DESCENT OF DIFFERENTIAL FORMS AND PRINCIPAL $\mathbb{C}^\times$-BUNDLES

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ABSTRACT. A constructive analysis is presented of the necessary and sufficient conditions for the descent of a principal $\mathbb{C}^\times$-bundle with a compatible connection to the base of a surjective submersion. Inspiration is drawn from the study of the descent of differential forms to quotient manifolds for free and proper group actions, subsequently generalised to that along arbitrary surjective submersions. The general results are specialised – in a boomerang move – to the case of the descent of flatly equivariant bundles of the said type to smooth orbispaces of Lie-group actions, of relevance to the gauging of rigid smooth symmetries in field theories with a topological charge.

CONTENTS

1. Introduction

Symmetry is one of the deepest and most powerful guiding principles in the study of physical models. In one of its instantiations – the Gauge Principle – we are confronted with the task of descending the dynamics from the configuration space to the space of orbits of the action of the (rigid) symmetry group, and so, whenever – as, *e.g.*, in the σ-model – there exists a higher-geometric object (*e.g.*, a bundle, a gerbe, an n-gerbe *etc.*) over the former space, which codetermines the original dynamics (in the said example: through a Cheeger–Simons differential character), we are prompted to answer the more challenging question:

Question 1: *Under what circumstances does a higher-geometric object descend from a manifold to the space of orbits of an action of a group on that manifold, or, more concretely, when is the object isomorphic to the pullback of an object (of the same type) along the projection to the orbispace if the latter is a manifold?*

The bonus of finding a structural answer to the above question is the following: It enables us to *model* the higher-geometric objects on an orbispace with the distinguished higher-geometric objects on the mother manifold *even if the orbispace is not a manifold*, and similarly for the field theory. Thus motivated, we shall approach the problem is steps, starting with the largely tractable issue of descending a differential form to the orbispace, which we subsequently generalise to that of descending a differential form to the base of an arbitrary surjective submersion. The generalisation shall provide us with a useful intuition that we shall employ towards solving the original problem.

2. An étude on differential forms – the emergence of a cohomology-free complex

The higher-geometric objects that co-determine σ-model-type field theories of interest (to us) arise as 'geometrisations' of Maxwell-type cocycles in the de Rham cohomology (possibly further refined, as in the Wess–Zumino–Witten or Green–Schwarz case), and so it seems natural to start our journey by carrying out a thorough investigation of the descent of these tnsorial objects. Prior to taking up the challenge of elucidating the descent in the case of an arbitrary surjective submersion, we first deal with

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the familiar (and physically much relevant) setting: Let G be a Lie group[1] of dimension d, and let M be a smooth manifold of dimension D equipped with a smooth (left) action

$$\lambda_{\cdot} \; : \; \mathrm{G} \times M \longrightarrow M \; : \; (g,m) \longmapsto g \triangleright m \equiv \lambda_g(m) \, .$$

The latter induces on M an integrable distribution spanned by the **fundamental vector fields** in the image of the G-equivariant Lie-algebra homomorphism

$$\mathcal{K}_{\cdot_1}(\cdot_2) \; : \; \mathfrak{g} \longrightarrow \Gamma(\mathsf{T}M) \; : \; X \longmapsto -\mathsf{T}_{((e,\cdot_2))}\big(X, 0_{\mathsf{T}M}(\cdot_2)\big) \equiv \mathcal{K}_X(\cdot_2) \, ,$$

with values

$$\mathcal{K}_X(m) = \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \lambda_{\exp(-t \triangleright X)}(m) \in \mathsf{T}_m M \, .$$

The integral leaves of the distribution are the orbits of the action, and if – as we shall assume henceforth – the latter is free and proper, The Quotient Manifold Theorem[2] ensures that there exists a(n essentially unique) smooth structure on the orbispace

$$M/\mathrm{G} = \{ \; \mathrm{G} \triangleright m \quad | \quad m \in M \; \}$$

with respect to which the projection

$$\pi_{M/\mathrm{G}} \; : \; M \longrightarrow M/\mathrm{G} \; : \; m \longmapsto \mathrm{G} \triangleright m$$

is a surjective submersion. In this setting, we arrive at a counterpart of the question from the Introduction:

Question 2: *Which differential forms on M are pullbacks of differential forms along the projection to the orbispace if the latter is a manifold?*

It is completely straightforward to solve the problem thus posed. Indeed, let $\mathcal{O} \subset M$ be the domain of a coordinate chart

$$\begin{aligned} \kappa \equiv \big(x^\mu\big)^{\mu \in \overline{1,D}} \equiv \big(\underline{x}^a, v^A\big)^{(a,A) \in \overline{1,D-d} \times \overline{1,d}} \quad &: \quad \mathcal{O} \xrightarrow{\cong} \mathcal{U} \subset \mathbb{R}^{\times D-d} \times \mathbb{R}^{\times d} \\ &: \quad m \longmapsto \big(\underline{x}^a(m), v^A(m)\big)^{(a,A) \in \overline{1,D-d} \times \overline{1,d}} \end{aligned}$$

in which the v^A coordinatise the integral leaves, whereas the $\underline{x}^a$ – the transverse directions (such adapted coordinates are explicitly constructed in the proof of the said theorem presented in the notes). The condition that a p-form $\omega \in \Omega^p(M)$ be the pullback of a p-form $\underline{\omega} \in \Omega^p(M/\mathrm{G})$ transcribes into the identity

$$\kappa^* \omega = \omega_{\mu_1 \mu_2 \ldots \mu_p}\big(x^\mu\big) \, \mathsf{d}x^{\mu_1} \wedge \mathsf{d}x^{\mu_2} \wedge \cdots \wedge \mathsf{d}x^{\mu_p} \stackrel{!}{=} \underline{\omega}_{a_1 a_2 \ldots a_p}\big(\underline{x}^a\big) \, \mathsf{d}\underline{x}^{a_1} \wedge \mathsf{d}\underline{x}^{a_2} \wedge \cdots \wedge \mathsf{d}\underline{x}^{a_p}$$

(to be satisfied in a vicinity of every point in M in the respective local adapted coordinates). The identity can be rewritten in the form

$$\forall_{A \in \overline{1,d}} \; : \; \Big(\; \tfrac{\partial}{\partial v^A} \lrcorner \, \kappa^* \omega \stackrel{!}{=} 0 \quad \wedge \quad \mathscr{L}_{\frac{\partial}{\partial v^A}} \kappa^* \omega \stackrel{!}{=} 0 \; \Big) \, ,$$

which is amenable to an obvious 'globalisation':

$$\forall_{X \in \mathfrak{g}} \; : \; \mathcal{K}_X \lrcorner \, \omega \stackrel{!}{=} 0 \qquad \wedge \qquad \forall_{g \in \mathrm{G}} \; : \; \lambda_g^* \omega \stackrel{!}{=} \omega \, .$$

The latter identifies the descendable forms as those which are **$\mathfrak{g}$-horizontal** (the first condition) and **G-invariant** (the second condition), and so, altogether, **G-basic**.

Here, our analysis reaches an early crossroads – from this point, we may take it in one of the two natural directions: Either we replace differential forms with (physically inspired) de Rham cohomology classes, or we replace $\pi_{M/\mathrm{G}}$ with an arbitrary surjective submersion. The first path leads to the highly structured realm of equivariant cohomology, whereas the second one takes us rather directly to the theory of higher-geometric descent, circumnavigating the wuthering heights of Cartan's cohomological model. Each of them is interesting in its own right, and each carries its share of relevance to the subject matter of interest to us – indeed, they reconverge at a structural solution to our problem. We choose the former path for the sake of brevity, and with the tranquillising foreknowledge that the path chosen ultimately does, with a touch ingenuity, take us to the physically motivated goal defined in the

[1] Most of our conclusions remain valid in the more general setting of topological (or even discrete) group actions, the generalisation affecting essentially only the tangential structure.

[2] *Cf.*, *e.g.*, Ref. [Sus21] for a hands-on proof.

introduction, whereupon we rediscover equivariance from a new, non-axiomatic angle, more geometric (although no less algebraic) than the alternative one offered by the second approach.

Let us first take a step 'away' from the original problem by considering an arbitrary surjective submersion (for which there is, *a priori*, no structural choice of the vertical distribution) and adapting the question from the earlier part of the section to the new, more general context:

Question 3: *Which differential forms on the total space of a surjective submersion are pullbacks of differential forms along the projection to its base?*

By way of setting up the scene and developing the language for the statement of the solution (and for the subsequent considerations), we give, with hindsight, the following

Definition 1. Let M and X be smooth manifolds, and let $\varpi : M \longrightarrow X$ be a surjective submersion. Denote the cartesian powers of M fibred over X as

$$M^{[n]} \equiv \underbrace{M \times_X M \times_X \cdots \times_X M}_{n \text{ times}} = \big\{ (m_1, m_2, \ldots, m_n) \in M^{\times n} \quad | \quad \forall_{i,j \in \overline{1,n}} \ : \ \varpi(m_i) = \varpi(m_j) \big\} .$$

The **nerve of the surjective submersion** ϖ is the simplicial submanifold

$$\mathsf{N}^{(\bullet)}\big(\mathrm{Pair}_\varpi(M)\big) \equiv \big(M(\varpi)^{(\bullet)}, d^{(\bullet)}_\cdot \equiv \widehat{\mathrm{pr}}^{(\bullet+1)}_{\cdot+1}, s^{(\bullet)}_\cdot \equiv \mathrm{id}_{M^{[\cdot]}} \times \delta \times \mathrm{id}_{M^{[\bullet-\cdot-1]}}\big)$$

$$: \quad \cdots \xrightarrow{\widehat{\mathrm{pr}}^{(4)}_i = \mathrm{pr}_{1,2,\widehat{\ldots}_i,4}} M^{[3]} \xrightarrow{\widehat{\mathrm{pr}}^{(3)}_i = \mathrm{pr}_{1,\widehat{\ldots}_i,3}} M^{[2]} \xrightarrow{\widehat{\mathrm{pr}}^{(2)}_i = \mathrm{pr}_{3-i}} M$$

of the nerve[3] of the pair groupoid (written in terms of $\tau : M^{\times 2} \circlearrowleft : (m_1, m_2) \longmapsto (m_2, m_1)$)

$$\mathrm{Pair}(M) = (M, M \times M, s = \mathrm{pr}_1, t = \mathrm{pr}_2, \mathrm{Id.} = (\mathrm{id}_M, \mathrm{id}_M), \circ = \mathrm{pr}_{1,3}, \mathrm{Inv} = \tau) ,$$

with the face maps

$$d^{(n)}_i \equiv \widehat{\mathrm{pr}}^{(n+1)}_{i+1} \ : \ M^{[n+1]} \longrightarrow M^{[n]} \ : \ (m_1, m_2, \ldots, m_{n+1}) \longmapsto (m_1, m_2, \underbrace{\ldots}_{\widehat{m_{i+1}}}, m_{n+1}) , \qquad i \in \overline{0,n}$$

related in an obvious manner to canonical projections, and the degeneracy maps

$$s^{(n)}_i \equiv \mathrm{id}_{M^{[i]}} \times \delta \times \mathrm{id}_{M^{[n-i-1]}} \quad : \quad M^{[n]} \longrightarrow M^{[n+1]}$$

$$: \quad (m_1, m_2, \ldots, m_n) \longmapsto (m_1, m_2, \ldots, m_i, m_{i+1}, m_{i+1}, m_{i+2}, \ldots, m_n) , \qquad i \in \overline{0, n-1} .$$

In other words, it is the nerve of the (sub)groupoid

$$\mathrm{Pair}_\varpi(M) = (M, M \times_X M, s = \mathrm{pr}_1, t = \mathrm{pr}_2, \mathrm{Id.} = (\mathrm{id}_M, \mathrm{id}_M), \circ = \mathrm{pr}_{1,3}, \mathrm{Inv} = \tau) ,$$

which we shall refer to by the name of the **ϖ-fibred pair groupoid**.

In what follows, we denote the pullback of a geometric object O (from a category with pullbacks) along the face map $\mathrm{pr}_{i_1,i_2,\ldots,i_n} : M^{[n+k]} \longrightarrow M^{[n]}$ (with $1 \le i_1 < i_2 < \ldots < i_n \le n+k, \ k > 0$) as

$$O_{[i_1,i_2,\ldots,i_n]} \equiv \mathrm{pr}^*_{i_1,i_2,\ldots,i_n} O .$$

◇

We need one last formal step prior to stating the solution to the problem in hand. This we take in

Definition 2. In the notation of Def. 1, and for any $p \in \mathbb{N}$, the **descent** (**cochain**) **complex of** ϖ **in degree** p is the semi-bounded cochain complex $(\Omega^p(\varpi)^{(\bullet)}, \Delta^{(\bullet)})$:

$$\mathbf{0} \dashrightarrow \Omega^p(X) \xrightarrow{\Delta^p_{(0)}} \Omega^p(M) \xrightarrow{\Delta^p_{(1)}} \Omega^p\big(M^{[2]}\big) \xrightarrow{\Delta^p_{(2)}} \Omega^p\big(M^{[3]}\big) \xrightarrow{\Delta^p_{(3)}} \cdots \xrightarrow{\Delta^p_{(q-1)}} \Omega^p\big(M^{[q]}\big) \xrightarrow{\Delta^p_{(q)}} \cdots$$

with the coboundary homomorphisms

$$\Delta^p_{(0)} \coloneqq \varpi^* , \qquad \Delta^p_{(q)} \coloneqq \sum_{i=1}^{q+1} (-1)^{i+1} d^{(q)\,*}_{i-1} \equiv \sum_{i=1}^{q+1} (-1)^{i+1} \widehat{\mathrm{pr}}^{(q+1)\,*}_i , \qquad q \in \mathbb{N}^\times .$$

◇

[3] *In order not to lose your nerve when reading on, consult App. A.*

Remark 3. The identities

$$\Delta^p_{(n+1)} \circ \Delta^p_{(n)} = 0\,, \qquad n \in \mathbb{N}$$

are readily checked by hand. The first of them derives as

$$\Delta^p_{(1)} \circ \Delta^p_{(0)} \equiv \big(\mathrm{pr}_2^* - \mathrm{pr}_1^*\big)\,\varpi^* \equiv (\varpi \circ \mathrm{pr}_2)^* - (\varpi \circ \mathrm{pr}_1)^* \equiv 0\,,$$

whereas the remaining ones follow from the so-called simplicial identities

$$d_i^{(n-1)} \circ d_j^{(n)} = d_{j-1}^{(n-1)} \circ d_i^{(n)}\,, \quad i < j\,,$$

readily verified directly. Indeed, we obtain, for $n > 0$,

$$\begin{aligned}
\Delta^p_{(n+1)} \circ \Delta^p_{(n)} &\equiv \sum_{k=0}^{n+2} \sum_{l=0}^{n+1} (-1)^{k+l}\, d_k^{(n+2)\,*} \circ d_l^{(n+1)\,*} = \sum_{k=0}^{n+2} \sum_{l=0}^{n+1} (-1)^{k+l} \big(d_l^{(n+1)} \circ d_k^{(n+2)}\big)^* \\
&= \sum_{k=1}^{n+2} \sum_{l=0}^{k-1} (-1)^{k+l} \big(d_{k-1}^{(n+1)} \circ d_l^{(n+2)}\big)^* + \sum_{k=0}^{n+1} \big(d_k^{(n+1)} \circ d_k^{(n+2)}\big)^* + \sum_{k=0}^{n} \sum_{l=k+1}^{n+1} (-1)^{k+l} \big(d_l^{(n+1)} \circ d_k^{(n+2)}\big)^* \\
&= -\sum_{k=0}^{n+1} \sum_{l=0}^{k} (-1)^{k+l} \big(d_k^{(n+1)} \circ d_l^{(n+2)}\big)^* + \sum_{k=0}^{n+1} \big(d_k^{(n+1)} \circ d_k^{(n+2)}\big)^* + \sum_{k=0}^{n} \sum_{l=k+1}^{n+1} (-1)^{k+l} \big(d_l^{(n+1)} \circ d_k^{(n+2)}\big)^* \\
&= -\sum_{k=1}^{n+1} \sum_{l=0}^{k-1} (-1)^{k+l} \big(d_k^{(n+1)} \circ d_l^{(n+2)}\big)^* + \sum_{k=0}^{n} \sum_{l=k+1}^{n+1} (-1)^{k+l} \big(d_l^{(n+1)} \circ d_k^{(n+2)}\big)^* \equiv 0\,.
\end{aligned}$$

The significance of the cochain complex introduced above is clarified in the following observation, due to Murray, *cf.* Ref. [Mur96].

Proposition 4. *In the notation of Def. 2. For any $p \in \mathbb{N}$, the cohomology of the descent complex for ϖ vanishes identically,*

$$H^\bullet\big(\Omega^p(\varpi)^{(\bullet)}, \Delta^{(\bullet)}\big) := \frac{\mathrm{Ker}\,\Delta^p_{(\bullet)}}{\mathrm{Im}\,\Delta^p_{(\bullet-1)}} \equiv \mathbf{0}\,.$$

In particular, p-forms on the total space M that are pullbacks of p-forms on the base X are precisely those from the kernel of $\Delta^p_{(1)}$,

$$\forall_{\omega \in \Omega^p(M)} \;:\; \big(\; \exists_{\underline{\omega} \in \Omega^p(X)} \;:\; \omega = \varpi^* \underline{\omega} \qquad \Longleftrightarrow \qquad \big(\mathrm{pr}_2^* - \mathrm{pr}_1^*\big)\omega{\restriction}_{M^{[2]}} = 0 \;\big)\,.$$

Proof. Let $\mathcal{U} \equiv \{\mathcal{U}_\alpha\}_{\alpha \in \mathscr{A}}$ be an open cover of $X \equiv M^{[0]}$ whose elements support respective sections

$$\varsigma_\alpha \;:\; M_\alpha^{[0]} \equiv \mathcal{U}_\alpha \longrightarrow M \equiv M^{[1]}\,, \qquad \varpi \circ \varsigma_\alpha = \mathrm{id}_{\mathcal{U}_\alpha}\,.$$

First of all, we trivialise the cohomology in restriction to the $M_\alpha^{[1]} \equiv M_\alpha \equiv \varpi^{-1}(\mathcal{U}_\alpha)$ and their higher fibred powers $M_\alpha^{[n]} \equiv M^{[n]} \cap M_\alpha^{\times n}$, $n \geq 2$. To this end, consider the smooth mappings

$$\varsigma_\alpha^{(q)} \;:\; M_\alpha^{[q]} \longrightarrow M_\alpha^{[q+1]} \;:\; (m_1, m_2, \ldots, m_q) \longmapsto \big(\varsigma_\alpha \circ \varpi(m_1), m_1, m_2, \ldots, m_q\big)\,, \qquad q \in \mathbb{N}^\times\,.$$

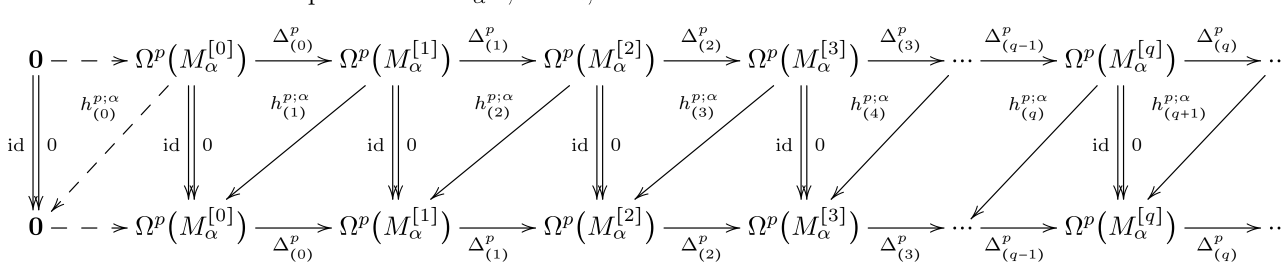
The corresponding pullback operators

$$h_{(1)}^{p;\alpha} := \varsigma_\alpha^* \;:\; \Omega^p\big(M_\alpha^{[1]}\big) \longrightarrow \Omega^p\big(M_\alpha^{[0]}\big)\,, \qquad h_{(q+1)}^{p;\alpha} := \varsigma_\alpha^{(q)\,*} \;:\; \Omega^p\big(M_\alpha^{[q+1]}\big) \longrightarrow \Omega^p\big(M_\alpha^{[q]}\big)\,, \qquad q \in \mathbb{N}^\times\,,$$

together with the zero map

$$h_{(0)}^{p;\alpha} \equiv 0 \;:\; \Omega^p\big(M_\alpha^{[0]}\big) \longrightarrow \mathbf{0}\,,$$

compose a cochain homotopy between the identity and zero cochain maps on the restriction of the descent cochain complex to the $M_\alpha^{[n]}$, $n \in \mathbb{N}$,



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that is, we have

$$h_{(q+1)}^{p;\alpha} \circ \Delta^p_{(q)} + \Delta^p_{(q-1)} \circ h_{(q)}^{p;\alpha} = \mathrm{id}_{\Omega^p(M_\alpha^{[q]})} - 0 = \mathrm{id}_{\Omega^p(M_\alpha^{[q]})}\,, \qquad q \in \mathbb{N}\,.$$

Indeed, for $q=0$, we obtain

$$h^{p;\alpha}_{(1)}\circ\Delta^p_{(0)}+0\circ h^{p;\alpha}_{(0)}\equiv\varsigma_\alpha^*\circ\varpi^*=(\varpi\circ\varsigma_\alpha)^*={\rm id}^*_{M^{[0]}_\alpha}\equiv{\rm id}_{\Omega^p(M^{[0]}_\alpha)}\,;$$

for $q=1$,

$$\begin{aligned} h^{p;\alpha}_{(2)}\circ\Delta^p_{(1)}+\Delta^p_{(0)}\circ h^{p;\alpha}_{(1)} &\equiv \varsigma_\alpha^{(1)\,*}\circ\big({\rm pr}_2^*-{\rm pr}_1^*\big)+\varpi^*\circ\varsigma_\alpha^*={\rm id}^*_{M^{[0]}_\alpha}-(\varsigma_\alpha\circ\varpi)^*+\varpi^*\circ\varsigma_\alpha^*={\rm id}^*_{M^{[0]}_\alpha}\\ &\equiv {\rm id}_{\Omega^p(M^{[1]}_\alpha)}\,;\end{aligned}$$

and, finally, for $q>1$,

$$\begin{aligned} h^{p;\alpha}_{(q+1)}\circ\Delta^p_{(q)}+\Delta^p_{(q-1)}\circ h^{p;\alpha}_{(q)} &\equiv \varsigma_\alpha^{(q)\,*}\circ\sum_{i=1}^{q+1}(-1)^{i+1}\,\widehat{\rm pr}_i^{(q+1)\,*}+\sum_{i=1}^{q}(-1)^{i+1}\,\widehat{\rm pr}_i^{(q)\,*}\circ\varsigma_\alpha^{(q-1)\,*}\\ &= \sum_{i=1}^{q+1}(-1)^{i+1}\big(\widehat{\rm pr}_i^{(q+1)}\circ\varsigma_\alpha^{(q)}\big)^*+\sum_{i=1}^{q}(-1)^{i+1}\big(\varsigma_\alpha^{(q-1)}\circ\widehat{\rm pr}_i^{(q)}\big)^*\\ &= \big(\widehat{\rm pr}_1^{(q+1)}\circ\varsigma_\alpha^{(q)}\big)^*+\sum_{i=2}^{q+1}(-1)^{i+1}\big(\widehat{\rm pr}_i^{(q+1)}\circ\varsigma_\alpha^{(q)}\big)^*+\sum_{i=1}^{q}(-1)^{i+1}\big(\varsigma_\alpha^{(q-1)}\circ\widehat{\rm pr}_i^{(q)}\big)^*\\ &= {\rm id}^*_{M^{[q]}_\alpha}+\sum_{i=2}^{q+1}(-1)^{i+1}\big(\widehat{\rm pr}_i^{(q+1)}\circ\varsigma_\alpha^{(q)}\big)^*+\sum_{i=1}^{q}(-1)^{i+1}\big(\varsigma_\alpha^{(q-1)}\circ\widehat{\rm pr}_i^{(q)}\big)^*\\ &= {\rm id}_{\Omega^p(M^{[0]}_\alpha)}+\sum_{i=1}^{q}(-1)^{i}\big(\widehat{\rm pr}_{i+1}^{(q+1)}\circ\varsigma_\alpha^{(q)}-\varsigma_\alpha^{(q-1)}\circ\widehat{\rm pr}_i^{(q)}\big)^*={\rm id}_{\Omega^p(M^{[0]}_\alpha)}\,,\end{aligned}$$

where the last equality follows from the identities:

$$\begin{aligned} &\widehat{\rm pr}_2^{(q+1)}\circ\varsigma_\alpha^{(q)}(m_1,m_2,\ldots,m_q)\equiv\widehat{\rm pr}_2^{(q+1)}\big(\varsigma_\alpha\circ\varpi(m_1),m_1,m_2,\ldots,m_q\big)=\big(\varsigma_\alpha\circ\varpi(m_1),m_2,m_3,\ldots,m_q\big)\\ =\ &\big(\varsigma_\alpha\circ\varpi(m_2),m_2,m_3,\ldots,m_q\big)\equiv\varsigma_\alpha^{(q-1)}(m_2,m_3,\ldots,m_q)\equiv\varsigma_\alpha^{(q-1)}\circ\widehat{\rm pr}_1^{(q)}(m_1,m_2,\ldots,m_q)\end{aligned}$$

and, for $i\in\overline{2,q}$,

$$\begin{aligned} &\widehat{\rm pr}_{i+1}^{(q+1)}\circ\varsigma_\alpha^{(q)}(m_1,m_2,\ldots,m_q)\equiv\widehat{\rm pr}_{i+1}^{(q+1)}\big(\varsigma_\alpha\circ\varpi(m_1),m_1,m_2,\ldots,m_q\big)=\big(\varsigma_\alpha\circ\varpi(m_1),m_1,m_2,\underbrace{\ldots}_{\widehat{m_i}},m_q\big)\\ \equiv\ &\varsigma_\alpha^{(q-1)}(m_1,m_2,\underbrace{\ldots}_{\widehat{m_i}},m_q)\equiv\varsigma_\alpha^{(q-1)}\circ\widehat{\rm pr}_i^{(q)}(m_1,m_2,\ldots,m_q)\,.\end{aligned}$$

Given $\omega_\alpha\in{\rm Ker}\,\Delta^p_{(q)}\cap\Omega^p(M^{[q]}_\alpha)$, the homotopy formula yields

$$\omega_\alpha=h^{p;\alpha}_{(q+1)}\circ\Delta^p_{(q)}(\omega_\alpha)+\Delta^p_{(q-1)}\circ h^{p;\alpha}_{(q)}(\omega_\alpha)=\Delta^p_{(q-1)}\big(h^{p;\alpha}_{(q)}(\omega_\alpha)\big)\in{\rm Im}\,\Delta^p_{(q-1)}\,,$$

so that the cohomology does vanish *locally* over X.

Passing to the global level, let $\omega\in{\rm Ker}\,\Delta^p_{(q)}$, so that, in particular, $\omega\restriction_{M^{[q]}_\alpha}\in{\rm Ker}\,\Delta^p_{(q)}\cap\Omega^p(M^{[q]}_\alpha)$ for each $\alpha\in\mathscr{A}$, whence

$$\omega\restriction_{M^{[q]}_\alpha}=\Delta^p_{(q-1)}\big(h^{p;\alpha}_{(q)}\big(\omega\restriction_{M^{[q]}_\alpha}\big)\big)\,.$$

Choose a partition of unity $\{\varrho^{(0)}_\alpha\equiv\varrho_\alpha\}_{\alpha\in\mathscr{A}}$ on X subordinate to $\mathcal{U}$, and induce from it the pullback partition of unity $\{\varrho^{(q)}_\alpha\equiv\varpi^*_{(q)}\varrho_\alpha\}_{\alpha\in\mathscr{A}}$ on $M^{[q]}$ (for each $q\in\mathbb{N}^\times$) using the projection

$$\varpi_{(q)}:=\varpi\circ{\rm pr}_1\ :\ M^{[q]}\longrightarrow X\,.$$

We have the obvious identities

$$\varpi^*\varrho^{(0)}_\alpha\equiv\varrho^{(1)}_\alpha\,,\qquad\widehat{\rm pr}_i^{(q+1)\,*}\varrho^{(q)}_\alpha=\varrho^{(q+1)}_\alpha\,,\qquad q\in\mathbb{N}^\times\,,$$

and so may write

$$\omega=\sum_{\alpha\in\mathscr{A}}\varrho^{(q)}_\alpha\,\omega\equiv\sum_{\alpha\in\mathscr{A}}\varrho^{(q)}_\alpha\,\omega\restriction_{M^{[q]}_\alpha}=\sum_{\alpha\in\mathscr{A}}\varrho^{(q)}_\alpha\,\Delta^p_{(q-1)}\big(h^{p;\alpha}_{(q)}\big(\omega\restriction_{M^{[q]}_\alpha}\big)\big)=\Delta^p_{(q-1)}\big(\sum_{\alpha\in\mathscr{A}}\varrho^{(q-1)}_\alpha\,h^{p;\alpha}_{(q)}\big(\omega\restriction_{M^{[q]}_\alpha}\big)\big)\,,$$

which yields the desired conclusion

$$\omega\in{\rm Im}\,\Delta^p_{(q-1)}\,.$$

□

We shall, next, lift the intuitions developed hereabove to the higher-geometric objects of interest.

3. The de(s)cent category

In this section, we discuss the descent of principal $\mathbb{C}^\times$-bundles with compatible connections along surjective submersions, drawing inspiration from Danny Stevenson's PhD Thesis [Ste00], in which a closely related issue was addressed in the context of gerbe theory. More specifically, we intend to answer, in a structured manner, the following question:

Question 4: *Under what circumstances is a principal $\mathbb{C}^\times$-bundle with a compatible connection $(\mathsf{P}, M, \pi_{\mathsf{P}}, \mathbb{C}^\times, \mathcal{A})$ over the total space M of a surjective submersion $\varpi : M \longrightarrow X$ isomorphic to the pullback along ϖ of a principal $\mathbb{C}^\times$-bundle with a compatible connection over the base X of the surjective submersion?*

We first identify, on the basis of the hitherto considerations, the object of our chief interest (*cf.* Refs. [GSW10, GSW13]), which – as shall turn out presently – yields a succinct answer to a suitably refined variant of the question posed above.

Definition 5. Adopt the notation of Def. 1. The **principal $\mathbb{C}^\times$-bundle descent category**

$$\mathbb{C}^\times\text{–}\mathfrak{BunDes}^\nabla(\varpi)$$

is composed of

- the object class with elements, termed **ϖ-descendable principal $\mathbb{C}^\times$-bundles with a compatible connection**, given by simplicial principal $\mathbb{C}^\times$-bundles with a compatible connection $\mathcal{P} \equiv ((\mathsf{P}, M, \pi_{\mathsf{P}}, \mathbb{C}^\times, \mathcal{A}), \chi)$ over $M(\varpi)^{(\bullet)}$, *i.e.*, pairs made up of a principal $\mathbb{C}^\times$-bundles over $M(\varpi)^{(0)}$,

$$\begin{array}{ccc} \mathbb{C}^\times & \rightsquigarrow & \mathsf{P} \\ & & \big\downarrow{\scriptstyle \pi_{\mathsf{P}}} \\ & & M \end{array} ,$$

  with a principal $\mathbb{C}^\times$-connection $\mathcal{A} \in \Omega^1(\mathsf{P})$, and of a connection-preserving isomorphism

$$\begin{array}{ccc} \mathsf{P}_{[1]} \equiv d_1^{(1)\,*}\mathsf{P} \equiv M^{[2]}\ {}_{\mathrm{pr}_1}\!\times_{\pi_{\mathsf{P}}} \mathsf{P} & \xrightarrow{\ \chi\ } & M^{[2]}\ {}_{\mathrm{pr}_2}\!\times_{\pi_{\mathsf{P}}} \mathsf{P} \equiv d_0^{(1)\,*}\mathsf{P} \equiv \mathsf{P}_{[2]} \\ \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ M^{[2]} & \overset{}{\underset{\mathrm{id}_{M^{[2]}}}{=\!=\!=\!=}} & M^{[2]} \end{array} , \tag{3.1}$$

  of principal $\mathbb{C}^\times$-bundles over $M(\varpi)^{(1)}$, subject to the coherence constraint over $M(\varpi)^{(2)}$ expressed by the commutative diagram

$$\begin{array}{ccccc} \mathsf{P}_{[1]} \equiv d_2^{(2)\,*}d_1^{(1)\,*}\mathsf{P} & \xrightarrow{d_2^{(2)\,*}\chi \equiv \chi_{[1,2]}} & d_2^{(2)\,*}d_0^{(1)\,*}\mathsf{P} & =\!=\!=\!= & d_0^{(2)\,*}d_1^{(1)\,*}\mathsf{P} \equiv \mathsf{P}_{[2]} \\ \big\| & & & & \big\downarrow{\scriptstyle d_0^{(2)\,*}\chi \equiv \chi_{[2,3]}} \\ d_1^{(2)\,*}d_1^{(1)\,*}\mathsf{P} & \xrightarrow[d_1^{(2)\,*}\chi \equiv \chi_{[1,3]}]{} & d_1^{(2)\,*}d_0^{(1)\,*}\mathsf{P} & =\!=\!=\!= & d_0^{(2)\,*}d_0^{(1)\,*}\mathsf{P} \equiv \mathsf{P}_{[3]} \end{array} ; \tag{3.2}$$

- for any pair $\mathcal{P}_K \equiv ((\mathsf{P}_K, M, \pi_{\mathsf{P}_K}, \mathbb{C}^\times, \mathcal{A}_K), \chi_K),\ K \in \{1,2\}$ of objects, a morphism class

$$\mathrm{Hom}_{\mathbb{C}^\times\text{–}\mathfrak{BunDes}^\nabla(\varpi)}(\mathcal{P}_1, \mathcal{P}_2)$$

with elements, termed (**connection-preserving**) ϖ**-descendable** (**principal** $\mathbb{C}^\times$**-bundle**) **morphisms**, given by connection-preserving isomorphisms

$$(\Phi, \mathrm{id}_M) \quad : \quad \begin{array}{ccc} \mathsf{P}_1 & \xrightarrow{\ \Phi\ } & \mathsf{P}_2 \\ {\scriptstyle \pi_{\mathsf{P}_1}} \downarrow & & \downarrow {\scriptstyle \pi_{\mathsf{P}_2}} \\ M & \underset{\mathrm{id}_M}{=\!=\!=} & M \end{array}$$

of principal $\mathbb{C}^\times$-bundles over $M(\varpi)^{(0)}$, subject to the coherence constraint over $M(\varpi)^{(1)}$ expressed by the commutative diagram

$$\begin{array}{ccc} \mathsf{P}_{1\,[1]} \equiv d_1^{(1)\,*}\mathsf{P}_1 & \xrightarrow{\ \chi_1\ } & d_0^{(1)\,*}\mathsf{P}_1 \equiv \mathsf{P}_{1\,[2]} \\ {\scriptstyle \Phi_{[1]} \equiv d_1^{(1)\,*}\Phi} \downarrow & & \downarrow {\scriptstyle d_0^{(1)\,*}\Phi \equiv \Phi_{[2]}} \\ \mathsf{P}_{2\,[1]} \equiv d_1^{(1)\,*}\mathsf{P}_2 & \xrightarrow[\ \chi_2\ ]{} & d_0^{(1)\,*}\mathsf{P}_2 \equiv \mathsf{P}_{2\,[2]} \end{array} . \tag{3.3}$$

◇

Remark 6. Let us unwrap the above definition and, in so doing, work out formulæ that will come in handy presently. First off, we look at the definition of the bundle isomorphism χ. Given that it covers the identity on the common base of the two principal $\mathbb{C}^\times$-bundles, we associate with it a smooth map

$$H \ : \ M^{[2]}\ {}_{\mathrm{pr}_1}\!\times_{\pi_{\mathsf{P}}} \mathsf{P} \longrightarrow \mathsf{P}$$

with the property expressed by the commutative diagram

$$\begin{array}{ccc} M^{[2]}\ {}_{\mathrm{pr}_1}\!\times_{\pi_{\mathsf{P}}} \mathsf{P} & \xrightarrow{\ H\ } & \mathsf{P} \\ {\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle \pi_{\mathsf{P}}} \\ M^{[2]} & \xrightarrow[\ \mathrm{pr}_2\ ]{} & M \end{array} ,$$

which enables us to rewrite χ as

$$\chi(m_1, m_2, p) = \big(m_1, m_2, H(m_1, m_2, p)\big)$$

for any $(m_1, m_2) \in M^{\times 2}$ such that $\varpi(m_1) = \varpi(m_2)$ and $p \in \mathsf{P}$ such that $m_1 = \pi_{\mathsf{P}}(p)$. Let us, next, establish a presentation of the isomorphism in local trivialisations of its domain and codomain. To this end, we fix a cover $\{\mathcal{O}_i\}_{i\in I}$ of M whose elements support the respective local trivialisations

$$\tau_i \ : \ \pi_{\mathsf{P}}^{-1}(\mathcal{O}_i) \xrightarrow{\ \cong\ } \mathcal{O}_i \times \mathbb{C}^\times ,$$

and the corresponding local potentials

$$A_i := \sigma_i^* \mathcal{A} \in \Omega^1(\mathcal{O}_i) , \qquad\qquad \sigma_i \equiv \tau_i^{-1}(\cdot, 1)$$

of the curvature of P. Subsequently, we define a subset

$$I^{[2]} := \big\{ \ (i_1, i_2) \in I^{\times 2} \quad | \quad \widetilde{\mathcal{O}}_{(i_1,i_2)} := \big(\mathcal{O}_{i_1} \times \mathcal{O}_{i_2}\big) \cap M^{[2]} \neq \varnothing \ \big\}$$

and use it index elements of an open cover $\{\widetilde{\mathcal{O}}_{(i_1,i_2)}\}_{(i_1,i_2)\in I^{[2]}}$ of the fibred square $M^{[2]}$ over which both the domain and codomain of χ trivialise as

$$\widetilde{\tau}^{[A]}_{(i_1,i_2)} \ : \ \mathsf{P}_{[A]}\restriction_{\widetilde{\mathcal{O}}_{(i_1,i_2)}} \xrightarrow{\ \cong\ } \widetilde{\mathcal{O}}_{(i_1,i_2)} \times \mathbb{C}^\times \ : \ (m_1, m_2, p) \longmapsto \big(m_1, m_2, \mathrm{pr}_2 \circ \tau_{i_A}(p)\big) , \qquad A \in \{1, 2\} .$$

The latter maps have the obvious inverses

$$\widetilde{\tau}^{[A]\,-1}_{(i_1,i_2)} \ : \ \widetilde{\mathcal{O}}_{(i_1,i_2)} \times \mathbb{C}^\times \longrightarrow \mathsf{P}_{[A]}\restriction_{\widetilde{\mathcal{O}}_{(i_1,i_2)}} \ : \ (m_1, m_2, z) \longmapsto \big(m_1, m_2, \tau_{i_A}^{-1}(m_A, z)\big) .$$

Using these, we readily compute local potentials of the pullback connection 1-forms

$$\mathcal{A}_{[A]} = \mathrm{pr}_2^* \mathcal{A} ,$$

to wit,

$$A^{[A]}_{(i_1,i_2)} = \widetilde{\sigma}^{[A]\,*}_{(i_1,i_2)}\mathcal{A}_{[A]} = \mathrm{pr}_A^*\sigma_{i_A}^*\mathcal{A} \equiv \mathrm{pr}_A^* A_{i_A} \in \Omega^1\big(\widetilde{\mathcal{O}}_{(i_1,i_2)}\big)\,, \qquad \widetilde{\sigma}^{[A]}_{(i_1,i_2)} \equiv \widetilde{\tau}^{[A]\,-1}_{(i_1,i_2)}(\cdot,1)\,.$$

We also obtain, for any $(m_1,m_2) \in \widetilde{\mathcal{O}}_{(i_1,i_2)}$ and $z \in \mathbb{C}^\times$, the local data

$$\widetilde{\tau}^{[2]}_{(i_1,i_2)} \circ \chi \circ \widetilde{\tau}^{[1]\,-1}_{(i_1,i_2)}(m_1,m_2,z) = \big(m_1,m_2,h_{(i_1,i_2)}(m_1,m_2)\cdot z\big)$$

of χ, expressed in terms of some $h_{(i_1,i_2)} \in C^\infty\big(\widetilde{\mathcal{O}}_{(i_1,i_2)},\mathrm{U}(1)\big)$. In other words, we have

$$H\big(m_1,m_2,\tau_{i_1}^{-1}(m_1,z)\big) = \tau_{i_2}^{-1}\big(m_2,h_{(i_1,i_2)}(m_1,m_2)\cdot z\big) \equiv \tau_{i_2}^{-1}\big(m_2,1\big) \triangleleft h_{(i_1,i_2)}(m_1,m_2)\cdot z\,, \tag{3.4}$$

where in the last transition the $\mathbb{C}^\times$-equivariance of the local trivialisation was used. The assumption that connection be preserved by χ,

$$\chi^*\mathcal{A}_{[2]} = \mathcal{A}_{[1]}\,, \tag{3.5}$$

now translates into the local statement

$$A_{i_2}(m_2) = A_{i_1}(m_1) + \mathsf{i}\,\mathsf{d}\log h_{(i_1,i_2)}(m_1,m_2)\,.$$

Passing to the coherence constraint satisfied by χ, we establish, for any $(m_1,m_2,m_3) \in M^{\times 3}$ with $\varpi(m_A) = \varpi(m_B)$, $A,B \in \{1,2,3\}$ and $p \in \mathsf{P}$ such that $m_1 = \pi_{\mathsf{P}}(p)$,

$$\begin{aligned}\big(m_1,m_2,m_3,H\big(m_2,m_3,H(m_1,m_2,p)\big)\big) &= \chi_{[2,3]}\big(m_1,m_2,m_3,H(m_1,m_2,p)\big)\\ &= \chi_{[2,3]}\circ\chi_{[1,2]}(m_1,m_2,m_3,p) \overset{!}{=} \chi_{[1,3]}(m_1,m_2,m_3,p) = \big(m_1,m_2,m_3,H(m_1,m_3,p)\big)\,,\end{aligned}$$

and, consequently, derive the useful ‘telescoping’ identity

$$H\big(m_2,m_3,H(m_1,m_2,p)\big) = H(m_1,m_3,p)\,. \tag{3.6}$$

Upon specialising the above identity to the case $m_1 = m_2 = m_3 =: m$ and invoking the injectivity of χ, we derive the identity

$$H(m,m,p) = p\,, \tag{3.7}$$

and so also – for $m_1 = m_3$ –

$$H\big(m_2,m_1,H(m_1,m_2,p)\big) = H(m_1,m_1,p) = p\,. \tag{3.8}$$

The latter description also has a local counterpart. Indeed, define

$$I^{[3]} := \big\{\ (i_1,i_2,i_3) \in I^{\times 3} \quad | \quad \widetilde{\mathcal{O}}_{(i_1,i_2,i_3)} := \big(\mathcal{O}_{i_1}\times\mathcal{O}_{i_2}\times\mathcal{O}_{i_3}\big)\cap M^{[3]} \neq \varnothing\ \big\}$$

and set

$$\begin{aligned}\tau^{[B]}_{(i_1,i_2,i_3)} \quad &: \quad \mathsf{P}_{[B]}\restriction_{\widetilde{\mathcal{O}}_{(i_1,i_2,i_3)}} \xrightarrow{\ \cong\ } \widetilde{\mathcal{O}}_{(i_1,i_2,i_3)}\times\mathbb{C}^\times\\ &: \quad (m_1,m_2,m_3,p) \longmapsto \big(m_1,m_2,m_3,\mathrm{pr}_2\circ\tau_{i_B}(p)\big)\,, \qquad B \in \{1,2,3\}\,,\end{aligned}$$

to obtain – for $(B,C) \in \{(1,2),(2,3),(1,3)\}$ –

$$\chi_{[B,C]}\circ\tau^{[B]\,-1}_{(i_1,i_2,i_3)}(m_1,m_2,m_3,z) = \tau^{[C]\,-1}_{(i_1,i_2,i_3)}\big(m_1,m_2,m_3,h_{(i_A,i_B)}(m_A,m_B)\cdot z\big)\,,$$

and hence the anticipated local form

$$h_{(i_2,i_3)}(m_2,m_3)\cdot h_{(i_1,i_2)}(m_1,m_2) = h_{(i_1,i_3)}(m_1,m_3)\,, \tag{3.9}$$

from which we deduce the local counterparts of Eqs. (3.7) and (3.8),

$$h_{(i,i)}(m,m) = 1\,, \qquad h_{(i_2,i_1)}(m_2,m_1) = h_{(i_1,i_2)}(m_1,m_2)^{-1}\,.$$

Finally, we consider the local presentation of morphisms Φ. Upon fixing a *common* trivialising cover $\{\mathcal{O}_i\}_{i\in I}$ of M for P_1 *and* P_2, with the respective local trivialisations

$$\tau_i^K\ :\ \pi_{\mathsf{P}_K}^{-1}(\mathcal{O}_i) \xrightarrow{\ \cong\ } \mathcal{O}_i\times\mathbb{C}^\times\,, \qquad K \in \{1,2\}$$

and the corresponding local connection potentials

$$A_i^K := \sigma_i^{K\,*}\mathcal{A}_K \in \Omega^1(\mathcal{O}_i)\,, \qquad \sigma_i^K \equiv \tau_i^{K\,-1}(\cdot,1)\,,$$

we arrive at a local presentation

$$\widetilde{\tau}_i^2\circ\Phi\circ\widetilde{\tau}_i^{1\,-1}(m,z) = \big(m,f_i(m)\cdot z\big)$$

for some $f_i \in C^\infty(\mathcal{O}_i, \mathrm{U}(1))$, which enters the relations

$$A_i^2 = A_i^1 + \mathsf{i}\,\mathsf{d}\log f_i$$

encoding the condition of preservation of connection,

$$\Phi^* \mathcal{A}_2 = \mathcal{A}_1 \,.$$

The coherence constraint of Diag. (3.3) imposed upon Φ over $M(\varpi)^{(1)}$ now reads, in the previous notation adapted to the situation in hand (through attachment of indices 1 (for χ_1) and 2 (for χ_2) to H and the h_{i_1,i_2}),

$$H_2\big(m_1, m_2, \Phi(p)\big) = \Phi \circ H_1(m_1, m_2, p)\,, \qquad p \in \pi_{\mathsf{P}_1}^{-1}\big(\{m_1\}\big)\,, \tag{3.10}$$

and, locally,

$$h^2_{(i_1,i_2)} \cdot \mathrm{pr}_1^* f_{i_1} = \mathrm{pr}_2^* f_{i_2} \cdot h^1_{(i_1,i_2)}\,.$$

We have the desired

Theorem 7. *Adopt the notation of Def. 5 and denote by* $\mathbb{C}^\times\text{–}\mathfrak{Bun}^\nabla(X;\mathrm{id}_X)$ *the category of principal* $\mathbb{C}^\times$*-bundles with a compatible connection over the base* X *of the surjective submersion* ϖ*, with morphisms covering the identity on the base. The pullback functor*

$$\widehat{\varpi}^* \;:\; \mathbb{C}^\times\text{–}\mathfrak{Bun}^\nabla(X;\mathrm{id}_X) \longrightarrow \mathbb{C}^\times\text{–}\mathfrak{BunDes}^\nabla(\varpi)\,,$$

with components defined by the formulæ

$$\begin{aligned}
\widehat{\varpi}^* \quad &: \quad \mathrm{Ob}\,\mathbb{C}^\times\text{–}\mathfrak{Bun}^\nabla(X;\mathrm{id}_X) \longrightarrow \mathrm{Ob}\,\mathbb{C}^\times\text{–}\mathfrak{BunDes}^\nabla(\varpi)\\
&: \quad (\underline{\mathsf{P}}, X, \pi_{\underline{\mathsf{P}}}, \mathbb{C}^\times, \underline{\mathcal{A}}) \longmapsto \big((\varpi^*\underline{\mathsf{P}}, M, \pi_{\varpi^*\underline{\mathsf{P}}}, \mathbb{C}^\times, \underline{\mathsf{P}}\varpi^*\underline{\mathcal{A}}), \mathrm{id}_{\mathrm{pr}_1^*\varpi^*\underline{\mathsf{P}}}\big)\,,
\end{aligned}$$

$$\begin{aligned}
\widehat{\varpi}^* \quad &: \quad \mathrm{Mor}\,\mathbb{C}^\times\text{–}\mathfrak{Bun}^\nabla(X;\mathrm{id}_X) \longrightarrow \mathrm{Mor}\,\mathbb{C}^\times\text{–}\mathfrak{BunDes}^\nabla(\varpi)\\
&: \quad \Big(\,\underline{\mathscr{P}}_1 \equiv (\underline{\mathsf{P}}_1, X, \pi_{\underline{\mathsf{P}}_1}, \mathbb{C}^\times, \underline{\mathcal{A}}_1) \xrightarrow{(\underline{\Phi},\mathrm{id}_X)} (\underline{\mathsf{P}}_2, X, \pi_{\underline{\mathsf{P}}_2}, \mathbb{C}^\times, \underline{\mathcal{A}}_2)\big) \equiv \underline{\mathscr{P}}_2\,\Big) \longmapsto \Big(\,\widehat{\varpi}^*\underline{\mathscr{P}}_1 \xrightarrow{(\varpi^*\underline{\Phi},\mathrm{id}_M)} \widehat{\varpi}^*\underline{\mathscr{P}}_2\,\Big)\,,
\end{aligned}$$

in which $\pi_{\varpi^*\underline{\mathsf{P}}} \equiv \mathrm{pr}_1$ *and* $\underline{\mathsf{P}}\varpi \equiv \mathrm{pr}_2$ *is to be understood as the covering map of* ϖ *in the commutative diagram*

$$\begin{array}{ccc}
\varpi^*\underline{\mathsf{P}} \equiv M\,{}_\varpi\!\times_{\pi_{\underline{\mathsf{P}}}}\underline{\mathsf{P}} & \xrightarrow{\;\underline{\mathsf{P}}\varpi \equiv \mathrm{pr}_2\;} & \underline{\mathsf{P}}\\
{\scriptstyle \pi_{\varpi^*\underline{\mathsf{P}}} \equiv \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\underline{\mathsf{P}}}}\\
M & \xrightarrow[\;\varpi\;]{} & X
\end{array}\,, \tag{3.11}$$

and in which

$$(\varpi^*\underline{\Phi}, \mathrm{id}_M) \quad : \quad \begin{array}{ccc}
\varpi^*\underline{\mathsf{P}}_1 \equiv M\,{}_\varpi\!\times_{\pi_{\underline{\mathsf{P}}_1}}\underline{\mathsf{P}}_1 & \xrightarrow{\;\varpi^*\underline{\Phi} \equiv \mathrm{id}_M \times \underline{\Phi}\;} & M\,{}_\varpi\!\times_{\pi_{\underline{\mathsf{P}}_2}}\underline{\mathsf{P}}_2 \equiv \varpi^*\underline{\mathsf{P}}_2\\
{\scriptstyle \pi_{\varpi^*\underline{\mathsf{P}}_1} \equiv \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1 \equiv \pi_{\varpi^*\underline{\mathsf{P}}_2}}\\
M & \underset{\mathrm{id}_M}{=\!=\!=\!=} & M
\end{array}\,, \tag{3.12}$$

is an equivalence of categories.

Proof. We begin by checking the well-definedness of the pullback functor $\widehat{\varpi}^*$. Thus, consider the pullback of a principal $\mathbb{C}^\times$-bundle with a compatible connection $\underline{\mathscr{P}} \equiv (\underline{\mathsf{P}}, X, \pi_{\underline{\mathsf{P}}}, \mathbb{C}^\times, \underline{\mathcal{A}})$ along ϖ, *i.e.*, the principal $\mathbb{C}^\times$-bundle with the total space given in Diag. (3.11) and the connection 1-form

$$\underline{\mathsf{P}}\varpi^*\underline{\mathcal{A}} \equiv \mathrm{pr}_2^*\underline{\mathcal{A}}\,.$$

The relevant pullbacks to $M^{[2]}$ by the canonical projections pr_A, $A \in \{1,2\}$ may now be written as

$$\mathrm{pr}_A^*\varpi^*\underline{\mathsf{P}} \equiv M^{[2]}\,{}_{\mathrm{pr}_A}\!\times_{\pi_{\varpi^*\underline{\mathsf{P}}}}\varpi^*\underline{\mathsf{P}} \equiv M^{[2]}\,{}_{\mathrm{pr}_A}\!\times_{\mathrm{pr}_1}\big(M\,{}_\varpi\!\times_{\pi_{\underline{\mathsf{P}}}}\underline{\mathsf{P}}\big) \equiv M^{[2]}\,{}_{\varpi\circ\mathrm{pr}_A}\!\times_{\pi_{\underline{\mathsf{P}}}}\underline{\mathsf{P}} \equiv \big(\varpi \circ \mathrm{pr}_A\big)^*\underline{\mathsf{P}}\,,$$

with – under this identification –

$$\big(\underline{\mathsf{P}}\varpi^*\underline{\mathcal{A}}\big)_{[A]} = \mathrm{pr}_2^*\underline{\mathcal{A}}$$

But by definition

$$\varpi \circ \mathrm{pr}_2 \restriction_{M^{[2]}} = \varpi \circ \mathrm{pr}_1 \restriction_{M^{[2]}} ,$$

and so we may take

$$\big(\varpi^*\underline{\mathsf{P}}\big)_{[2]} \equiv \mathrm{pr}_2^*\varpi^*\underline{\mathsf{P}} \equiv \mathrm{pr}_1^*\varpi^*\underline{\mathsf{P}} \equiv \big(\varpi^*\underline{\mathsf{P}}\big)_{[1]} ,$$

with

$$\big(\underline{\mathsf{P}}\varpi^*\underline{\mathcal{A}}\big)_{[2]} = \big(\underline{\mathsf{P}}\varpi^*\underline{\mathcal{A}}\big)_{[1]} ,$$

which justifies the choice of the trivial connection-preserving isomorphism

$$\mathrm{id}_{\mathrm{pr}_1^*\varpi^*\underline{\mathsf{P}}} \;:\; \big(\varpi^*\underline{\mathsf{P}}\big)_{[1]} \xrightarrow{\cong} \big(\varpi^*\underline{\mathsf{P}}\big)_{[2]}$$

as the remaining datum in the definition of an object of $\mathbb{C}^\times\text{–}\mathfrak{BunDes}^\nabla(\varpi)$ induced from $\underline{\mathcal{P}}$.

Next, take a connection-preserving isomorphism of principal $\mathbb{C}^\times$-bundles over X

$$\big(\underline{\Phi}, \mathrm{id}_X\big) \qquad : \qquad \begin{array}{ccc} \underline{\mathsf{P}}_1 & \xrightarrow{\;\;\underline{\Phi}\;\;} & \underline{\mathsf{P}}_2 \\ {\scriptstyle \pi_{\underline{\mathsf{P}}_1}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\underline{\mathsf{P}}_2}} \\ X & \underset{\mathrm{id}_X}{=\!=\!=\!=} & X \end{array}$$

and consider the pullback isomorphism (3.12). Taking into account the triviality of the isomorphism datum for both pullback bundles alongside the former identifications for the (double-)pullback bundles, we arrive at the coherence condition of Diag. (3.3) in the form

$$\begin{array}{ccc} \big(\varpi^*\underline{\mathsf{P}}_1\big)_{[1]} \equiv M^{[2]}\,{}_{\varpi\circ\mathrm{pr}_1}\!\times_{\pi_{\underline{\mathsf{P}}_1}} \underline{\mathsf{P}}_1 & \xrightarrow{\;\mathrm{id}_{\mathrm{pr}_1^*\varpi^*\underline{\mathsf{P}}_1}\;} & M^{[2]}\,{}_{\varpi\circ\mathrm{pr}_1}\!\times_{\pi_{\underline{\mathsf{P}}_1}} \underline{\mathsf{P}}_1 \equiv \big(\varpi^*\underline{\mathsf{P}}_1\big)_{[2]} \\ {\scriptstyle \mathrm{pr}_1^*\varpi^*\underline{\Phi}\equiv\mathrm{id}_{M^{[2]}}\times\Phi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{M^{[2]}}\times\Phi\equiv\mathrm{pr}_2^*\varpi^*\underline{\Phi}} \\ \big(\varpi^*\underline{\mathsf{P}}_2\big)_{[1]} \equiv M^{[2]}\,{}_{\varpi\circ\mathrm{pr}_1}\!\times_{\pi_{\underline{\mathsf{P}}_2}} \underline{\mathsf{P}}_2 & \underset{\mathrm{id}_{\mathrm{pr}_1^*\varpi^*\underline{\mathsf{P}}_2}}{\longrightarrow} & M^{[2]}\,{}_{\varpi\circ\mathrm{pr}_1}\!\times_{\pi_{\underline{\mathsf{P}}_2}} \underline{\mathsf{P}}_2 \equiv \big(\varpi^*\underline{\mathsf{P}}_2\big)_{[2]} \end{array} ,$$

putting on display its triviality. Thus, the functor $\widehat{\varpi}^*$ is, indeed, well-defined.

Next, we induce a principal $\mathbb{C}^\times$-bundle over X from a descendable principal $\mathbb{C}^\times$-bundle $(\mathcal{P}, \chi) \equiv ((\mathsf{P}, M, \pi_{\mathsf{P}}, \mathbb{C}^\times, \mathcal{A}), \chi)$. To this end, consider an open cover $\mathcal{U} \equiv \{\mathcal{U}_\alpha\}_{\alpha\in\mathscr{A}}$ of X whose elements support the respective local sections

$$\varsigma_\alpha \;:\; \mathcal{U}_\alpha \longrightarrow M$$

of the surjective submersion ϖ, *i.e.*, we have

$$\varpi \circ \varsigma_\alpha = \mathrm{id}_{\mathcal{U}_\alpha} .$$

With these, we may associate the family of pullback principal $\mathbb{C}^\times$-bundles

$$\begin{array}{ccc} \underline{\mathsf{P}}_\alpha \coloneqq \varsigma_\alpha^*\mathsf{P} \equiv \mathcal{U}_\alpha\,{}_{\varsigma_\alpha}\!\times_{\pi_{\mathsf{P}}} \mathsf{P} & \xrightarrow{\;\mathsf{P}\varsigma_\alpha\equiv\mathrm{pr}_2\;} & \mathsf{P} \\ {\scriptstyle \pi_{\underline{\mathsf{P}}_\alpha}\equiv\mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathsf{P}}} \\ \mathcal{U}_\alpha & \underset{\varsigma_\alpha}{\longrightarrow} & M \end{array} ,$$

which over $\mathcal{U}_{\alpha\beta} \equiv \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ (assumed non-empty), with the corresponding sections

$$\varsigma_{\alpha,\beta} \coloneqq (\varsigma_\alpha, \varsigma_\beta) \;:\; \mathcal{U}_{\alpha\beta} \longrightarrow M^{[2]} ,$$

become related as

$$\chi_{\alpha,\beta} \coloneqq \varsigma_{\alpha,\beta}^*\chi \quad : \quad \varsigma_{\alpha,\beta}^*\mathrm{pr}_1^*\mathsf{P} \equiv \mathcal{U}_{\alpha\beta}\,{}_{\varsigma_{\alpha,\beta}}\!\times_{\mathrm{pr}_1} \big(M^{[2]}\,{}_{\mathrm{pr}_1}\!\times_{\pi_{\mathsf{P}}} \mathsf{P}\big) \equiv \mathcal{U}_{\alpha\beta}\,{}_{\varsigma_\alpha}\!\times_{\pi_{\mathsf{P}}} \mathsf{P} \equiv \underline{\mathsf{P}}_\alpha \restriction_{\mathcal{U}_{\alpha\beta}}$$

$$\xrightarrow{\cong} \underline{\mathsf{P}}_\beta\upharpoonright_{\mathcal{U}_{\alpha\beta}} \equiv \mathcal{U}_{\alpha\beta}\ {}_{\varsigma_\beta}\times_{\pi_{\mathsf{P}}} \mathsf{P} \equiv \mathcal{U}_{\alpha\beta}\ {}_{\varsigma_{\alpha,\beta}}\times_{\mathrm{pr}_1} \big(M^{[2]}\ {}_{\mathrm{pr}_2}\times_{\pi_{\mathsf{P}}} \mathsf{P}\big) \equiv \varsigma_{\alpha,\beta}^*\mathrm{pr}_2^*\mathsf{P}$$

$$: \quad \big(x, \big(\varsigma_\alpha(x), \varsigma_\beta(x), p\big)\big) \equiv (x,p) \longmapsto \big(x, H\big(\varsigma_\alpha(x), \varsigma_\beta(x), p\big)\big)$$

$$\equiv \big(x, \big(\varsigma_\alpha(x), \varsigma_\beta(x), H\big(\varsigma_\alpha(x), \varsigma_\beta(x), p\big)\big)\big)\,,$$

i.e., under the above identifications,

$$\chi_{\alpha,\beta} \equiv \big(\mathrm{pr}_1, \mathrm{pr}_2 \circ \chi \circ (\varsigma_{\alpha,\beta} \times \mathrm{id}_{\mathsf{P}})\big)\,,$$

so that over $\mathcal{U}_{\alpha\beta\gamma} \equiv \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$ (assumed non-empty), with the corresponding sections

$$\varsigma_{\alpha,\beta,\gamma} := (\varsigma_\alpha, \varsigma_\beta, \varsigma_\gamma) \; : \; \mathcal{U}_{\alpha\beta\gamma} \longrightarrow M^{[3]}\,,$$

we obtain

$$\begin{array}{ccc}
\varsigma_{\alpha,\beta,\gamma}^*\mathrm{pr}_1^*\mathsf{P} \equiv \underline{\mathsf{P}}_\alpha\upharpoonright_{\mathcal{U}_{\alpha\beta\gamma}} & \xrightarrow{\varsigma_{\alpha,\beta,\gamma}^*\mathrm{pr}_{1,2}^*\chi \equiv \chi_{\alpha,\beta}\upharpoonright_{\mathcal{U}_{\alpha\beta\gamma}}} & \underline{\mathsf{P}}_\beta\upharpoonright_{\mathcal{U}_{\alpha\beta\gamma}} \equiv \varsigma_{\alpha,\beta,\gamma}^*\mathrm{pr}_2^*\mathsf{P} \\
\Big\downarrow {\scriptstyle \varsigma_{\alpha,\beta,\gamma}^*\mathrm{pr}_{1,3}^*\chi \equiv \chi_{\alpha,\gamma}\upharpoonright_{\mathcal{U}_{\alpha\beta\gamma}}} & & \Big\downarrow {\scriptstyle \varsigma_{\alpha,\beta,\gamma}^*\mathrm{pr}_{2,3}^*\chi \equiv \chi_{\beta,\gamma}\upharpoonright_{\mathcal{U}_{\alpha\beta\gamma}}} \\
\varsigma_{\alpha,\beta,\gamma}^*\mathrm{pr}_3^*\mathsf{P} \equiv \underline{\mathsf{P}}_\gamma\upharpoonright_{\mathcal{U}_{\alpha\beta\gamma}} & \underset{\mathrm{id}_{\underline{\mathsf{P}}_\gamma\upharpoonright_{\mathcal{U}_{\alpha\beta\gamma}}}}{=\!=\!=\!=\!=} & \underline{\mathsf{P}}_\gamma\upharpoonright_{\mathcal{U}_{\alpha\beta\gamma}} \equiv \varsigma_{\alpha,\beta,\gamma}^*\mathrm{pr}_3^*\mathsf{P}
\end{array}\,.$$

The disjoint union of local bundles:

$$\underline{\mathsf{P}}_\sqcup := \bigsqcup_{\alpha\in\mathscr{A}} \underline{\mathsf{P}}_\alpha$$

clearly is *not* a bundle (unless $|\mathscr{A}| = 1$), but it may be turned into one with the help of a construction similar in spirit to that used in the constructive proof of The Clutching Theorem for principal $\mathbb{C}^\times$-bundles. Indeed, we may consider on $\underline{\mathsf{P}}_\sqcup$ a relation

$$(x_1, p_1, \alpha_1) \sim_{\chi_{\cdot,\cdot}} (x_2, p_2, \alpha_2) \qquad \Longleftrightarrow \qquad \left\{ \begin{array}{c} x_1 = x_2 \in \mathcal{U}_{\alpha_1\alpha_2} \\ (x_2, p_2) = \chi_{\alpha_1,\alpha_2}(x_1, p_1) \equiv \big(x_1, H\big(\varsigma_\alpha(x_1), \varsigma_\beta(x_1), p_1\big)\big) \end{array} \right.$$

We readily check that it is actually an equivalence relation. Indeed, its reflexivity is ensured, for $x \in \mathcal{U}_\alpha$, by property (3.7),

$$x = x \in \mathcal{U}_\alpha \equiv \mathcal{U}_{\alpha\alpha} \qquad \wedge \qquad \chi_{\alpha,\alpha}(x,p) \equiv \big(x, H\big(\varsigma_\alpha(x), \varsigma_\alpha(x), p\big)\big) = (x,p)\,;$$

symmetry follows, for $y \in \mathcal{U}_{\alpha\beta}$, from property (3.8) (and $\mathcal{U}_{\beta\alpha} \equiv \mathcal{U}_{\alpha\beta}$),

$$(y, p_2) = \chi_{\alpha,\beta}(y, p_1) \equiv \big(y, H\big(\varsigma_\alpha(y), \varsigma_\beta(y), p_1\big)\big)$$

$$\Longrightarrow \qquad (y, p_1) = \big(y, H\big(\varsigma_\beta(y), \varsigma_\alpha(y), H\big(\varsigma_\alpha(y), \varsigma_\beta(y), p_1\big)\big)\big) = \big(y, H\big(\varsigma_\beta(y), \varsigma_\alpha(y), p_2\big)\big) \equiv \chi_{\beta,\alpha}(y, p_2)\,;$$

and transitivity is, for $u \in \mathcal{U}_{\alpha\beta\gamma}$, a consequence of the telescoping identity (3.6) in its full form,

$$\left. \begin{array}{l} (u, p_2) = \chi_{\alpha,\beta}(u, p_1) \equiv \big(y, H\big(\varsigma_\alpha(y), \varsigma_\beta(y), p_1\big)\big) \\ (u, p_3) = \chi_{\beta,\gamma}(u, p_2) \equiv \big(y, H\big(\varsigma_\beta(y), \varsigma_\gamma(y), p_2\big)\big) \end{array} \right\}$$

$$\Longrightarrow \qquad (u, p_3) = \big(y, H\big(\varsigma_\beta(y), \varsigma_\gamma(y), H\big(\varsigma_\alpha(y), \varsigma_\beta(y), p_1\big)\big)\big) = \big(y, H\big(\varsigma_\alpha(y), \varsigma_\gamma(y), p_1\big)\big) \equiv \chi_{\alpha,\gamma}(u, p_1)\,.$$

Thus, the set of equivalence classes

$$\underline{\mathsf{P}}^\chi := \big(\bigsqcup_{\alpha\in\mathscr{A}} \underline{\mathsf{P}}_\alpha\big)/_{\sim_{\chi_{\cdot,\cdot}}}$$

can be formed, with the redundancy of the assignment of fibres to a given point in an intersection of distinct elements of the open cover $\mathcal{U}$ removed completely owing to the bijective character of the $\chi_{\alpha,\beta}$. We induce on $\underline{\mathsf{P}}^\chi$ the quotient topology along the projection

$$\pi_\sim \; : \; \underline{\mathsf{P}}_\sqcup \longrightarrow \underline{\mathsf{P}}^\chi \; : \; (x, p, \alpha) \longmapsto [(x, p, \alpha)]\,,$$

i.e., we declare a subset $\mathcal{W} \subset \underline{\mathsf{P}}^\chi$ open if its preimage under $\pi_\sim$ is open in the disjoint-sum topology on $\underline{\mathsf{P}}_\sqcup$. The topology is readily seen to be Hausdorff due to the following fact: Each class $[(x,p,\alpha)]$ contains at most one element with a given index as

$$(3.13) \qquad (y, q, \alpha) \in [(x, p, \alpha)] \qquad \Longrightarrow \qquad (y, q) = \chi_{\alpha,\alpha}(x, p) = (x, p)\,.$$

Consequently, whenever $[(x_1,p_1,\alpha_1)]\neq[(x_2,p_2,\alpha_2)]$, we have the disjunction: Either $x_1\neq x_2$, in which case the two base points have[4] open neighbourhoods $\mathcal{U}_{x_A}\ni x_A,\ A\in\{1,2\}$ which separate them, $\mathcal{U}_{x_1}\cap\mathcal{U}_{x_2}=\varnothing$, and hence give us separating open neighbourhoods $\pi_\sim(\mathcal{U}_{x_A}\,{}_{\varsigma_{\alpha_A}}\!\times_{\pi_{\mathsf{P}}}\mathsf{P}\times\{\alpha_A\})\ni[(x_A,p_A,\alpha_A)]$ in the quotient, or $x_1=x_2\in\mathcal{U}_{\alpha_1\alpha_2}$, in which case we may rewrite $[(x_2,p_2,\alpha_2)]\equiv[(x_1,p_2,\alpha_2)]=[(\chi_{\alpha_2,\alpha_1}(x_1,p_2),\alpha_1)]=[(x_1,H(\varsigma_{\alpha_2,\alpha_1}(x_1),p_2),\alpha_1)]$, with $\widetilde{p}_2\equiv H(\varsigma_{\alpha_2,\alpha_1}(x_1),p_2)\neq p_1\equiv\widetilde{p}_1$ by the above argument, so that there exist separating open neighbourhoods $\mathcal{V}_{\widetilde{p}_A}\ni\widetilde{p}_A,\ A\in\{1,2\}$ in the Hausdorff space P that yield separating open neighbourhoods $\pi_\sim(\mathcal{U}^A_\alpha\,{}_{\varsigma_{\alpha_A}}\!\times_{\pi_{\mathsf{P}}}\mathcal{V}_{\widetilde{p}_A}\times\{\alpha\})\ni[(x_A,p_A,\alpha_A)]$ with $\mathcal{U}^A_\alpha=\mathcal{U}_\alpha\cap\varsigma_{\alpha_A}^{-1}(\pi_{\mathsf{P}}(\mathcal{V}_{\widetilde{p}_A}))$. Having established the structure of a Hausdorff topological space on $\underline{\mathsf{P}}^\chi$, we may, next, identify the anticipated principal $\mathbb{C}^\times$-fibration over X. In order to be able to invoke essentially the same arguments as in the aforementioned proof (*i.e.*, employ the local structure on the local models $\underline{\mathsf{P}}_\alpha$), we need to refine the original cover $\mathcal{U}$ relative to the pullback covers of its elements induced by any trivialising cover $\mathcal{O}\equiv\{\mathcal{O}_i\}_{i\in I}$ for (the base M of) P, coming with the respective local trivialisations

$$\tau_i\ :\ \pi_{\mathsf{P}}^{-1}(\mathcal{O}_i)\xrightarrow{\ \cong\ }\mathcal{O}_i\times\mathbb{C}^\times\,,$$

that is, we consider the open sets

$$\mathcal{U}_{(\alpha,i_\alpha)}:=\varsigma_\alpha^{-1}(\mathcal{O}_{i_\alpha})\subset\mathcal{U}_\alpha\,,\qquad\qquad i_\alpha\in\{\ i\in I\quad|\quad\varsigma_\alpha(\mathcal{U}_\alpha)\cap\mathcal{O}_i\neq\varnothing\ \}\equiv I_\alpha\,,$$

obtaining the cover

$$X=\bigcup_{\alpha\in\mathscr{A}}\ \bigcup_{i_\alpha\in I_\alpha}\mathcal{U}_{(\alpha,i_\alpha)}\,.$$

Over elements of the refined cover, we have mappings

$$\underline{\tau}_{(\alpha,i_\alpha)}\ :\ \underline{\mathsf{P}}^\chi\restriction_{\mathcal{U}_{(\alpha,i_\alpha)}}\longrightarrow\mathcal{U}_{(\alpha,i_\alpha)}\times\mathbb{C}^\times\ :\ [(x,p,\alpha)]\longmapsto\big(x,\mathrm{pr}_2\circ\tau_{i_\alpha}(p)\big)\,,$$

which – by (the argument leading up to) (3.13) – are bijections with inverses

$$\underline{\tau}^{-1}_{(\alpha,i_\alpha)}\ :\ \mathcal{U}_{(\alpha,i_\alpha)}\times\mathbb{C}^\times\longrightarrow\underline{\mathsf{P}}^\chi\restriction_{\mathcal{U}_{(\alpha,i_\alpha)}}\ :\ (x,z)\longmapsto\big[\big(x,\tau_{i_\alpha}^{-1}(x,z),\alpha\big)\big]\,,$$

and so – tautologically – homeomorphisms. These can subsequently be used to import the smooth structure from the local models $\underline{\mathsf{P}}_\alpha$, whereby they are promoted (tautologically, again) to the rank of diffeomorphisms. As such, they become local trivialisations of the ensuing principal $\mathbb{C}^\times$-bundle

$$\begin{array}{ccc}\mathbb{C}^\times & \rightsquigarrow & \underline{\mathsf{P}}^\chi\\ & & \big\downarrow{\scriptstyle \pi_{\underline{\mathsf{P}}^\chi}}\\ & & X\end{array}\ ,\qquad\qquad\pi_{\underline{\mathsf{P}}^\chi}\big([(x,p,\alpha)]\big)=x\,.$$

At this stage, we still need to derive the transition mappings associated with the above trivialisations and verify their smoothness. To this end, pick up a point $x\in\mathcal{U}_{(\alpha,i_\alpha)}\cap\mathcal{U}_{(\beta,j_\beta)}\equiv\mathcal{U}_{(\alpha,i_\alpha)(\beta,j_\beta)}$ and $p\in\mathsf{P}_{\varsigma_\alpha(x)}\subset\pi_{\mathsf{P}}^{-1}(\mathcal{O}_{i_\alpha})$, so that $H(\varsigma_\alpha(x),\varsigma_\beta(x),p)\in\mathsf{P}_{\varsigma_\beta(x)}\subset\pi_{\mathsf{P}}^{-1}(\mathcal{O}_{j_\beta})$. We then find

$$\begin{aligned}&\underline{\tau}_{(\alpha,i_\alpha)}\circ\underline{\tau}^{-1}_{(\beta,j_\beta)}(x,z)=\underline{\tau}_{(\alpha,i_\alpha)}\big(\big[\big(x,\tau_{j_\beta}^{-1}(x,z),\beta\big)\big]\big)=\underline{\tau}_{(\alpha,i_\alpha)}\big(\big[\big(\chi_{\beta,\alpha}\big(x,\tau_{j_\beta}^{-1}(x,z)\big),\alpha\big)\big]\big)\\ =\ &\underline{\tau}_{(\alpha,i_\alpha)}\big(\big[\big(x,H\big(\varsigma_\beta(x),\varsigma_\alpha(x),\tau_{j_\beta}^{-1}(x,z)\big),\alpha\big)\big]\big)=\big(x,\mathrm{pr}_2\circ\tau_{i_\alpha}\circ H\big(\varsigma_\beta(x),\varsigma_\alpha(x),\tau_{j_\beta}^{-1}(x,z)\big)\big)\,,\end{aligned}$$

with

$$\mathrm{pr}_2\circ\tau_{i_\alpha}\circ H\big(\varsigma_\beta(x),\varsigma_\alpha(x),\tau_{j_\beta}^{-1}(x,z)\big)=\mathrm{pr}_2\big(\varsigma_\alpha(x),h_{(j_\beta,i_\alpha)}\big(\varsigma_\beta(x),\varsigma_\alpha(x)\big)\cdot z\big)=h_{(j_\beta,i_\alpha)}\big(\varsigma_\beta(x),\varsigma_\alpha(x)\big)\cdot z\,,$$

cf. Eq. (3.4), and so conclude that the transition mappings take the manifestly smooth form

$$\underline{g}_{(\alpha,i_\alpha)(\beta,j_\beta)}=\varsigma_{\beta,\alpha}^*h_{(j_\beta,i_\alpha)}\ :\ \mathcal{U}_{(\alpha,i_\alpha)(\beta,j_\beta)}\longrightarrow\mathrm{U}(1)\,.$$

Note that the relevant 1-cocycle condition is ensured by identity (3.9).

Having descended P to a principal $\mathbb{C}^\times$-bundle over X, we may, next, endow it with a connection. Let us start with the pullback connection 1-form

$$\underline{\mathcal{A}}_\alpha:=\mathsf{P}\varsigma_\alpha^*\mathcal{A}\equiv\mathrm{pr}_2^*\mathcal{A}$$

on $\underline{\mathsf{P}}_\alpha$. Over $\mathcal{U}_{\alpha\beta}$, we find the identity

$$\chi_{\alpha,\beta}^*\underline{\mathcal{A}}_\beta\ \ \equiv\ \ \chi_{\alpha,\beta}^*\mathsf{P}\varsigma_\beta^*\mathcal{A}\equiv\chi_{\alpha,\beta}^*\mathrm{pr}_2^*\mathcal{A}=\big(\mathrm{pr}_1,\mathrm{pr}_2\circ\chi\circ(\varsigma_{\alpha,\beta}\times\mathrm{id}_{\mathsf{P}})\big)^*\mathrm{pr}_2^*\mathcal{A}=\big(\mathrm{pr}_2\circ\chi\circ(\varsigma_{\alpha,\beta}\times\mathrm{id}_{\mathsf{P}})\big)^*\mathcal{A}$$

[4]The base is a manifold, and so it is Hausdorff.

$$= \quad (\varsigma_{\alpha,\beta} \times \mathrm{id}_{\mathsf{P}})^* \chi^* \mathcal{A}_{[2]} = (\varsigma_{\alpha,\beta} \times \mathrm{id}_{\mathsf{P}})^* \mathcal{A}_{[1]} \equiv (\varsigma_{\alpha,\beta} \times \mathrm{id}_{\mathsf{P}})^* \mathsf{P}\varsigma_\alpha^* \mathcal{A} = \mathsf{P}\varsigma_\alpha^* \mathcal{A} = \underline{\mathcal{A}}_\alpha \,,$$

cf. Eq. (3.5). The last result means that the $\underline{\mathcal{A}}_\alpha$ descend from the disjoint components $\underline{\mathsf{P}}_\sqcup$ to induce a smooth connection 1-form

$$\underline{\mathcal{A}}^\chi \in \Omega^1(\underline{\mathsf{P}}^\chi)$$

on the quotient $\underline{\mathsf{P}}$, with the property

$$\underline{\mathcal{A}}^\chi \restriction_{\mathcal{U}_\alpha} \equiv \underline{\mathcal{A}}_\alpha \,.$$

Thus, altogether, we obtain a principal $\mathbb{C}^\times$-bundle with a compatible connection

$$\underline{\mathcal{P}}^\chi \equiv \big(\underline{\mathsf{P}}^\chi, X, \pi_{\underline{\mathsf{P}}^\chi}, \mathbb{C}^\times, \underline{\mathcal{A}}^\chi\big) \,.$$

We shall now pull back $\underline{\mathcal{P}}^\chi$ along ϖ and compare the resultant bundle over M with the original bundle $\mathcal{P}$. For that, we shall first formalise the obvious (canonical) connection-preserving isomorphisms:

$$\iota_\alpha \; : \; \underline{\mathsf{P}}^\chi \restriction_{\mathcal{U}_\alpha} \xrightarrow{\cong} \underline{\mathsf{P}}_\alpha \; : \; [(x,p,\alpha)] \longmapsto (x,p) \,, \qquad \alpha \in \mathscr{A} \,.$$

Their diffeomorphic character (with respect to the above-induced smooth structure on the domain) is implied by the following observations: In any neighbourhood $\mathcal{U}_{(\alpha,i_\alpha)} \subset \mathcal{U}_\alpha$, the (global) map ι_α decomposes as

$$\iota_\alpha = \big(\mathrm{id}_X \times \tau_{i_\alpha}^{-1}\big) \circ (\mathrm{pr}_1, \varsigma_\alpha \times \mathrm{id}_{\mathbb{C}^\times}) \circ \underline{\tau}_{(\alpha,i_\alpha)}$$

in terms of smooth maps. Its (global) inverse

$$\iota_\alpha^{-1}(x,p) = [(x,p,\alpha)] \,,$$

on the other hand, factorises, over the same neighbourhood $\mathcal{U}_{(\alpha,i_\alpha)}$, as

$$\iota_\alpha^{-1} = \underline{\tau}_{(\alpha,i_\alpha)}^{-1} \circ \big(\mathrm{id}_X \times \mathrm{pr}_2 \circ \tau_{i_\alpha}\big)$$

in terms of smooth maps. Clearly, the ι_α preserve the connections. These properties are inherited by the pullbacks

$$\widehat{\iota}_\alpha \equiv \varpi^* \iota_\alpha \; : \; \varpi^* \underline{\mathsf{P}}^\chi \restriction_{\varpi^{-1}(\mathcal{U}_\alpha)} \equiv \varpi^* \big(\underline{\mathsf{P}}^\chi \restriction_{\mathcal{U}_\alpha}\big) \xrightarrow{\cong} \varpi^* \underline{\mathsf{P}}_\alpha \,.$$

Secondly, we need the smooth maps

$$\widehat{\varsigma}_\alpha \; : \; \varpi^{-1}(\mathcal{U}_\alpha) \longrightarrow M^{[2]} \; : \; m \longmapsto \big(m, \varsigma_\alpha \circ \varpi(m)\big) \,, \qquad \alpha \in \mathscr{A}$$

to define the pullback bundles

$$\widehat{\varsigma}_\alpha^* \mathsf{P}_{[1]} \equiv \varpi^{-1}(\mathcal{U}_\alpha) \,{}_{\widehat{\varsigma}_{\alpha A}}\!\times_{\mathrm{pr}_1} \big(M^{[2]} \,{}_{\mathrm{pr}_1}\!\times_{\pi_{\mathsf{P}}} \mathsf{P}\big) \equiv \varpi^{-1}(\mathcal{U}_\alpha) \,{}_{\mathrm{id}_M}\!\times_{\pi_{\mathsf{P}}} \mathsf{P} \equiv \mathsf{P} \restriction_{\varpi^{-1}(\mathcal{U}_\alpha)} \,,$$

$$\widehat{\varsigma}_\alpha^* \mathsf{P}_{[2]} \equiv \varpi^{-1}(\mathcal{U}_\alpha) \,{}_{\widehat{\varsigma}_{\alpha A}}\!\times_{\mathrm{pr}_1} \big(M^{[2]} \,{}_{\mathrm{pr}_2}\!\times_{\pi_{\mathsf{P}}} \mathsf{P}\big) \equiv \varpi^{-1}(\mathcal{U}_\alpha) \,{}_{\varsigma_\alpha \circ \varpi}\!\times_{\pi_{\mathsf{P}}} \mathsf{P} \equiv M \,{}_{\varpi}\!\times_{\mathrm{pr}_1} \big(\mathcal{U}_\alpha \,{}_{\varsigma_\alpha}\!\times_{\pi_{\mathsf{P}}} \mathsf{P}\big) \equiv \varpi^* \underline{\mathsf{P}}_\alpha \,,$$

related by the (connection-preserving) pullback isomorphisms

$$\widehat{\chi}_\alpha \equiv \widehat{\varsigma}_\alpha^* \chi \; : \; \mathsf{P} \restriction_{\varpi^{-1}(\mathcal{U}_\alpha)} \equiv \widehat{\varsigma}_\alpha^* \mathsf{P}_{[1]} \xrightarrow{\cong} \widehat{\varsigma}_\alpha^* \mathsf{P}_{[2]} \equiv \varpi^* \underline{\mathsf{P}}_\alpha \,.$$

With all requisites in hand, we may finally define the composite connection-preserving isomorphisms

$$\eta_\alpha := \widehat{\iota}_\alpha^{-1} \circ \widehat{\chi}_\alpha \; : \; \mathsf{P} \restriction_{\varpi^{-1}(\mathcal{U}_\alpha)} \xrightarrow{\cong} \varpi^* \underline{\mathsf{P}}^\chi \restriction_{\varpi^{-1}(\mathcal{U}_\alpha)} \,.$$

In the remainder of this part of the proof, we demonstrate that the η_α are restrictions of a globally smooth connection-preserving ϖ-descendable principal $\mathbb{C}^\times$-bundle isomorphism

$$\eta_{(\mathcal{P},\chi)} \; : \; \mathsf{P} \xrightarrow{\cong} \varpi^* \underline{\mathsf{P}}^\chi \,. \tag{3.14}$$

In order to attain our goal, we need to work out the explicit form assumed by the $\widehat{\chi}_\alpha$ under the above identifications. We have, for any $m \in M$ such that $\varpi(m) \in \mathcal{U}_\alpha$ and $p \in \mathsf{P}_m$,

$$\begin{aligned} \widehat{\chi}_\alpha \quad : \quad p \equiv \big(m, \big(m, \varsigma_\alpha \circ \varpi(m), p\big)\big) &\longmapsto \big(m, \big(m, \varsigma_\alpha \circ \varpi(m), H\big(m, \varsigma_\alpha \circ \varpi(m), p\big)\big)\big) \\ &\equiv \big(m, \big(\varpi(m), H\big(m, \varsigma_\alpha \circ \varpi(m), p\big)\big)\big) \,, \end{aligned}$$

whence

$$\eta_\alpha(p) = \big(\pi_{\mathsf{P}}(p), \big[\big(\varpi \circ \pi_{\mathsf{P}}(p), H\big(\pi_{\mathsf{P}}(p), \varsigma_\alpha \circ \varpi \circ \pi_{\mathsf{P}}(p), p\big), \alpha\big)\big]\big) \,.$$

Let, now, $m \in \varpi^{-1}(\mathcal{U}_{\alpha\beta})$ and $p \in \mathsf{P}_m$ as before. Using the definition of the quotient $\underline{\mathsf{P}}^\chi$ in conjunction with the telescoping identity (3.6), we then obtain the desired equality

$$\begin{aligned} \eta_\beta(p) &= \big(\pi_{\mathsf{P}}(p), \big[\big(\varpi \circ \pi_{\mathsf{P}}(p), H\big(\pi_{\mathsf{P}}(p), \varsigma_\beta \circ \varpi \circ \pi_{\mathsf{P}}(p), p\big), \beta\big)\big]\big) \\ &= \big(\pi_{\mathsf{P}}(p), \big[\big(\varpi \circ \pi_{\mathsf{P}}(p), H\big(\varsigma_\beta \circ \varpi \circ \pi_{\mathsf{P}}(p), \varsigma_\alpha \circ \varpi \circ \pi_{\mathsf{P}}(p), H\big(\pi_{\mathsf{P}}(p), \varsigma_\beta \circ \varpi \circ \pi_{\mathsf{P}}(p), p\big)\big), \alpha\big)\big]\big) \\ &= \big(\pi_{\mathsf{P}}(p), \big[\big(\varpi \circ \pi_{\mathsf{P}}(p), H\big(\pi_{\mathsf{P}}(p), \varsigma_\alpha \circ \varpi \circ \pi_{\mathsf{P}}(p), p\big), \alpha\big)\big]\big) \equiv \eta_\alpha(p)\,, \end{aligned}$$

which proves the existence of an isomorphism (3.14). It remains to verify that the latter satisfies the identity expressed by the commutative diagram

$$\begin{array}{ccc} \mathsf{P}_{[1]} & \xrightarrow{\qquad\qquad \chi \qquad\qquad} & \mathsf{P}_{[2]} \\ {\scriptstyle \eta_{(\mathcal{P},\chi)\,[1]}}\big\downarrow & & \big\downarrow{\scriptstyle \eta_{(\mathcal{P},\chi)\,[2]}} \\ \big(\varpi^*\underline{\mathsf{P}}^\chi\big)_{[1]} \equiv \big(\varpi \circ {\rm pr}_1\big)^*\underline{\mathsf{P}}^\chi & \xrightarrow[{\rm id}_{{\rm pr}_1^*\varpi^*\underline{\mathsf{P}}^\chi}]{} & \big(\varpi \circ {\rm pr}_1\big)^*\underline{\mathsf{P}}^\chi \equiv \big(\varpi^*\underline{\mathsf{P}}^\chi\big)_{[2]} \end{array} .$$

This we do by computing, for any $(m_1, m_2) \in M^{[2]}$ with $\varpi(m_1) = \varpi(m_2) \in \mathcal{U}_\alpha$ and $p \in \mathsf{P}_{m_1}$, and with the above identifications in mind,

$$\begin{aligned} &\eta_{(\mathcal{P},\chi)\,[2]} \circ \chi(m_1, m_2, p) = \eta_{(\mathcal{P},\chi)\,[2]}\big(m_1, m_2, H(m_1, m_2, p)\big) = \big(m_1, m_2, \eta_\alpha \circ H(m_1, m_2, p)\big) \\ &= \big(m_1, m_2, \big[\big(\varpi \circ \pi_{\mathsf{P}} \circ H(m_1, m_2, p), H\big(\pi_{\mathsf{P}} \circ H(m_1, m_2, p), \varsigma_\alpha \circ \varpi \circ \pi_{\mathsf{P}} \circ H(m_1, m_2, p), H(m_1, m_2, p)\big), \alpha\big)\big]\big) \\ &= \big(m_1, m_2, \big[\big(\varpi(m_2), H\big(m_2, \varsigma_\alpha \circ \varpi(m_2), H(m_1, m_2, p)\big), \alpha\big)\big]\big) \\ &= \big(m_1, m_2, \big[\big(\varpi(m_2), H\big(m_1, \varsigma_\alpha \circ \varpi(m_2), p\big), \alpha\big)\big]\big) \end{aligned}$$

(where we have taken identity (3.6) into account once more), and comparing it with

$$\begin{aligned} &\eta_{(\mathcal{P},\chi)\,[1]}(m_1, m_2, p) = \big(m_1, m_2, \eta_\alpha(p)\big) = \big(m_1, m_2, \big[\big(\varpi \circ \pi_{\mathsf{P}}(p), H\big(\pi_{\mathsf{P}}(p), \varsigma_\alpha \circ \varpi \circ \pi_{\mathsf{P}}(p), p\big), \alpha\big)\big]\big) \\ &= \big(m_1, m_2, \big[\big(\varpi(m_1), H\big(m_1, \varsigma_\alpha \circ \varpi(m_1), p\big), \alpha\big)\big]\big)\,. \end{aligned}$$

Thus, the equality $\varpi(m_1) = \varpi(m_2)$ ensures the commutativity of the diagram. We conclude that

$$\eta_{(\mathcal{P},\chi)} \in {\rm Hom}_{\mathbb{C}^\times-\mathfrak{BunDes}^\nabla(\varpi)}\big((\mathcal{P}, \chi), \widehat{\varpi}^*\underline{\mathcal{P}}^\chi\big)\,.$$

We now pass to the morphism component of the inverse of the pullback functor $\widehat{\varpi}^*$ under reconstruction. Thus, we consider a connection-preserving isomorphism $(\Phi, {\rm id}_M) \,:\, \mathsf{P}_1 \xrightarrow{\cong} \mathsf{P}_2$ subject to the coherence condition expressed by Diag. (3.3), encoded in the functional relation (3.10). We commence its descent by defining the local pullback (connection-preserving) isomorphisms

$$\underline{\Phi}_\alpha \equiv \varsigma_\alpha^*\Phi \;:\; \underline{\mathsf{P}}_{1\,\alpha} \equiv \mathcal{U}_\alpha\, {}_{\varsigma_\alpha}\!\times_{\pi_{\mathsf{P}_1}} \mathsf{P}_1 \xrightarrow{{\rm id}_{\mathcal{U}_\alpha}\times\Phi} \mathcal{U}_\alpha\, {}_{\varsigma_\alpha}\!\times_{\pi_{\mathsf{P}_2}} \mathsf{P}_2 \equiv \underline{\mathsf{P}}_{2\,\alpha}\,.$$

These compose a connection-preserving isomorphism

$$\bigsqcup_{\alpha\in\mathscr{A}} \underline{\Phi}_\alpha \;:\; \underline{\mathsf{P}}_{1\,\sqcup} \xrightarrow{\cong} \underline{\mathsf{P}}_{2\,\sqcup}$$

and satisfy the identity expressed by the commutative diagram

$$\begin{array}{ccc} \underline{\mathsf{P}}_{1\,\alpha}\restriction_{\mathcal{U}_{\alpha\beta}} & \xrightarrow{\quad \chi_{1\,\alpha,\beta} \quad} & \underline{\mathsf{P}}_{1\,\beta}\restriction_{\mathcal{U}_{\alpha\beta}} \\ {\scriptstyle \underline{\Phi}_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \underline{\Phi}_\beta} \\ \underline{\mathsf{P}}_{2\,\alpha}\restriction_{\mathcal{U}_{\alpha\beta}} & \xrightarrow[\quad \chi_{2\,\alpha,\beta} \quad]{} & \underline{\mathsf{P}}_{2\,\beta}\restriction_{\mathcal{U}_{\alpha\beta}} \end{array} .$$

Indeed, we compute, for $x \in \mathcal{U}_{\alpha\beta}$ and $p \in (\varpi \circ \pi_{\mathsf{P}_1})^{-1}(\{x\})$ and using Eq. (3.10),

$$\begin{aligned} &\underline{\Phi}_\beta \circ \chi_{1\,\alpha,\beta}(x, p) = \underline{\Phi}_\beta\big(x, \big(\varsigma_\alpha(x), \varsigma_\beta(x), H_1\big(\varsigma_\alpha(x), \varsigma_\beta(x), p\big)\big)\big) \\ &= \big(x, \big(\varsigma_\alpha(x), \varsigma_\beta(x), \Phi \circ H_1\big(\varsigma_\alpha(x), \varsigma_\beta(x), p\big)\big)\big) = \big(x, \big(\varsigma_\alpha(x), \varsigma_\beta(x), H_2\big(\varsigma_\alpha(x), \varsigma_\beta(x), \Phi(p)\big)\big)\big) \end{aligned}$$

$$\equiv \quad \chi_{2\,\alpha,\beta}\big(x,\Phi(p)\big) \equiv \chi_{2\,\alpha,\beta}\circ\underline{\Phi}_\alpha(x,p)\,.$$

Consequently, we may define the descended connection-preserving isomorphism

$$\underline{\Phi}^{\chi_1,\chi_2}\ :\ \underline{\mathsf{P}}_1^{\chi_1}\xrightarrow{\cong}\underline{\mathsf{P}}_2^{\chi_2}\ :\ [(x,p,\alpha)]\longmapsto\big[\big(x,\Phi(p),\alpha\big)\big]\,.$$

The definition makes sense as for any other representative of the argument class we obtain

$$\underline{\Phi}^{\chi_1,\chi_2}\big(\big[\big(x,H_1\big(\varsigma_\beta(x),\varsigma_\alpha(x),p\big),\beta\big)\big]\big)=\big[\big(x,\Phi\circ H_1\big(\varsigma_\beta(x),\varsigma_\alpha(x),p\big),\beta\big)\big]$$

$$=\quad\big[\big(x,H_2\big(\varsigma_\beta(x),\varsigma_\alpha(x),\Phi(p)\big),\beta\big)\big]=\big[\big(x,\Phi(p),\alpha\big)\big]\equiv\underline{\Phi}^{\chi_1,\chi_2}\big([(x,p,\alpha)]\big)\,.$$

Clearly, we have, for any $\mathcal{P}$ as above,

$$\underline{{\rm id}_{\mathsf{P}}}^{\chi,\chi}={\rm id}_{\underline{\mathsf{P}}^\chi}$$

and, for any $(\Phi_A,{\rm id}_M)\in_{\mathbb{C}^\times-\mathfrak{BunDes}^\nabla(\varpi)}(\mathcal{P}_A,\mathcal{P}_{A+1}),\ A\in\{1,2\}$

$$\underline{\Phi_2\circ\Phi_1}^{\chi_1,\chi_3}=\underline{\Phi}_2^{\chi_2,\chi_3}\circ\underline{\Phi}_1^{\chi_1,\chi_2}\,.$$

Thus, altogether, we end up with a covariant functor

$${\rm Desc}\ :\ \mathbb{C}^\times-\mathfrak{BunDes}^\nabla(\varpi)\longrightarrow\mathbb{C}^\times-\mathfrak{Bun}^\nabla(X;{\rm id}_X)$$

with the object component

$${\rm Desc}\restriction_{{\rm Ob}\,\mathbb{C}^\times-\mathfrak{BunDes}^\nabla(\varpi)}\ :\ (\mathcal{P},\chi)\longmapsto\underline{\mathcal{P}}^\chi$$

and the morphism component

$${\rm Desc}\restriction_{{\rm Mor}\,\mathbb{C}^\times-\mathfrak{BunDes}^\nabla(\varpi)}\ :\ \Big(\ (\mathcal{P}_1,\chi_1)\xrightarrow{(\Phi,{\rm id}_M)}(\mathcal{P}_2,\chi_2)\ \Big)\longmapsto\Big(\ \underline{\mathcal{P}}_1^{\chi_1}\xrightarrow{(\underline{\Phi}^{\chi_1,\chi_2},{\rm id}_X)}\underline{\mathcal{P}}_2^{\chi_2}\ \Big)\,.$$

Upon pulling back the descended isomorphisms to the total space of the surjective submersion,

$$\widehat{\varpi}^*\big(\underline{\Phi}^{\chi_1,\chi_2},{\rm id}_X\big)\ :\ \widehat{\varpi}^*\underline{\mathcal{P}}_1^{\chi_1}\xrightarrow{\cong}\widehat{\varpi}^*\underline{\mathcal{P}}_2^{\chi_2}\,,$$

we may, next, ask the natural question as to the commutativity of the diagram

$$\begin{array}{ccc}\mathsf{P}_1 & \xrightarrow{\ \ \Phi\ \ } & \mathsf{P}_2\\ {\scriptstyle\eta_{(\mathcal{P}_1,\chi_1)}}\big\downarrow & & \big\downarrow{\scriptstyle\eta_{(\mathcal{P}_2,\chi_2)}}\\ \varpi^*\underline{\mathsf{P}}_1^{\chi_1} & \xrightarrow[\varpi^*\underline{\Phi}^{\chi_1,\chi_2}]{} & \varpi^*\underline{\mathsf{P}}_2^{\chi_2}\end{array}\,.$$

This can be checked in a direct computation,

$$\varpi^*\underline{\Phi}^{\chi_1,\chi_2}\circ\eta_{(\mathcal{P}_1,\chi_1)}(p)=\varpi^*\underline{\Phi}^{\chi_1,\chi_2}\big(\pi_{\mathsf{P}_1}(p),\big[\big(\varpi\circ\pi_{\mathsf{P}_1}(p),H_1\big(\pi_{\mathsf{P}_1}(p),\varsigma_\alpha\circ\varpi\circ\pi_{\mathsf{P}_1}(p),p\big),\alpha\big)\big]\big)$$

$$=\quad\big(\pi_{\mathsf{P}_1}(p),\big[\big(\varpi\circ\pi_{\mathsf{P}_1}(p),\Phi\circ H_1\big(\pi_{\mathsf{P}_1}(p),\varsigma_\alpha\circ\varpi\circ\pi_{\mathsf{P}_1}(p),p\big),\alpha\big)\big]\big)$$

$$=\quad\big(\pi_{\mathsf{P}_1}(p),\big[\big(\varpi\circ\pi_{\mathsf{P}_1}(p),H_2\big(\pi_{\mathsf{P}_1}(p),\varsigma_\alpha\circ\varpi\circ\pi_{\mathsf{P}_1}(p),\Phi(p)\big),\alpha\big)\big]\big)$$

$$=\quad\big(\pi_{\mathsf{P}_2}\circ\Phi(p),\big[\big(\varpi\circ\pi_{\mathsf{P}_2}\circ\Phi(p),H_2\big(\pi_{\mathsf{P}_2}\circ\Phi(p),\varsigma_\alpha\circ\varpi\circ\pi_{\mathsf{P}_2}\circ\Phi(p),\Phi(p)\big),\alpha\big)\big]\big)$$

$$\equiv\quad\eta_{(\mathcal{P}_2,\chi_2)}\circ\Phi(p)\,,$$

carried out for an arbitrary $p\in(\varpi\circ\pi_{\mathsf{P}_1})^{-1}(\mathcal{U}_\alpha)$. All in all, the $\eta_{(\mathcal{P},\chi)}$ compose a natural isomorphism

$$\eta_\cdot\equiv\{\eta_{(\mathcal{P},\chi)}\}_{(\mathcal{P},\chi)\in{\rm Ob}\,\mathbb{C}^\times-\mathfrak{BunDes}^\nabla(\varpi)}\ :\ {\rm Id}_{\mathbb{C}^\times-\mathfrak{BunDes}^\nabla(\varpi)}\xRightarrow{\cong}\widehat{\varpi}^*\circ{\rm Desc}\,.$$

We complete the proof by showing that Desc is also a *left* functorial inverse of $\widehat{\varpi}^*$.

Given a principal $\mathbb{C}^\times$-bundle $\underline{\mathcal{P}}\equiv(\underline{\mathsf{P}},X,\pi_{\underline{\mathsf{P}}},\mathbb{C}^\times,\underline{\mathcal{A}})$, we consider the pullback descendable bundle

$$\widehat{\varpi}^*\underline{\mathcal{P}}=\big(\big(\varpi^*\underline{\mathsf{P}}\equiv M\,{}_\varpi\times_{\pi_{\underline{\mathsf{P}}}}\underline{\mathsf{P}},M,{\rm pr}_1,\mathbb{C}^\times,{\rm pr}_2^*\underline{\mathcal{A}}\big),{\rm id}_{(\varpi\circ{\rm pr}_1)^*\underline{\mathsf{P}}}\big)\,,$$

and subsequently identify

$$\underline{\big(\varpi^*\underline{\mathsf{P}}\big)_\alpha}\equiv\mathcal{U}_\alpha\,{}_{\varsigma_\alpha}\times_{{\rm pr}_1}\big(M\,{}_\varpi\times_{\pi_{\underline{\mathsf{P}}}}\underline{\mathsf{P}}\big)\equiv\mathcal{U}_\alpha\,{}_{\varsigma_\alpha}\times_{\pi_{\underline{\mathsf{P}}}}\underline{\mathsf{P}}\equiv\underline{\mathsf{P}}\restriction_{\mathcal{U}_\alpha}\,,$$

so that

$$\chi_{\alpha,\beta} \equiv \mathrm{id}_{\underline{\mathsf{P}}\restriction_{\mathcal{U}_{\alpha,\beta}}}\,,$$

and, accordingly,

$$\underline{\big(\varpi^*\underline{\mathsf{P}}\big)}^{\mathrm{id}_{(\varpi\circ\mathrm{pr}_1)^*\underline{\mathsf{P}}}} \equiv \underline{\mathsf{P}}\,,\qquad \underline{\big(\mathrm{pr}_2^*\underline{\mathcal{A}}\big)}^{\mathrm{id}_{(\varpi\circ\mathrm{pr}_1)^*\underline{\mathsf{P}}}} \equiv \underline{\mathcal{A}}\,,$$

or, quite simply,

$$\mathrm{Desc}\circ\widehat{\varpi}^*(\underline{\mathcal{P}}) \equiv \underline{\mathcal{P}}\,.$$

Thus, we obtain the (trivial) connection-preserving isomorphism

$$\vartheta_{\underline{\mathcal{P}}} \equiv \mathrm{id}_{\underline{\mathcal{P}}}\ :\ \mathrm{Id}_{\mathbb{C}^\times-\mathfrak{Bun}^\nabla(X;\mathrm{id}_X)}(\underline{\mathcal{P}}) \xrightarrow{\cong} \mathrm{Desc}\circ\widehat{\varpi}^*(\underline{\mathcal{P}})\,.$$

Passing to (iso)morphisms,

$$(\underline{\Phi},\mathrm{id}_X)\ :\ \underline{\mathcal{P}}_1 \xrightarrow{\cong} \underline{\mathcal{P}}_2\,,$$

we find, under the above identifications,

$$\underline{\big(\widehat{\varpi}^*(\underline{\Phi},\mathrm{id}_X)\big)}^{\mathrm{id}_{(\varpi\circ\mathrm{pr}_1)^*\underline{\mathsf{P}}},\mathrm{id}_{(\varpi\circ\mathrm{pr}_1)^*\underline{\mathsf{P}}}} \equiv (\underline{\Phi},\mathrm{id}_X)\,,$$

and so

$$\begin{aligned}\mathrm{Desc}\circ\widehat{\varpi}^*(\underline{\Phi},\mathrm{id}_X)\big)\circ\vartheta_{\underline{\mathcal{P}}_1} &\equiv (\underline{\Phi},\mathrm{id}_X)\circ\mathrm{id}_{\underline{\mathcal{P}}_1} \equiv (\underline{\Phi},\mathrm{id}_X) \equiv \mathrm{id}_{\underline{\mathcal{P}}_2}\circ(\underline{\Phi},\mathrm{id}_X)\\ &\equiv \vartheta_{\underline{\mathcal{P}}_2}\circ\mathrm{Id}_{\mathbb{C}^\times-\mathfrak{Bun}^\nabla(X;\mathrm{id}_X)}(\underline{\Phi},\mathrm{id}_X)\,.\end{aligned}$$

We conclude that the $\vartheta_{\underline{\mathcal{P}}}$ make up a natural isomorphism

$$\vartheta_\cdot \equiv \{\vartheta_{\underline{\mathcal{P}}}\}_{\underline{\mathcal{P}}\in\mathrm{Ob}\,\mathbb{C}^\times-\mathfrak{Bun}^\nabla(X;\mathrm{id}_X)}\ :\ \mathrm{Id}_{\mathbb{C}^\times-\mathfrak{Bun}^\nabla(X;\mathrm{id}_X)} \xRightarrow{\cong} \mathrm{Desc}\circ\widehat{\varpi}^*\,.$$

□

4. Descent through equivariance

Now that we have derived a perfectly hands-on answer to Question 4, we may return to the original Question 1 and extract an answer to the latter from a specialisation of the former. Thus, we enquire

Question 1': *Under what circumstances is a principal $\mathbb{C}^\times$-bundle with a compatible connection $(\mathsf{P},M,\pi_{\mathsf{P}},\mathbb{C}^\times,\mathcal{A})$ over the total space M of the surjective submersion $\pi_{M/\mathrm{G}}\ :\ M\longrightarrow M/\mathrm{G}$ isomorphic to the pullback along $\pi_{M/\mathrm{G}}$ of a principal $\mathbb{C}^\times$-bundle with a compatible connection over the quotient manifold M/G (whenever the latter exists)?*

The answer can be read off from Thm. 7: It is quantified (and the question itself is made precise) by the equivalence of categories

$$\widehat{\pi}^*_{M/\mathrm{G}}\ :\ \mathbb{C}^\times-\mathfrak{Bun}^\nabla(M/\mathrm{G};\mathrm{id}_{M/\mathrm{G}}) \xrightarrow{\cong} \mathbb{C}^\times-\mathfrak{BunDes}^\nabla(\pi_{M/\mathrm{G}})\,,$$

which, however, we want to massage into an equivalent form based on a 'more natural' description of the nerve $M(\pi_{M/\mathrm{G}})$ of the projection to the orbispace. To this end, note the existence of diffeomorphisms

$$\begin{aligned}\varepsilon^{(n)}\ &:\ \mathrm{G}^{\times n}\times M \xrightarrow{\cong} M(\pi_{M/\mathrm{G}})^{[n]}\,,\qquad n\in\mathbb{N}^\times\\ &:\ (g_n,g_{n-1},\ldots,g_1,m)\longmapsto\big((m,g_1\rhd m),(g_1\rhd m,g_{2:1}\rhd m),\ldots,(g_{n-1:1}\rhd m,g_{n:1}\rhd m)\big)\,,\end{aligned}$$

written in the shorthand notation

$$g_{k:1}\equiv g_k\cdot g_{k-1}\cdot\dots\cdot g_1\,,$$

which together with the identity map $\varepsilon^{(0)}\equiv\mathrm{id}_M$ compose a simplicial diffeomorphism

$$\varepsilon^{(\bullet)}\ :\ \mathsf{N}^{(\bullet)}\big(\mathrm{G}\ltimes_\lambda M\big) \xrightarrow{\cong} \mathsf{N}^{(\bullet)}\big(\mathrm{Pair}_{\pi_{M/\mathrm{G}}}(M)\big)$$

based on the equivalence of Lie groupoids

$$\big(\varepsilon^{(0)},\varepsilon^{(1)}\big)\ :\ \mathrm{G}\ltimes_\lambda M \xrightarrow{\cong} \mathrm{Pair}_{\pi_{M/\mathrm{G}}}(M)$$

between the **action groupoid**

$$\mathrm{G}\ltimes_\lambda M \equiv \big(M,\mathrm{G}\times M,s=\mathrm{pr}_2,t=\lambda,\mathrm{Id}_\cdot=(e,\cdot),\circ=\big(\mathrm{m}_{\mathrm{G}}\circ\mathrm{pr}_{1,3},\mathrm{pr}_4\big),\mathrm{Inv}=\big(\mathrm{Inv}_{\mathrm{G}}\circ\mathrm{pr}_1,\lambda\big)\big)$$

and the $\pi_{M/\mathrm{G}}$-fibred pair groupoid of Def. 1. Invertibility of the latter functor,

$$\left(\varepsilon^{(0)},\varepsilon^{(1)}\right)^{-1}\equiv\left(\varepsilon^{(0)\,-1},\varepsilon^{(1)\,-1}\right),$$

immediately yields an equivalent solution to our original problem: It suffices to 'pull back' the descent category $\mathbb{C}^\times\text{-}\mathfrak{BunDes}^\nabla(\pi_{M/\mathrm{G}})$ from $\mathsf{N}^{(\bullet)}(\mathrm{Pair}_{\pi_{M/\mathrm{G}}}(M))$ to $\mathsf{N}^{(\bullet)}(\mathrm{G}\ltimes_\lambda M)$ along the diffeomorphism $\varepsilon^{(\bullet)}$. The intuition is made precise and concretised in

Theorem 8. *Adopt the hitherto notation, let ϱ be a 1-form on $\mathrm{G}\times M$ satisfying the identity*

$$\Delta^1_{(2)}\varrho\equiv\left(\mathrm{pr}_{2,3}^*+(\mathrm{id}_\mathrm{G}\times\lambda)^*-(\mathrm{m}_\mathrm{G}\times\mathrm{id}_M)^*\right)\varrho=0 \tag{4.1}$$

over $\mathrm{G}^{\times 2}\times M$, and let $\mathbb{C}^\times\text{-}\mathfrak{Bun}^\nabla(M)^\mathrm{G}_\varrho$ be the **category of (G,ϱ)-equivariant principal $\mathbb{C}^\times$-bundles with a compatible connection over M**, *composed of*

- *the object class with elements, termed* (G,ϱ)**-equivariant principal $\mathbb{C}^\times$-bundles with a compatible connection**, *given by simplicial principal $\mathbb{C}^\times$-bundles with a compatible connection $\mathcal{P}\equiv((\mathsf{P},M,\pi_\mathsf{P},\mathbb{C}^\times,\mathcal{A}),\gamma)$ over $\mathsf{N}^{(\bullet)}(\mathrm{G}\ltimes_\lambda M)$, i.e., pairs made up of a principal $\mathbb{C}^\times$-bundles over $\mathsf{N}^{(0)}(\mathrm{G}\ltimes_\lambda M)\equiv M$,*

$$\begin{array}{ccc}\mathbb{C}^\times & \rightsquigarrow & \mathsf{P}\\ & & \big\downarrow{\scriptstyle \pi_\mathsf{P}}\\ & & M\end{array},$$

with a principal $\mathbb{C}^\times$-connection $\mathcal{A}\in\Omega^1(\mathsf{P})$, and of a connection-preserving isomorphism

$$\begin{array}{ccc}\lambda^*\mathsf{P}\equiv(\mathrm{G}\times M)\,{}_\lambda{\times}_{\pi_\mathsf{P}}\,\mathsf{P} & \xrightarrow{\ \gamma\ } & (\mathrm{G}\times M)\,{}_{\mathrm{pr}_2}{\times}_{\pi_\mathsf{P}}\,\mathsf{P}\equiv\mathrm{pr}_2^*\mathsf{P}\otimes\mathcal{I}_\varrho\\ \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \mathrm{pr}_1}\\ \mathrm{G}\times M & \underset{\mathrm{id}_{\mathrm{G}\times M}}{=\!=\!=} & \mathrm{G}\times M\end{array},$$

of principal $\mathbb{C}^\times$-bundles over $\mathsf{N}^{(1)}(\mathrm{G}\ltimes_\lambda M)\equiv\mathrm{G}\times M$ ($\mathcal{I}_\varrho$ is the trivial bundle of the said type equipped with the connection 1-form $\mathrm{pr}_2^*\vartheta+\varrho$, where $\vartheta\in\Omega^1(\mathbb{C}^\times)$ is the (left-)invariant Maurer–Cartan 1-form on the Lie group $\mathbb{C}^\times$), subject to the coherence constraint over $\mathsf{N}^{(2)}(\mathrm{G}\ltimes_\lambda M)\equiv\mathrm{G}^{\times 2}\times M$ expressed by the commutative diagram

$$\begin{array}{ccccc}\mathrm{pr}_{2,3}^*\lambda^*\mathsf{P} & \xrightarrow{\ \mathrm{pr}_{2,3}^*\gamma\ } & \mathrm{pr}_{2,3}^*\mathrm{pr}_2^*\mathsf{P}\otimes\mathcal{I}_{\mathrm{pr}_{2,3}^*\varrho} & =\!=\!= & (\mathrm{m}_\mathrm{G}\times\mathrm{id}_M)^*\mathrm{pr}_2^*\mathsf{P}\otimes\mathcal{I}_{(\mathrm{m}_\mathrm{G}\times\mathrm{id}_M)^*\varrho}\otimes\mathcal{I}_{(\mathrm{pr}_{2,3}^*-(\mathrm{m}_\mathrm{G}\times\mathrm{id}_M)^*)\varrho}\\ \Vert & & & & \big\uparrow{\scriptstyle (\mathrm{m}_\mathrm{G}\times\mathrm{id}_M)^*\gamma\otimes\mathrm{id}}\\ (\mathrm{id}_\mathrm{G}\times\lambda)^*\mathrm{pr}_2^*\mathsf{P}\otimes\mathcal{I}_{\Delta^1_{(2)}\varrho} & \xleftarrow[(\mathrm{id}_\mathrm{G}\times\lambda)^*\gamma\otimes\mathrm{id}]{} & (\mathrm{id}_\mathrm{G}\times\lambda)^*\lambda^*\mathsf{P}\otimes\mathcal{I}_{(\mathrm{pr}_{2,3}^*-(\mathrm{m}_\mathrm{G}\times\mathrm{id}_M)^*)\varrho} & =\!=\!= & (\mathrm{m}_\mathrm{G}\times\mathrm{id}_M)^*\lambda^*\mathsf{P}\otimes\mathcal{I}_{(\mathrm{pr}_{2,3}^*-(\mathrm{m}_\mathrm{G}\times\mathrm{id}_M)^*)\varrho}\end{array}; \tag{4.2}$$

- *for any pair $\mathcal{P}_K\equiv((\mathsf{P}_K,M,\pi_{\mathsf{P}_K},\mathbb{C}^\times,\mathcal{A}_K),\gamma_K)$, $K\in\{1,2\}$ of objects, a morphism class*

$$\mathrm{Hom}_{\mathbb{C}^\times\text{-}\mathfrak{Bun}^\nabla(M)^\mathrm{G}_\varrho}(\mathcal{P}_1,\mathcal{P}_2)$$

*with elements, termed (***connection-preserving***)* (G,ϱ)**-equivariant** *(***principal $\mathbb{C}^\times$-bundle***)* **morphisms**, *given by connection-preserving isomorphisms*

$$(\Phi,\mathrm{id}_M)\quad:\quad\begin{array}{ccc}\mathsf{P}_1 & \xrightarrow{\ \Phi\ } & \mathsf{P}_2\\ \big\downarrow{\scriptstyle \pi_{\mathsf{P}_1}} & & \big\downarrow{\scriptstyle \pi_{\mathsf{P}_2}}\\ M & \underset{\mathrm{id}_M}{=\!=\!=} & M\end{array}$$

of principal $\mathbb{C}^\times$-bundles over M, subject to the coherence constraint over $\mathrm{G}\times M$ expressed by the commutative diagram

$$\begin{array}{ccc} \lambda^*\mathsf{P}_1 & \xrightarrow{\ \gamma_1\ } & \mathrm{pr}_2^*\mathsf{P}_1\otimes\mathcal{I}_\varrho \\ {\scriptstyle \lambda^*\Phi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_2^*\Phi\otimes\mathrm{id}} \\ \lambda^*\mathsf{P}_2 & \xrightarrow[\ \gamma_2\ ]{} & \mathrm{pr}_2^*\mathsf{P}_2\otimes\mathcal{I}_\varrho \end{array} \ . \tag{4.3}$$

The equivalence $(\varepsilon^{(0)},\varepsilon^{(1)})$ induces an equivalence of categories

$$\mathbb{C}^\times\text{-}\mathfrak{Bun}^\nabla(M)_0^{\mathrm{G}}\cong\mathbb{C}^\times\text{-}\mathfrak{Bun}\mathfrak{Des}^\nabla(\pi_{M/\mathrm{G}})\,.$$

Proof. First of all, note that $\varrho=0$ automatically satisfies condition (4.1), and so it makes sense to consider this possibility, which leads us to look for principal $\mathbb{C}^\times$-bundles endowed with connection-preserving isomorphisms

$$\begin{array}{ccc} \lambda^*\mathsf{P} & \xrightarrow{\ \gamma\ } & \mathrm{pr}_2^*\mathsf{P} \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathrm{G}\times M & \overset{}{\underset{\mathrm{id}_{\mathrm{G}\times M}}{=\!=\!=\!=}} & \mathrm{G}\times M \end{array}$$

subject to the coherence condition

$$\begin{array}{ccccc} \mathrm{pr}_{2,3}^*\lambda^*\mathsf{P} & \xrightarrow{\ \mathrm{pr}_{2,3}^*\gamma\ } & \mathrm{pr}_{2,3}^*\mathrm{pr}_2^*\mathsf{P} & =\!=\!=\!= & (\mathrm{m}_{\mathrm{G}}\times\mathrm{id}_M)^*\mathrm{pr}_2^*\mathsf{P} \\ \Big\| & & & & \big\uparrow{\scriptstyle (\mathrm{m}_{\mathrm{G}}\times\mathrm{id}_M)^*\gamma} \\ (\mathrm{id}_{\mathrm{G}}\times\lambda)^*\mathrm{pr}_2^*\mathsf{P} & \xleftarrow[\ (\mathrm{id}_{\mathrm{G}}\times\lambda)^*\gamma\ ]{} & (\mathrm{id}_{\mathrm{G}}\times\lambda)^*\lambda^*\mathsf{P} & =\!=\!=\!= & (\mathrm{m}_{\mathrm{G}}\times\mathrm{id}_M)^*\lambda^*\mathsf{P} \end{array}\ ,$$

and, for any pair thereof, connection-preserving isomorphisms between them subject to the coherence condition

$$\begin{array}{ccc} \lambda^*\mathsf{P}_1 & \xrightarrow{\ \gamma_1\ } & \mathrm{pr}_2^*\mathsf{P}_1 \\ {\scriptstyle \lambda^*\Phi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_2^*\Phi} \\ \lambda^*\mathsf{P}_2 & \xrightarrow[\ \gamma_2\ ]{} & \mathrm{pr}_2^*\mathsf{P}_2 \end{array}$$

among pullbacks of objects of the descent category along the diffeomorphism $\varepsilon^{(1)}$ and – respectively – pullbacks of morphisms of that category along the diffeomorphism $\varepsilon^{(0)}$. But under the former pullback, Diag. (3.1) is readily seen to transform into

$$\begin{array}{ccc} \varepsilon^{(1)\,*}d_1^{(1)\,*}\mathsf{P}\equiv\mathrm{pr}_2^*\mathsf{P} & \xrightarrow{\ \varepsilon^{(1)\,*}\chi\ } & \lambda^*\mathsf{P}\equiv\varepsilon^{(1)\,*}d_0^{(1)\,*}\mathsf{P} \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathrm{G}\times M & \underset{\mathrm{id}_{\mathrm{G}\times M}}{=\!=\!=\!=} & \mathrm{G}\times M \end{array}\ ,$$

whereas the pullback of Diag. (3.2) along $\varepsilon^{(2)}$ takes the form

$$\begin{array}{ccccc}
\varepsilon^{(2)\,*}d_2^{(2)\,*}d_1^{(1)\,*}\mathsf{P}\equiv\mathrm{pr}_{2,3}^*\mathrm{pr}_2^*\mathsf{P} & \xrightarrow{\varepsilon^{(2)\,*}d_2^{(2)\,*}\chi\equiv\mathrm{pr}_{2,3}^*\varepsilon^{(1)\,*}\chi} & \varepsilon^{(2)\,*}d_2^{(2)\,*}d_0^{(1)\,*}\mathsf{P}\equiv\mathrm{pr}_{2,3}^*\lambda^*\mathsf{P} & = & \varepsilon^{(2)\,*}d_0^{(2)\,*}d_1^{(1)\,*}\mathsf{P}\equiv(\mathrm{id}_{\mathrm{G}}\times\lambda)^*\varepsilon^{(1)\,*}\mathrm{pr}_2^*\mathsf{P} \\
\| & & & & \big\downarrow{\scriptstyle \varepsilon^{(2)\,*}d_0^{(2)\,*}\chi\equiv(\mathrm{id}_{\mathrm{G}}\times\lambda)^*\varepsilon^{(1)\,*}\chi} \\
\varepsilon^{(2)\,*}d_1^{(2)\,*}d_1^{(1)\,*}\mathsf{P}\equiv(\mathrm{m}_{\mathrm{G}}\times\mathrm{id}_M)^*\mathrm{pr}_2^*\mathsf{P} & \xrightarrow[\varepsilon^{(2)\,*}d_1^{(2)\,*}\gamma\equiv(\mathrm{m}_{\mathrm{G}}\times\mathrm{id}_M)^*\varepsilon^{(1)\,*}\gamma]{} & \varepsilon^{(2)\,*}d_1^{(2)\,*}d_0^{(1)\,*}\mathsf{P}\equiv(\mathrm{m}_{\mathrm{G}}\times\mathrm{id}_M)^*\lambda^*\mathsf{P} & = & \varepsilon^{(2)\,*}d_0^{(2)\,*}d_0^{(1)\,*}\mathsf{P}\equiv(\mathrm{id}_{\mathrm{G}}\times\lambda)^*\lambda^*\mathsf{P}
\end{array},$$

so that we are led to the *i*nvertible postulate

$$\varepsilon^{(1)\,*}\chi=\gamma^{-1}.$$

The ultimate confirmation comes from inspection of the pullback of Diag. (3.3) along $\varepsilon^{(1)}$,

$$\begin{array}{ccc}
\varepsilon^{(1)\,*}d_1^{(1)\,*}\mathsf{P}_1\equiv\mathrm{pr}_2^*\mathsf{P}_1 & \xrightarrow{\varepsilon^{(1)\,*}\chi_1} & \varepsilon^{(1)\,*}d_0^{(1)\,*}\mathsf{P}_1\equiv\lambda^*\mathsf{P}_1 \\
{\scriptstyle \varepsilon^{(1)\,*}d_1^{(1)\,*}\Phi\equiv\mathrm{pr}_2^*\Phi}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon^{(1)\,*}d_0^{(1)\,*}\Phi\equiv\lambda^*\Phi} \\
\varepsilon^{(1)\,*}d_1^{(1)\,*}\mathsf{P}_2\equiv\mathrm{pr}_2^*\mathsf{P}_2 & \xrightarrow[\varepsilon^{(1)\,*}\chi_2]{} & \varepsilon^{(1)\,*}d_0^{(1)\,*}\mathsf{P}_2\equiv\lambda^*\mathsf{P}_2
\end{array},$$

which completes the proof. □

Putting the two main results of the present exposition together, we arrive at the final answer to the question from the Introduction:

Corollary 9. *Under the previous assumptions, and in the hitherto notation, there exists a canonical equivalence of categories*

$$\mathbb{C}^\times\text{-}\mathfrak{Bun}^\nabla(M)_0^{\mathrm{G}}\cong\mathbb{C}^\times\text{-}\mathfrak{Bun}^\nabla(M/\mathrm{G};\mathrm{id}_{M/\mathrm{G}})\,.$$

Appendix A. Never lose your nerve

Some useful phraseology, lest the Avid Reader should lose it…

Definition 10. Let $\mathcal{C}$ be a category with the set of objects $\mathrm{Ob}(\mathcal{C})$. A **simplicial object** $(X_\bullet,d_\cdot^{(\bullet)},s_\cdot^{(\bullet)})$ **in** $\mathcal{C}$ is a collection of objects $X_n\in\mathrm{Ob}(\mathcal{C})$, $n\in\mathbb{N}$, together with distinguished morphisms: the **face maps** $d_i^{(n)}\in\mathrm{Hom}_{\mathcal{C}}(X_n,X_{n-1})$ and the **degeneracy maps** $s_i^{(n)}\in\mathrm{Hom}_{\mathcal{C}}(X_n,X_{n+1})$, defined for all $0\le i\le n$ and satisfying the **simplicial identities**:

$$\begin{aligned}
d_i^{(n-1)}\circ d_j^{(n)} &= d_{j-1}^{(n-1)}\circ d_i^{(n)}\,, \quad i<j\,,\\
s_i^{(n+1)}\circ s_j^{(n)} &= s_{j+1}^{(n+1)}\circ s_i^{(n)}\,, \quad i\le j\,,\\
d_i^{(n+1)}\circ s_j^{(n)} &= \begin{cases} s_{j-1}^{(n-1)}\circ d_i^{(n)} & \text{if}\quad i<j\,,\\ \mathrm{id}_{X_n} & \text{if}\quad i=j \text{ or } i=j+1\,,\\ s_j^{(n-1)}\circ d_{i-1}^{(n)} & \text{if}\quad i>j+1\,.\end{cases}
\end{aligned}$$

A simplicial object in the category **Set** (resp. **Top**, **(s)Man** *etc.*) is termed a **simplicial set** (resp. **space**, (**super**)**manifold** *etc.*).

⋄

A fundamental class of examples is provided by nerves of categories (*cf.* Ref. [Seg68]).

Definition 11. Let $\mathcal{C}$ be a small category with the set of objects $\mathrm{Ob}(\mathcal{C})$, the set of morphisms $\mathrm{Mor}(\mathcal{C})$ and structure maps $s:\mathrm{Mor}(\mathcal{C})\longrightarrow\mathrm{Ob}(\mathcal{C})$ (the source map) and $t:\mathrm{Mor}(\mathcal{C})\longrightarrow\mathrm{Ob}(\mathcal{C})$ (the target map), respectively[5]. The **nerve of** $\mathcal{C}$ is the simplicial set $\mathsf{N}_\bullet(\mathcal{C})$ with the following data: $\mathsf{N}_0(\mathcal{C})=\mathrm{Ob}(\mathcal{C})$ and, for $n\ge1$,

$$\mathsf{N}_n(\mathcal{C})=\{\ (f_1,f_2,\ldots,f_n)\in\mathrm{Mor}(\mathcal{C})^{\times n}\quad|\quad t(f_i)=s(f_{i+1})\ \}\,,$$

[5] One should not confuse the source map s with a degeneracy map $s_i^{(n)}$.

i.e., $\mathsf{N}_n(\mathcal{C})$ is the set of all n-tuples of composable morphisms (note that in this ordering convention f_i and f_{i+1} are composable if $f_{i+1}\circ f_i$ makes sense). The degeneracy maps are: $s_0(a) = \mathrm{id}_a$ for $a \in \mathrm{Ob}(\mathcal{C})$, and, for $n \geq 1$,

$$s_i^{(n)}(f_1, f_2, \ldots, f_n) = (f_1, f_2, \ldots, f_i, \mathrm{id}_{t(f_i)}, f_{i+1}, \ldots, f_n)\,.$$

The face maps are: $d_0(f) = t(f)$ and $d_1(f) = s(f)$ for $f \in \mathrm{Mor}(\mathcal{C})$, and, for $n \geq 2$,

$$d_i^{(n)}(f_1, f_2, \ldots, f_n) = \begin{cases} (f_2, f_3, \ldots, f_n) & \text{for} \quad i = 0\,, \\ (f_1, f_2, \ldots, f_{i+1}\circ f_i, \ldots, f_n) & \text{for} \quad 0 < i < n\,, \\ (f_1, f_2, \ldots, f_{n-1}) & \text{for} \quad i = n\,. \end{cases}\,,$$

◇

A natural context for physical applications of nerves and higher-geometric structures over them is provided by the study of (super-)σ-models of dynamics of extended distributions of (super-)charge in ambient geometries in the presence of defects (and so, in particular, symmetries, including the gauged ones, and more general dualities of the underlying field theories), *cf.*, *e.g.*, Refs. [GSW10, GSW13, Sus22].

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