ORDER TO DEVELOP SOME INTUITION, REPHRASE THE BD DEFINITION
 of a 1-GERBE $\mathcal{G}$ from 9.43. IN THE 2-CATEGORIAL LANGUAGE
 IN A STRUCTURED MANNER. DEFINE LOCAL CELLS:

$$\mathcal{G}_i := (\mathcal{B}_i, 0_i) (= \mathbb{I}_{\mathcal{B}_i}) \in \mathcal{G}rb_{\nabla}(\mathcal{O}_i)_{(0)} \quad , \quad \mathcal{F}_{ij} := (A_{ij}, 1) \in \mathcal{G}rb_{\nabla}(\mathcal{O}_{ij})_{(1)} \quad , \quad \varphi_{ijk} := (g_{ijk}) \in \mathcal{G}rb_{\nabla}(\mathcal{O}_{ijk})_{(2)}.$$

THE LOCAL GERBES $\mathcal{G}_i$ DETERMINE A GLOBAL GERBE $\mathcal{G}$ s.t. $\mathcal{G}|_{\mathcal{O}_i} \cong \mathcal{G}_i$

through THE FOLLOWING RELATIONS:

(DO)

$$\mathcal{F}_{ij} : \mathcal{G}|_{\mathcal{O}_{ij}} \xrightarrow{\cong} \mathcal{G}_i|_{\mathcal{O}_{ij}}$$

$$\varphi_{ijk} : \mathcal{F}_{jk} \circ \mathcal{F}_{ij}|_{\mathcal{O}_{ijk}} \xrightarrow{\cong} \mathcal{F}_{ik}|_{\mathcal{O}_{ijk}}$$

$$\varphi_{ikl} \cdot (\text{id}_{\mathcal{F}_{kl}} \circ \varphi_{ijk}) = \varphi_{ijl} \cdot (\varphi_{ikl} \circ \text{id}_{\mathcal{F}_{ij}})$$

NEXT, WE REFORMULATE THE DESCRIPTION of 1- & 2-CELLS of $\mathcal{G}rb_{\nabla}(M)$
 in A SIMILAR VEIN...

DEFINE LOCAL CELLS: $\Psi_i := (P_i, 1) \in \text{Grb}_{\triangleright}(\mathcal{O}_i)_{(1)}$, $\psi_{ij} := (k_{ij}) \in \text{Grb}_{\triangleright}(\mathcal{O}_{ij})_{(2)}$

to CONCLUDE THAT THE FORMER LOCAL 1-ISOMORPHISMS

$$\Psi_i : G_i \xrightarrow{\sim} G'_i$$

DETERMINE A GLOBAL 1-ISOMORPHISM $\Psi : G \xrightarrow{\sim} G'$

between THE GLOBAL GERBES G & G' DETERMINED by $(G_i, \Phi_{ij}, \varphi_{ijk})$
& $(G'_i, \Phi'_{ij}, \varphi'_{ijk})$, RESPECTIVELY, through $\Psi_j \circ \Phi_{ij}|_{\mathcal{O}_{ij}} \xrightarrow{\sim} \Phi'_{ij} \circ \Psi_i|_{\mathcal{O}_{ij}}$

$$(D1) \quad \psi_{ik} \cdot (\text{id}_{\Psi_k} \circ \varphi_{ijk})|_{\mathcal{O}_{ijk}} = (\varphi'_{ijk} \circ \text{id}_{\Psi_i}) \cdot (\text{id}_{\Phi'_{jk}} \circ \psi_{ij}) \cdot (\psi_{jk} \circ \text{id}_{\Phi_{ij}})|_{\mathcal{O}_{ijk}}$$

FINALLY, NOTE THAT CELLS $\xi_i := (h_i)$ MODELLING LOCAL 2-ISOMORPHISMS

$$\xi_i : \Psi_i \xrightarrow{\sim} \Psi'_i$$

DETERMINE A GLOBAL 2-ISOMORPHISM $\xi : \Psi \xrightarrow{\sim} \Psi'$

through $(\text{id}_{\Phi'_{ij}} \circ \xi_i) \cdot \psi_{ij}|_{\mathcal{O}_{ij}} = \psi'_{ij} \cdot (\xi_j \circ \text{id}_{\Phi_{ij}})|_{\mathcal{O}_{ij}}$. (D2)

ALTOGETHER, THIS IS A (TRIVIAL) MANIFESTATION OF A GENERAL PROPERTY

Th^m 7. 1-GERBES FORM A 2-STACK over Man , i.e., THERE EXISTS CANONICAL EQUIVALENCE OF 2-CATEGORIES $\forall M \in \text{Man} \forall \mathcal{O} = \{\theta_i\}_{i \in I}$

$$G_{\text{Ger}} b_{\triangleright}(M) \simeq \text{Desc}(\mathcal{O}), \quad \bigcup_{i \in I} \theta_i = M$$

THE LATTER HAVING

2-CATEGORY of DESCENT DATA

(*) 0-CELLS $(\{C_i\}_{i \in I}, \{\Phi_{ij}\}_{i,j \in I}, \{\varphi_{ijk}\}_{i,j,k \in I})$ SATISFYING (D0);

(**) 1-CELLS $(\{\Psi_i\}_{i \in I}, \{\psi_{ij}\}_{i,j \in I})$ SATISFYING (D1);

(***) 2-CELLS $\{\xi_i\}_{i \in I}$ SATISFYING (D2),

EL OBVIOUS REMAINING COMPONENTS OF STRUCTURE.

9. [Stevenson; Nikolaus-Schweigert] $\square$

END of THE TACTICAL PAUSE
&
BACK to PAC

WE HAVE LOCAL A_i -AUGMENTED CERBERES $G_j[A_i] := \mathcal{F}c_i^*(p_{\mathcal{I}}^* G_j \otimes I_{\omega[A_i]})$
 over $\pi_{\mathcal{F}}^{-1}(\partial_i)$,

with STRUCTURE ENSURING REALISATION OF ω -HOLONOMIC

Defⁿ 19.

$$\text{Gau}_{\omega}(\mathcal{F}) := \mathcal{F}_*(\text{Gau}_{\omega}(\mathcal{P}))$$

GAUGE
GROUP

$$\text{Gau}_{\omega}(\mathcal{P}) := \{ \mathcal{E} \in \text{Gau}(\mathcal{P}) \mid \mathcal{P}c_i \circ \mathcal{E} \circ \mathcal{P}c_i^{-1}(\sigma_i, g) = (\sigma_i, L_{\gamma_i(\sigma)}(g)), \gamma_i: \partial_i \rightarrow B_{\omega}(G), i \in \mathcal{I} \}$$

for TRIVIALISATIONS $\mathcal{P}c_i: \pi_{\mathcal{P}}^{-1}(\partial_i) \xrightarrow{\cong} \partial_i \times G$ with $\beta_{ij}: \partial_{ij} \rightarrow B_{\omega}(G)$

Propⁿ 14. ω -TWISTED G -EQUIVARIANT STRUCTURE ON G_j INDUCES

$$\forall \mathcal{E} \in \text{Gau}_{\omega}(\mathcal{P}): \quad \Psi_i: \mathcal{F}_*(\mathcal{E})^* G_j[A_i^{\mathcal{E}}] \xrightarrow{\cong} G_j[A_i], i \in \mathcal{I}.$$

9: WE HAVE $\Psi_i = \mathcal{F}c_i^*(\gamma_i^* \mathcal{I} \otimes \text{Id}_{I_{\Gamma_i^* \omega[A_i^{\mathcal{E}}]}})$ for $\gamma_i: \partial_i \times M \rightarrow G, (\sigma, x) \mapsto t_*(\gamma_i(\sigma))(x)$
 (EXERCISE!) & $\Gamma_i = \mathcal{F}c_i \circ \mathcal{F}_*(\mathcal{E}) \circ \mathcal{F}c_i^{-1} \dots \square$ (60)

THE PROOF USES

LEMMA 2. MASTER FORMULA $\omega[A] - B^* \omega[A_\beta] = \tilde{\beta}^* \omega$ HOLDS TRUE

FOR ALL $\tilde{\beta} : \mathcal{O} \times M \rightarrow \mathcal{G}, (\sigma, x) \mapsto \beta(\sigma)(x)$ WITH $\beta : \mathcal{O} \rightarrow B_\omega(\mathcal{G})$ & $B = (\mathfrak{p}_1, t \circ \tilde{\beta})$,

AS WELL AS $A_\beta(B(\sigma, x)) = T_{\text{Id}_x} C_{\beta(\sigma)} \circ A(\sigma, x) + \Theta \circ T_\sigma \tilde{\beta}(\cdot, x)$.

9: EXERCISE. (USE THE SPLITTING FROM 9.16) $\square$

THE LATTER GIVES US ALSO THE FUNDAMENTAL

Thm 8. WHENEVER $\mathcal{G}_j$ CARRIES AN ω -TWISTED $\mathcal{G}$ -EQUIVARIANT STRUCTURE,

THE FAMILY $\{\mathcal{G}_j[A_i]\}_{i \in I}$ CANONICALLY EXTENDS TO DESCENT DATUM.

9: WE HAVE $\mathcal{E}_{ij} := \mathcal{F}_{\mathcal{G}_j}^*(\tilde{\beta}_{ij}^* \gamma \otimes \text{Id}_{\mathcal{I}_{B_{ij}^* \omega[A_i]}}) \Big|_{\pi_{\mathcal{G}}^{-1}(\mathcal{O}_i)} : \mathcal{G}_j[A_i] \Big|_{\pi_{\mathcal{G}}^{-1}(\mathcal{O}_i)} \xrightarrow{\cong} \mathcal{G}_j[A_j] \Big|_{\pi_{\mathcal{G}}^{-1}(\mathcal{O}_i)}$

FOR $\tilde{\beta}_{ij} : \mathcal{O}_i \times M \rightarrow \mathcal{G}, (\sigma, x) \mapsto \beta_{ij}(\sigma)(x)$ & $B_{ij} = (\mathfrak{p}_1, t \circ \tilde{\beta})$. $\square$

THESE FORM 2-COCYCLE UP TO $\varphi_{ijk} := \mathcal{F}_{\mathcal{G}_k}^*(\tilde{\beta}_{ik}^* \gamma \otimes \text{Id}_{\mathcal{I}_{B_{ik}^* \omega[A_i]}}) \Big|_{\pi_{\mathcal{G}}^{-1}(\mathcal{O}_{jk})}$ FOR $\tilde{\beta}_{jk} = (\tilde{\beta}_{ij} \circ B_{jk}, \tilde{\beta}_{jk}) \Big|_{\mathcal{O}_{jk} \times M}$. (61)

Cor. 2. Under ASSUMPTIONS of Th^m 8. THE LOCAL A_i -AUGMENTED GERBES DESCEND to A GLOBAL $G_{\mathcal{F}} \in \text{Grb}_{\nabla}(\mathcal{F})_{(0)}$: $\tilde{\pi}^* G_{\mathcal{F}} = \bigsqcup_{i \in I} G_i[A_i]$.

MORAL: A G -EQUIVARIANT STRUCTURE on G — ITSELF A PREQUANTISATION OF THE DE RHAM CLASS over $M//G$ ASSOCIATED WITH THE GERBE'S CURVATURE — EFFECTS AN ENRICHMENT OF THE 1-STACKY CLUTCHING CONSTRUCTION OF THE SHADOW $\mathcal{F} : \begin{pmatrix} \mathcal{O}_i \times M \\ \downarrow \beta_{ij} \\ \bigwedge_{(i,j)} \beta_{ijk} = \text{Id} \end{pmatrix} \nearrow \begin{pmatrix} G[A_i] \\ \mathbb{F}_{ij} \\ \varphi_{ijk} \\ \bigwedge_{(i,j,k)} \varphi_{ijk} = \text{id} \\ \text{(BY COHERENCE)} \end{pmatrix}$ to A 2-STACKY CLUTCHING CONSTRUCTION OF THE DESCENDED GERBE $G_{\mathcal{F}}$.

IT REMAINS to CHECK STABILITY of ISOCASSES of $G_{\mathcal{F}}$ under $\text{Gau}_{\omega}(\mathcal{F})$...

RECALL THAT WE HAVE THE 1-ISOMORPHISMS Ψ_i (SEE: Propⁿ 14.)

$\Rightarrow$ In VIRTUE of Th^m 7., WE MERELY NEED THE INTERTWINING 2-ISOMORPHISMS ψ_{ij} (SEE: q. 57.). (2)

Thm 9. WHENEVER G_j CARRIES AN ω -TWISTED G -EQUIVARIANT STRUCTURE, THE FAMILY $\{\Psi_i: \mathcal{F}_*(\mathcal{E})^* G_j[A_i^\#] \xrightarrow{\sim} G_j[A_i]\}_{i \in \bar{I}}$ CANONICALLY EXTENDS TO 1-ISOMORPHISM $(\mathcal{F}_*(\mathcal{E})^* G_j[A_i], \mathcal{F}_*(\mathcal{E})^* \mathcal{E}_j, \mathcal{F}_*(\mathcal{E})^* \varphi_{ijk}) \xrightarrow{\sim} (G_j[A_i], \mathcal{E}_j, \varphi_{ijk})$ in $\text{Desc}(\check{Y}_{\mathcal{F}})$.

9: WE HAVE $\psi_j := \mathcal{F}_{G_j}^* (\tilde{\gamma} \tilde{\beta}_j^* \gamma^{-1} \cdot \tilde{\beta} \tilde{\gamma}_j^* \gamma \otimes \text{id}_{\text{Id}_{\mathcal{F}_j^* B_j^* \omega[A_i^\#]}})$ for $\begin{matrix} \tilde{\gamma} \tilde{\beta}_j := (\tilde{\gamma}_i \circ B_{ij}, \tilde{\beta}_{ij}) \\ \tilde{\beta} \tilde{\gamma}_j := (\tilde{\beta}_j \circ \Gamma_j, \tilde{\gamma}_j) \end{matrix} : \mathcal{O}_j \times M \rightarrow G_2$. □

Cor 3. UNDER ASSUMPTIONS OF Thm 9, THE GLOBAL GERBE $\mathcal{F}_*(\mathcal{E})^* G_{j, \mathcal{F}}$ DESCENDED FROM $\mathcal{F}_*(\mathcal{E})^* (G_j[A_i], \mathcal{E}_j, \varphi_{ijk})$ IS CANONICALLY ISOMORPHIC WITH THE GLOBAL GERBE $G_{j, \mathcal{F}}$ DESCENDED FROM $(G_j[A_i], \mathcal{E}_j, \varphi_{ijk})$.

MORAL: A G -EQUIVARIANT STRUCTURE ON G_j EFFECTS AN ENRICHMENT OF THE 1-STACK CLUTCHING CONSTRUCTION OF A GAUGE AUTOMORPHISM OF $\mathcal{F}: \begin{pmatrix} t \circ \gamma_i \\ \gamma_i \cdot \beta_{ij}|_{\mathcal{O}_i} = \beta_{ij} \cdot \gamma_j|_{\mathcal{O}_i} \end{pmatrix} \nearrow \begin{pmatrix} \Psi_i \\ \psi_{ij} \end{pmatrix}$ + COHERENCE

↳ A 2-STACK CLUTCHING CONSTRUCTION OF GAUGE AUTOMORPHISM OF $G_{j, \mathcal{F}}$. 63

UPSHOT: THE $\mathfrak{g}$ -GAUGED 2d PAC σ -MODEL

$(\mathcal{F}, \Sigma, M, \pi_{\mathcal{F}}, \eta_{\Sigma}, A_{\text{PAC}}^{\mathfrak{g}})$ with GAUGE SYMMETRY $\text{Gau}_{\mathfrak{g}}(\mathcal{F})$

$\text{Gau}_{\mathfrak{g}_M}(\mathcal{F}) \cap \text{Gau}_{\omega}(\mathcal{F})$

HAS (Σ, η_{Σ}) : $\dim \Sigma = 2$, ORIENTABLE, $\partial \Sigma = \emptyset$, COMPACT
EUCLIDEAN or PIECEWISE LORENTZIAN

M : OBJECT MANIFOLD of $\mathfrak{g} \Rightarrow M$ with METRIC g_M

& MULTIPLICATIVELY ω -TWISTED $\mathfrak{g}$ -EQUIVARIANT GERBE $\mathcal{G}$

$\mathcal{F} \xrightarrow{\pi_{\mathcal{F}}} \Sigma$: SHADOW of PRINCIPALOID BUNDLE $\mathcal{P} \xrightarrow{\pi_{\mathcal{P}}} \Sigma$ (ARBITRARY)

with ω -HOLONOMICALLY REDUCED STRUCTURE GROUP $\mathbb{B}_{\mathfrak{r}}(\mathfrak{g})$

$\mathbb{B}_{\mathfrak{g}_M}(\mathfrak{g}) \cap \mathbb{B}_{\omega}(\mathfrak{g})$

& DF AMPLITUDE

$\Gamma(\mathcal{F}) \ni \phi \longmapsto \exp \left(-\frac{i}{2} \|\tau \phi\|_{\eta_{\Sigma}, g_{\mathcal{F}}}^2 \right) \cdot \text{Hol}_{\mathfrak{g}_{\mathcal{F}}}(\phi(\Sigma)) = \mathcal{A}_{\text{PAC}}^{\mathfrak{g}}[\phi]$ (64)