TOPOLOGICAL ACTIONS IN TWO-DIMENSIONAL QUANTUM FIELD THEORIES *

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ABSTRACT

A systematic approach to 2-dimensional quantum field theories with topological terms in the action is developed using as a mathematical tool the Deligne cohomology. As an application, it is shown how to bosonize the action of free fermions of arbitrary spin on a Riemann surface and how to find the spectrum of the Wess-Zumino-Witten sigma models without recurrence to modular invariance.

LIST OF TOPICS

- Dirac's monopole : an example of a topological action in quantum mechanics.
- Line bundles with connections and Deligne cohomologies.
- Cohomology of topological ambiguities in two-dimensional field theory.
- Bosonized actions for free fermions on a Riemann surface.
- Canonical quantization of the Wess-Zumino-Witten chiral model.
- Spectrum of the Wess-Zumino-Witten model with SU(2) and SO(3) groups.

1. INTRODUCTION.

These lectures are devoted to one topological aspect of quantum theory from the plethora of such phenomena which constitute the main topic of this Institute. It appears in theories which lack globally defined actions. The simplest and longest known system of this sort describes a particle in the field of Dirac's magnetic monopole[1]. In Section 2, we shall briefly recall how the lack of global action leads to Dirac's quantization condition for the monopole charge and to other topological effects. This will be done in the spirit of the so-called geometric quantization[2] representing the quantum mechanical states as sections of a line bundle and will serve as a natural occasion to introduce cohomological notions known to

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mathematicians as Deligne cohomology to which Section 3 is devoted.

In 2-dimensional field theories an example of a system without global action is provided by the Wess-Zumino-Witten (WZW) model discussed by Witten in his paper[3] on non-abelian bosonization. Its study was the main motivation for our interest in models without global action. Another system in this category is the free (euclidean) field bosonizing fermions of arbitrary spin on a Riemann surface[4]. In Section 4, we shall discuss path integral quantization of 2-dimensional field theories without global actions on a general 2-dimensional surface using the Deligne cohomology as a mathematical tool. We shall see how the relation of path-integral and canonical quantizations gives rise to natural appearence of line-bundles with connection over the loop spaces of manifolds of field values and to a "stringy" generalization of the notion of parallel transport. In Section 5, the general theory is applied to the bosonized version of free fermionic theories on Riemann surfaces. Section 6 studies with details the WZW model with fields taking values in a simply connected group G . We show how rich symmetries of the classical models give rise to the left-right action of the Kac-Moody group in the space of states of the quantized model represented by sections of the line bundle over the loop group of G . This action decomposes into irreducible representations whose spectrum conjectured by Gepner-Witten in [5] is found explicitly for G = SU(2) . Section 7 treats the complications appearing in the case of non-simply connected groups on example of SO(3) model. We find again the result coinciding with the Gepner-Witten spectrum derived originally from the SU(2) one by postulating the modular invariance of the partition function of the model on the torus[5].

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2. TOPOLOGICAL ACTIONS IN MECHANICS.

We start by discussing the case of quantum mechanics where the topological ambiguities may appear when we attempt to realize Dirac's or Feynman's quantization programmes[6,7] in topologically non-trivial backgrounds. The geometric aspects of this well known phenomenon will be briefly recalled here. Classical mechanics is usually formulated with the help of Lagrangians, but what really enters the classical equations of motion are specific combinations of their derivatives only. To be more concrete, consider a particle moving in magnetic field described by vector potential $\vec{A}(\vec{x})$. The action on the trajectory $\vec{x}(t)$ is given as

$$\frac{1}{2}\, m \int \dot{\vec{x}}^{\,2} dt + e \int \vec{A}\cdot d\vec{x} \qquad (1)$$

whereas the equations of motion involve only magnetic induction $\vec{B}$

$$m\ddot{\vec{x}} = e\vec{B} \wedge \vec{v} \ . \tag{2}$$

In other words, the action contains integral of 1-form $\eta = e\vec{A}\cdot d\vec{x}$ along the trajectory, whereas the equations of motion involve only 2-form $\omega = d\eta = \frac{1}{2} e\vec{B}\cdot d\vec{x} \wedge d\vec{x}$.

It is possible to consider classical systems involving a 2-form which is closed but not exact. Such is for example the case for classical particle moving in the field of magnetic monopoles[1,8] and, to make the problem more interesting, infinitely thin Bohm-Aharonov[9] solenoids carrying magnetic fluxes. Away from the singularities at the monopole locations and flux lines, the magnetic induction defines 2-form

$$\omega = \frac{1}{2} e\vec{B}\cdot d\vec{x} \wedge d\vec{x} = \frac{1}{2} \sum_{\alpha} e\mu_\alpha \sum_{i,j,k=1}^{3} \varepsilon^{ijk} \frac{x^i - x^i_\alpha}{|\vec{x}-\vec{x}_\alpha|^3} dx^j dx^k \tag{3}$$

where μ_α are charges of monopoles located at points $\vec{x}_\alpha$. ω is no more exact (vector potentials for magnetic monopoles have singularities along Dirac strings).

Lack of global form η such that $\omega = d\eta$, although no complication on classical level, obstructs the passage to quantum mechanics, where, following Feynman, we should sum the probability amplitudes given as exponentials of (i times) the action over all the trajectories. In some cases however, a global action can still be defined for closed trajectories modulo an ambiguity in $2\pi\mathbb{Z}$ removed by exponentiation. This situation has a nice geometric interpretation : the (topological parts of) the probability amplitudes are the loop holonomies in a line bundle with a hermitian connection of curvature ω . For open trajectories, the parallel transport in the bundle leads to "probability amplitudes" which are no more complex numbers but rather linear maps between the line bundle fibers, but they can still be used to define complex valued probability amplitudes between states represented by sections of the line bundle. That the wave functions should be bundle valued can also be seen from the fact that an important subalgebra of classical physical quantities can be naturally represented by differential operators acting in the space of sections of the line bundle so that the Poisson bracket corresponds to (i times) the commutator, realizing Dirac's quantization programme. This is one of the main constructions of geometric quantization which provided an effective tool in the theory of unitary representations of groups (obtained by quantization of classical group actions carried by orbits of the coadjoint representation of the group[10]). Later, we shall develop some of the concepts of geometric quantization in the infinite-dimensional context of two-dimensional field theories. Here we shall present the elements of the line bundle theory[11] in a cohomological language, specially useful in view of the later discussion.

Suppose that we are given a hermitian line bundle L over a (smooth) manifold M. If $\{\mathcal{O}_\alpha\}$ is a sufficiently fine open covering of M then by taking normalized sections s_α of L over $\mathcal{O}_\alpha$, we obtain U(1)-valued transition functions on non-empty intersections $\mathcal{O}_{\alpha_0\alpha_1} \equiv \mathcal{O}_{\alpha_0} \cap \mathcal{O}_{\alpha_1}$: $s_{\alpha_0} = g_{\alpha_0\alpha_1} s_{\alpha_1}$. $(g_{\alpha_0\alpha_1})$ form a 2-cocycle, i.e.

$$g_{\alpha_0\alpha_1} = g_{\alpha_1\alpha_0}^{-1} \quad , \quad g_{\alpha_0\alpha_1} g_{\alpha_1\alpha_2} = g_{\alpha_0\alpha_2} \quad \text{on} \quad \mathcal{O}_{\alpha_0\alpha_1\alpha_2} \; . \tag{4}$$

A hermitian connection on L can be locally described by real 1-forms η_α on $\mathcal{O}_\alpha$ transforming by the gauge transformations

$$\eta_{\alpha_1} = \eta_{\alpha_0} + \frac{1}{i} g_{\alpha_0\alpha_1}^{-1} dg_{\alpha_0\alpha_1} \quad \text{on} \quad \mathcal{O}_{\alpha_0\alpha_1} \; . \tag{5}$$

The curvature of the connection corresponding to (η_α) is the real closed 2-form on M equal on each $\mathcal{O}_\alpha$ to $d\eta_\alpha$.

If we choose other sections $\tilde{s}_\alpha = \chi_\alpha^{-1} s_\alpha$ of L where χ_α are U(1)-valued functions on $\mathcal{O}_\alpha$ then we obtain equivalent local data $(\tilde{g}_{\alpha_0\alpha_1}, \tilde{\eta}_\alpha)$:

$$\begin{aligned} \tilde{g}_{\alpha_0\alpha_1} &= g_{\alpha_0\alpha_1} \chi_{\alpha_1} \chi_{\alpha_0}^{-1} \quad , \\ \tilde{\eta}_\alpha &= \eta_\alpha + \frac{1}{i} \chi_\alpha^{-1} d\chi_\alpha \quad . \end{aligned} \tag{6}$$

Conversely, the equivalence class $[(g_{\alpha_0\alpha_1}, \eta_\alpha)]$ of local presentations defines the line bundle with connection up to isomorphism (projecting to identity on M). In particular, L can be taken as the set of elements (α,x,z), $x \in \mathcal{O}_\alpha$, $z \in \mathbb{C}^1$, with the identification of (α_0,x,z) and $(\alpha_1,x,g_{\alpha_0\alpha_1}(x)z)$ if $x \in \mathcal{O}_{\alpha_0\alpha_1}$. If $\{\mathcal{O}_\alpha\}$ has the property that all $\mathcal{O}_{\alpha_0\ldots\alpha_p}$ are contractible, we may identify the isomorphism classes of the line bundles equipped with hermitian connections and classes $q = [(g_{\alpha_0\alpha_1}, \eta_\alpha)]$.

From the cohomological point of view, the latter are the elements of the cohomology of a homomorphism of sheaves[12] $\mathbb{F}_M^1$

$$U(1)_M \xrightarrow{\frac{1}{i} d \log} \Omega_M^1 \tag{7}$$

where $U(1)_M$ is the sheaf of local smooth U(1) valued functions on M and Ω_M^1 the sheaf of local smooth real 1-forms on M. For a general complex of sheaves of abelian groups

$$\mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{p-1}} \mathcal{F}_p \quad , \tag{8}$$

$d_i \circ d_{i-1} = 0$, and open covering $\{\mathcal{O}_\alpha\}$ of M consider groups $C_i^p \equiv C^p(\mathcal{F}_i)$ of Čech p-cochains with values in $\mathcal{F}_i$. Elements of C_i^p are families $c_i^p \equiv (c^p_{i,\alpha_0\ldots\alpha_p})$, $c^p_{i,\alpha_0\ldots\alpha_p} \in \mathcal{F}_i(\mathcal{O}_{\alpha_0\ldots\alpha_p})$, antisymmetric in indices $\alpha_0,\ldots,\alpha_p$. The Čech coboundary $\delta^p : C_i^p \to C_i^{p+1}$,

$$(\delta^p c_i^p)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \, c^p_{i,\alpha_0\ldots\hat{\alpha}_q\ldots\alpha_{p+1}} \quad . \tag{9}$$

From the bi-complex of abelian groups

$$\begin{array}{ccccc} & \downarrow & & \downarrow & \\ \longrightarrow & C_i^p & \xrightarrow{d_i} & C_{i+1}^p & \longrightarrow \\ & \delta^p \downarrow & & \downarrow & \\ \longrightarrow & C_i^{p+1} & \longrightarrow & C_{i+1}^{p+1} & \longrightarrow \\ & \downarrow & & \downarrow & \end{array} \tag{10}$$

we may derive the diagonal complex

$$\longrightarrow E_k \xrightarrow{D_k} E_{k+1} \longrightarrow \tag{11}$$

where

$$E_k = \bigoplus_{\substack{(i,p) \\ i+p=k}} C_i^p \tag{12}$$

and for $j+q = k+1$

$$(D_k((c_i^p)_{i+p=k}))_j^q = (-1)^j \, \delta^{q-1} c_j^{q-1} + (-1)^{j-1} d_{j-1} c_{j-1}^q \quad . \tag{13}$$

The cohomology groups of the complex $\mathbb{F}_M^1$ of sheaves (8),

$$\mathbb{H}^k(\mathbb{F}_M^1) := \frac{\ker D_k}{\operatorname{Im} D_{k-1}} \quad . \tag{14}$$

More exactly, they are given by the inductive limit of the right hand side of (14) over the finer and finer coverings of M , as usually in Čech cohomology. Notice that our local description of isomorphism classes of line bundles with hermitian connection on M identifies them with elements of $\mathbb{H}^1(\mathbb{F}_M^1)$.

Given a local presentation $(g_{\alpha_0\alpha_1}, \eta_\alpha)$ of the bundle and a loop

$\phi : S^1 \to M$ in M, we may express the holonomy $P(\phi) \in U(1)$ corresponding to ϕ (given by the parallel transport around ϕ) in the following way. We break S^1 into a union of intervals b with common points v so that $\phi(b) \subset \mathcal{O}_{\alpha_b}$, see Fig. 1. We also choose for each v an α_v s.t. $v \in \mathcal{O}_{\alpha_v}$. Then

$$P(\phi) = \exp[i \sum_b \int_b \phi^* \eta_{\alpha_b}] \prod_v g_{\alpha_{b_v^-} \alpha_{b_v^+}}(\phi(v))$$

$$= \exp[i \sum_b \int_b \phi^* \eta_{\alpha_b}] \prod_{\substack{v,b \\ v \in \partial b}} g^{-1}_{\alpha_v \alpha_b}(\phi(v)) \tag{15}$$

where on the right hand side the convention is implied which inverts $g^{-1}_{\alpha_v \alpha_b}(\phi(v))$ if v inherits from b negative orientation (i.e. v is the starting point of b). In cohomological language, ϕ allows to pull back any element $q \in \mathbb{H}^1(\mathbb{F}^1_M)$ to an element $\phi^* q \in \mathbb{H}^1(\mathbb{F}^1_{S^1}) \cong U(1)$ and $P(\phi)$ realizes the last isomorphism. Of course, the right hand side of (15) does not depend of the choice of triangulation of S^1, the assignments α_b, α_v or the representative $(g_{\alpha_0 \alpha_1}, \eta_\alpha)$ of q. This is no more true if we apply the same formula to $\phi : I \to M$ where I is a closed interval $[v_i, v_f]$. Then the right hand side of (15) depends on α_{v_i} and α_{v_f} and transforms under the change of this data as an element of $L^*_{\phi(v_i)} \otimes L_{\phi(v_f)}$ giving the parallel transport along ϕ in bundle L which maps fiber $L_{\phi(v_i)}$ into fiber $L_{\phi(v_f)}$. Thus eq. (16) describes the parallel transport in L along both closed and open curves. We should also notice, that although defined for parametrized curves, it does not change under orientation preserving reparametrizations of them.

The knowledge of the curvature of the connection fixes the holonomy of the contractible loops. More generally, if $\phi : \Sigma \to M$ where Σ is a 2-dimensional compact oriented manifold with the boundary composed of loops $(\partial\Sigma)_i$, see Fig.2, then

$$\prod_i P(\phi\restriction_{(\partial\Sigma)_i}) = \exp[i \int_\Sigma \phi^* \omega] \tag{16}$$

where ω is the curvature form.

Given a real closed 2-form ω on M, we may wonder whether there exists a line bundle with a hermitian connection of curvature ω. This is the case if and only if the integrals of ω over 2-dimensional closed surfaces in M are in $2\pi\mathbb{Z}$ (that this is a necessary condition is easily seen from eq. (16)). In the cohomological language this means that the element $[\omega] \in H^2(M, \mathbb{R})$ defined by ω should be in the image of $H^2(M, 2\pi\mathbb{Z})$. We shall call such ω integral. The cohomology group $H^1(M, U(1))$ acts freely on $\mathbb{H}^1(\mathbb{F}^1_M)$ sending $[(g_{\alpha_0\alpha_1}, \eta_\alpha)]$ into $[(\lambda_{\alpha_0\alpha_1} g_{\alpha_0\alpha_1}, \eta_\alpha)]$ for $(\lambda_{\alpha_0\alpha_1})$ being a cocycle defining an element of $H^1(M, U(1))$. The orbits of this action are exactly the sets $Q(M,\omega)$ of isomorphism classes of line bundles with hermitian connection of curvature ω. Thus $H^1(M, U(1))$ enumerates the elements of non-empty $Q(M,\omega)$.

Coming back to our example of the particle moving in the field of magnetic monopoles and flux lines, the integrality of the form ω given by eq. (3) is equivalent to Dirac's quantization condition of magnetic charge

$$e\mu_\alpha \in \frac{1}{2}\mathbb{Z} \quad . \tag{17}$$

If this condition is satisfied, the possible choices of line bundle with curvature ω differ by holonomies of loops surrounding flux lines. For small loops these holonomies become $e^{i\theta}$ where $\frac{1}{e}\theta$ are the magnetic fluxes carried by the lines. These holonomy differences are seen in the quantum probability amplitudes (θ vacua [13,14]) although they are ignored by classical trajectories of the particles which is the essence of the Bohm-Aharonov effect.

Another important topological effect appears in the relation between classical and quantum symmetries. If D is a diffeomorphism of M preserving ω, it is a symmetry of classical mechanics. It does not have to give rise to a symmetry of quantum mechanics. One of the obstructions may be that the pull-back of the line bundle L with connection of curvature ω by D can be non-isomorphic to L. If this is not the case, e.g. if $H^1(M, U(1)) = 1$, then D can be lifted to an isomorphism $\hat{D}$ of L (projecting to D) preserving the hermitian structure and the connection. For connected M, the lift is defined up to multiplication by elements of $U(1)$. If G is a group acting on M by diffeomorphisms preserving ω liftable to L, we obtain an action on L of a central extension $\hat{G}$ of G by $U(1)$. This action, in general, gives rise only to projective representations of G on the space of quantum mechanical states. The parallel transports $P(\phi)$ transform covariantly under liftable maps, $P(D\circ\phi) = \hat{D}P(\phi)$ (the right hand side which is an element of $L^*_{D(\phi(v_i))} \otimes L_{D(\phi(v_f))}$ does not depend of the choice of the lift $\hat{D}$ of D).

For the particle moving in the field of a single monopole of charge μ, the classical mechanics has $SO(3)$ as the symmetry group which lifts to an action on L only for integer μe. For half-integer μe, it lifts to the action of $SU(2)$ on L and gives rise to a projective representation of SO(3) on the space of quantum states : half-integer magnetic charges carry half-integer spins[15].

Similar discussion applies to infinitesimal action of Lie algebras on M and L.

In certain cases, manifold M can be extended to a complex manifold $M^{\mathbb{C}}$ and curvature form ω to a (2,0) holomorphic form $\omega^{\mathbb{C}}$ on $M^{\mathbb{C}}$. It may be convenient then to extend line bundle L with hermitian connection of curvature ω on M to a holomorphic line bundle $L^{\mathbb{C}}$ with holomorphic connection with curvature $\omega^{\mathbb{C}}$ and to consider quantum states extending to holomorphic sections of $L^{\mathbb{C}}$. Isomorphism classes of holomorphic bundles with holomorphic connections may be identified with elements of $\mathbb{H}^1(A^1_{M^{\mathbb{C}}})$ where $A^1_{M^{\mathbb{C}}}$ is the sheaf homomorphism

$$\mathcal{O}^*_{M^{\mathbb{C}}} \xrightarrow{\frac{1}{i}\, d\log} A^1_{M^{\mathbb{C}}} \tag{18}$$

with $\mathcal{O}^*_{M^{\mathbb{C}}}$ denoting the sheaf of local holomorphic nowhere vanishing functions on $M^{\mathbb{C}}$ and $A^p_{M^{\mathbb{C}}}$ the sheaf of local holomorphic (p,0) forms on $M^{\mathbb{C}}$. Groups $\mathbb{H}^1(A^1_{M^{\mathbb{C}}})$ are known as Deligne cohomology (of degree 2), see[16]. More explicitly, elements of $\mathbb{H}^1(A^1_{M^{\mathbb{C}}})$ are given by classes of $(g_{\alpha_0\alpha_1}, \eta_\alpha)$ modulo $(\chi_{\alpha_1}\chi_{\alpha_0}^{-1}, \frac{1}{i}\chi_\alpha^{-1}d\chi_\alpha)$ where $g_{\alpha_0\alpha_1}$ are holomorphic nowhere vanishing functions on $\mathcal{O}_{\alpha_0\alpha_1}$ satisfying (4), η_α are holomorphic (1,0) forms on χ_α satisfying (5) and χ_α are nowhere vanishing holomorphic functions on $\mathcal{O}_\alpha$. Parallel transport along path in $M^{\mathbb{C}}$ is defined again by eq. (15) (the holonomy takes values in $\mathbb{C}^*$ now). Let $Q^a(M^{\mathbb{C}},\omega^{\mathbb{C}})$ denote the set of $w \in \mathbb{H}^1(A^1_{M^{\mathbb{C}}})$ with curvature $\omega^{\mathbb{C}}$. Again, $Q^a(M^{\mathbb{C}},\omega^{\mathbb{C}}) \neq \emptyset$ if and only if $\omega^{\mathbb{C}}$ is integral and for such $\omega^{\mathbb{C}}$, $H^1(M,\mathbb{C}^*)$ acts freely and transitively on $Q^a(M^{\mathbb{C}},\omega^{\mathbb{C}})$.

The gain from extending M to $M^{\mathbb{C}}$ may be the richer symmetry group. For example, in the case of $M = \mathbb{R}^3 \setminus \{0\}$ and ω given by the magnetic induction of a monopole sitting at the origin, we may take $M^{\mathbb{C}} = \{\vec{y} \in \mathbb{C}^3 | \vec{y}^2 \notin]-\infty,0]\}$. This allows to represent SO(3,$\mathbb{C}$) in the space of holomorphic sections of $L^{\mathbb{C}}$. In Appendix 1, we give an explicit local representative of the holomorphic monopole line bundle $L^{\mathbb{C}}$ over $M^{\mathbb{C}}$ which restricted to $\mathbb{R}^3 \setminus \{0\}$ reproduces the monopole bundle with hermitian connection.

3. TOPOLOGICAL AMBIGUITIES IN 2-DIMENSIONAL FIELD THEORY.

In 2-dimensional field theory, closed 3-forms play similar role as closed 2-forms in mechanics. To consider classical equations of motion, we need to know the action locally and only up to an integral of the derivative of a 1 form. Such an information is available if we start from a closed 3-form γ and define (a part of) the local action by taking integrals of local prime 2-forms of γ. On the quantum level, however, one needs the global action to be well defined modulo $2\pi\mathbb{Z}$. To achieve this, we shall employ again the cohomology of sheaf complexes. Consider complex $\mathbb{F}_M^2$

$$U(1)_M \xrightarrow{\frac{1}{i}\,d\log} \Omega_M^1 \xrightarrow{d} \Omega_M^2 \tag{1}$$

and its second cohomology group $\mathbb{H}^2(\mathbb{F}_M^2)$. The elements of $\mathbb{H}^2(\mathbb{F}_M^2)$ are equivalence classes of systems $(g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_{\alpha_0})$ such that $g_{\alpha_0\alpha_1\alpha_2}$ are U(1) valued functions on $\mathcal{O}_{\alpha_1\alpha_2\alpha_3}$ (multiplicatively) antisymmetric in indices α_i, $\eta_{\alpha_0\alpha_1} = -\eta_{\alpha_1\alpha_0}$ are real 1-forms on $\mathcal{O}_{\alpha_0\alpha_1}$ and ω_α are real 2-forms on $\mathcal{O}_\alpha$ s.t.

$$g_{\alpha_1\alpha_2\alpha_3} g^{-1}_{\alpha_0\alpha_2\alpha_3} g_{\alpha_0\alpha_1\alpha_3} g^{-1}_{\alpha_0\alpha_1\alpha_2} = 1 \quad \text{on} \quad \mathcal{O}_{\alpha_0\alpha_1\alpha_2\alpha_3}\ , \tag{2}$$

$$\eta_{\alpha_1\alpha_2} - \eta_{\alpha_0\alpha_2} + \eta_{\alpha_0\alpha_1} = \frac{1}{i}\, g^{-1}_{\alpha_0\alpha_1\alpha_2}\, dg_{\alpha_0\alpha_1\alpha_2} \quad \text{on} \quad \mathcal{O}_{\alpha_0\alpha_1\alpha_2}\ , \tag{3}$$

$$\omega_{\alpha_1} - \omega_{\alpha_0} = d\eta_{\alpha_0\alpha_1}\ . \tag{4}$$

The equivalence is defined modulo systems

$$(\chi_{\alpha_1\alpha_2}\chi^{-1}_{\alpha_0\alpha_2}\chi_{\alpha_0\alpha_1}\ ,\ \pi_{\alpha_0} - \pi_{\alpha_1} + \frac{1}{i}\,\chi^{-1}_{\alpha_0\alpha_1}\, d\chi_{\alpha_0\alpha_1}\ ,\ -d\pi_\alpha) \tag{5}$$

where $\chi_{\alpha_0\alpha_1} = \chi^{-1}_{\alpha_1\alpha_0}$ are U(1) valued function on $\mathcal{O}_{\alpha_0\alpha_1}$ and π_α are real 1-forms on $\mathcal{O}_\alpha$. A class $w = [(g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_\alpha)]$ defines uniquely a closed 3-form γ on M equal to $d\omega_\alpha$ on each $\mathcal{O}_\alpha$. We shall call γ the curvature of w and denote by $W(M,\gamma)$ the subset of $\mathbb{H}^2(\mathbb{F}_M^2)$ of elements with curvature equal to γ.

If $\phi : \Sigma \to M$ is a map of a connected compact oriented manifold without boundary into M, then, pulling back $w \in \mathbb{H}^2(\mathbb{F}_M^2)$, we obtain an element $\phi^* w \in \mathbb{H}^2(\mathbb{F}_\Sigma^2) \cong U(1)$. To realize the last isomorphism explicitly, let us triangulate Σ by 2-cells c, 1-cells b and vertices v in such a way that $\phi(c) \subset \mathcal{O}_{\alpha_c}$, $\phi(b) \subset \mathcal{O}_{\alpha_b}$, $\phi(v) \in \mathcal{O}_{\alpha_v}$ for some assignment $\alpha_c, \alpha_b, \alpha_v$. If $w = [(g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_\alpha)]$, then $\phi^* w$ is represented by the number $A(\phi) \in U(1)$,

$$A(\phi) = \exp\Big[i \sum_c \int_c \phi^*\omega_{\alpha_c} - i \sum_{\substack{b,c \\ b\subset\partial c}} \int_b \phi^*\eta_{\alpha_b\alpha_c}\Big] \prod_{\substack{v,b,c \\ v\in\partial b \\ b\subset\partial c}} g_{\alpha_v\alpha_b\alpha_c}(\phi(v))\ , \tag{6}$$

compare expression (2.15) for the loop holonomy in the line bundle case. In (6), $\int_b$ is performed with the orientation of b inherited from c and $g_{\alpha_v\alpha_b\alpha_c}(\phi(v))$ should be inverted if v inherits the negative orientation from c via b. The right hand side of (6) is independent of the choices of the triangulation of Σ, of the assignment $\alpha_c, \alpha_b, \alpha_v$ and of the representative of w (in any covering). It is also invariant under the composition of ϕ with orientation-preserving diffeomorphisms of Σ. We shall call $A(\phi)$ the amplitude of ϕ and interpret it as the exponential of (i times) the global action of ϕ as it is built of integrals of local prime forms of the curvature γ of w and of correction terms.

As in the case of the holonomy of line bundles, amplitudes $A(\phi)$ are partially fixed by curvature γ. If B is a 3-dimensional compact oriented manifold with boundary ∂B composed of connected components $(\partial B)_i$, then

$$\prod_i A(\phi\restriction_{(\partial B)_i}) = \exp[i \int_B \phi^* \gamma] \quad . \tag{7}$$

Witten used (7) to define probability amplitudes in the Wess-Zumino-Witten chiral sigma model where a part of the action is given by a closed but not exact 3-form, see Section 5. More general definition (6) was introduced in [17]. It makes sense also for maps $\phi : \Sigma \to M$ which do not extend to B such that $\Sigma = \partial B$. Similarly, in the line bundle case, the curvature did not fix the holonomies of non-contractible loops.

From relation (7), it follows easily that $w \in W(M,\gamma)$ exists only if integrals of γ over closed 3-dimensional surfaces in M are in $2\pi\mathbb{Z}$ or, in the cohomological language, if $[\gamma] \in H^3(M, \mathbb{R})$ is in the image of $H^3(M, 2\pi\mathbb{Z})$ i.e. if γ is integral. It is not difficult to see that this is also a sufficient condition. Moreover, similarly to the case of line bundles with connection, $H^2(M,U(1))$ acts freely on $\mathbb{H}^2(\mathbb{F}^2_M)$ sending $[(g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_\alpha)]$ into $[(\lambda_{\alpha_0\alpha_1\alpha_2} g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_\alpha)]$ for $(\lambda_{\alpha_0\alpha_1\alpha_2})$ representing an element of $H^2(M,U(1))$, the orbits coinciding with sets $W(M,\gamma)$ for different integral closed 3-forms γ.

In the case of line bundle, $P(\phi)$, as given by eq. (2.15), defined also the parallel transport along open paths. The natural question arises as to whether (6) possesses a meaning for $\phi : \Sigma \to M$ where Σ is a 2-dimensional surface with boundary (being a union of loops). This leads us to the consideration of the loop space LM of all smooth maps from $S^1 \to M$. LM, with the topology of uniform convergence with all derivatives, possesses a natural structure of a (Fréchet) manifold [18] to which we shall refer below. An element $w \in \mathbb{H}^2(\mathbb{F}^2_M)$ defines naturally an isomorphism class Q of line bundles with hermitian connection over LM. We shall describe a local presentation $(G_{A_0A_1}, E_{A_0})$ of Q for an open covering $\{\mathcal{U}_A\}$ of $L\mathcal{M}$ constructed as follows. Choose an open covering $\{\mathcal{O}_\alpha\}$ of $\mathcal{M}$ and a triangulation of S^1 by intervals b meeting at points v . Choose additionally an assignment α_b, α_v such that the set

$$\{\phi : S^1 \to \mathcal{M} \mid \phi(b) \subset \mathcal{O}_{\alpha_b}, \phi(v) \subset \mathcal{O}_{\alpha_v}\} \equiv \mathcal{U}_A \neq \emptyset \ . \tag{8}$$

$\mathcal{U}_A$ for various triangulations and assignments α_b, α_v cover $L\mathcal{M}$. For $\emptyset = \mathcal{U}_{A_0} \cap \mathcal{U}_{A_1} \equiv \mathcal{U}_{A_0A_1}$, where $\mathcal{U}_{A_i}$ comes from a triangulation of S^1 by intervals b_i and vertices v_i , and assignments $\alpha^i_{b_i}, \alpha^i_{v_i}$, $i = 0,1$, consider the triangulation of S^1 by non-empty intersections $\bar{b} = b_0 \cap b_1$ and vertices $\bar{v}$ of either triangulation. Put $\alpha^0_{\bar{b}} = \alpha^0_{b_0}$, $\alpha^1_{\bar{b}} = \alpha^1_{b_1}$. For a vertex $\bar{v}$ of the new triangulation, set $\alpha^0_{\bar{v}} = \alpha^0_{v_0}$ if $v = v_0$ and $\alpha_{\bar{v}} = \alpha^0_{b_0}$ if $\bar{v}$ is an interior point of interval b_0 otherwise. Similarly define $\alpha^1_{\bar{v}} = \alpha^1_{v_1}$ or $\alpha^1_{b_1}$. The transition functions of the line bundle defined by a system $(g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_\alpha)$ are given by

$$G_{A_0A_1}(\phi) = \exp[i\sum_{\bar{b}} \int_{\bar{b}} \phi^* \eta_{\alpha^0_{\bar{b}}\alpha^1_{\bar{b}}}] \prod_{\substack{\bar{v},\bar{b} \\ \bar{v}\in\partial\bar{b}}} \frac{g_{\alpha^0_{\bar{v}}\alpha^1_{\bar{v}}\alpha^1_{\bar{b}}}}{g_{\alpha^0_{\bar{v}}\alpha^0_{\bar{b}}\alpha^1_{\bar{b}}}} (\phi(\bar{v})) \ . \tag{9}$$

The connection 1-forms E_A on $\mathcal{U}_A$ are defined by their action on vectors X_ϕ tangent to $L\mathcal{M}$ at loop ϕ . X_ϕ is a vector field on ϕ (i.e. a smooth section of $\phi^*T\mathcal{M}$) .

$$\langle X_\phi, E_A \rangle = \sum_b \int_b \phi^*(X_\phi \rfloor \omega_{\alpha_b}) + \sum_{\substack{v,b \\ v\in\partial b}} X_\phi(v) \rfloor \eta_{\alpha_v\alpha_b}(\phi(v)) \ . \tag{10}$$

A somewhat tedious but straightforward check shows that $(G_{A_0A_1}, E_A)$ describes locally a line bundle with hermitian connection over $L\mathcal{M}$. If we change $(g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_\alpha)$ by system (5) then $(G_{\alpha_0\alpha_1}, E_A)$ changes by $(K_{A_1}K^{-1}_{A_0} , \frac{1}{i} K^{-1}_A dK_A)$ where K_A are U(1) valued functions on $\mathcal{U}_A$ defined by

$$K_A(\phi) = \exp[-i \sum_b \int_b \phi^* \pi_{\alpha_b}] \prod_{\substack{v,b \\ v\in\partial b}} \chi_{\alpha_v\alpha_b}(\phi(v)) \ . \tag{11}$$

Thus eqs. (9) and (10) define a map

$$L : \mathbb{H}^2(\mathcal{F}^2_{\mathcal{M}}) \to \mathbb{H}^1(\mathcal{F}^1_{L\mathcal{M}}) \ . \tag{12}$$

The curvature Ω of Lw is related to the curvature γ of w . More exactly

$$\langle X_\phi, X'_\phi, \Omega\rangle = \int_{S^1} \phi^*(X'_\phi \rfloor X_\phi \rfloor \gamma) \tag{13}$$

for $X_\phi, X'_\phi \in T_\phi LM$. Thus L maps $W(M,\gamma)$ into $Q(LM,\Omega)$.

If line bundle L is obtained by identification of triples (A_0,ϕ,z) and $(A_1,\phi,G_{A_0A_1}(\phi)z)$ for $\phi \in U_{A_0A_1}$, then the fibers of L over loops ϕ and ϕ' differing by an orientation preserving reparametrization of S^1 are canonically isomorphic : such a reparametrization sends U_A onto $U_{A'}$, where A' corresponds to image triangulation of S^1 and $G_{A_0A_1}(\phi) = G_{A'_0A'_1}(\phi')$. Similarly the fibers of L over ϕ and ϕ' differing by an orientation changing reparametrization of S^1 are naturally dual one to another. The fibers of L over constant loops ϕ_0 are canonically identified with $\mathbb{C}^1$ by choosing local representatives (A,ϕ_0,z) of their elements with A defined by a constant assignment of α's .

Now let $\phi : \Sigma \to M$, where Σ is a connected, oriented 2-dimensional compact manifold with boundary composed of loops $(\partial\Sigma)_i$ (parametrized by S^1) . A triangulation of Σ such that $\phi(c) \subset O_{\alpha_c}$, $\phi(b) \subset O_{\alpha_b}$, $\phi(v) \in O_{\alpha_v}$ for some assignments of α's induces triangulations of the boundary circles with (restricted) assigments of α's s.t. $\phi\restriction_{(\partial\Sigma)_i}$ lie in the correponding U_{A_i}'s . For such Σ , the right hand side of (6) is no more triangulation and α-assignment invariant. Instead it picks up a factor $\prod_i G_{A_{i_0}A_{i_1}}(\phi\restriction_{(\partial\Sigma_i)})$ if we change the latter. Thus eq. (6) defines $A(\phi)$ as an element $\otimes_i L_{\phi\restriction(\partial\Sigma)_i}$, generalizing the notion of the parallel transport in L . If we use in eq. (6) an equivalent representative of w , then its right hand side changes by factor $\prod_i K_{A_i}(\phi\restriction_{(\partial\Sigma)_i})$, i.e. transforms by the isomorphism of bundles L obtained from equivalent cocycles.

If $D : M \to M$ preserves the curvature form γ of w , it may, but need not, preserve w except when $H^2(M,U(1)) = 1$. If $D^*w = w$, then the induced map $LD : LM \to LM$, $LD(\phi) = D\circ\phi$ lifts to an isomorphism $\widehat{LD}$ of line bundle L associated to w . $\widehat{LD}$ may be canonically defined up to isomorphisms of L locally represented by U(1) valued functions K_A of eq. (11) with $\chi_{\alpha_1\alpha_2}\chi_{\alpha_0\alpha_2}^{-1}\chi_{\alpha_0\alpha_1} = 1$, $\pi_{\alpha_0} - \pi_{\alpha_1} + \frac{1}{i}\chi_{\alpha_0\alpha_1}^{-1} d\chi_{\alpha_0\alpha_1} = 0$, $d\pi_\alpha = 0$, i.e. isomorphisms multiplying each fiber L_ϕ by the holonomy of ϕ in a flat hermitian line bundle over M defined by $(\chi_{\alpha_1\alpha_2}^{-1}, -\pi_\alpha)$. In particular, if M is simply connected, the lift $\widehat{LD}$ may be canonically chosen in a unique way. In that case actions of groups of diffeomorphisms of M preserving w lifts canonically to bundle L . The amplitudes

$A(\phi)$ of eq. (6) transform covariantly

$$A(D\circ\phi) = \widehat{LD}\, A(\phi) \quad . \tag{14}$$

Notice that the right hand side does not depend of the choice of lift $\widehat{LD}$ since the multiplication in L by holonomies of flat bundles over M leaves $A(\phi)$ invariant due to relation (2.16).

Again, similar considerations apply to infinitesimal actions of vector fields on M, i.e. Lie algebras of vector fields on M.

If $M \subset M^{\mathbb{C}}$, where $M^{\mathbb{C}}$ is a complex manifold and γ extends to holomorphic (3,0) form $\gamma^{\mathbb{C}}$ on $M^{\mathbb{C}}$, we may consider classes $[(g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_\alpha)] \in \mathbb{H}^2(\mathcal{A}^2_{M^{\mathbb{C}}})$ (the Deligne cohomology of degree 3) where $\mathcal{A}^2_{M^{\mathbb{C}}}$ is the sheaf complex

$$\mathcal{O}^*_{M^{\mathbb{C}}} \xrightarrow{\frac{1}{i}\, d\log} A^1_{M^{\mathbb{C}}} \xrightarrow{d} A^2_{M^{\mathbb{C}}} \tag{15}$$

and $d\omega_\alpha = \gamma^{\mathbb{C}}$ (denote by $W^a(M^{\mathbb{C}},\gamma^{\mathbb{C}})$ the set of such classes). $W^a(M^{\mathbb{C}},\gamma^{\mathbb{C}}) \neq \emptyset$ if and only if $\gamma^{\mathbb{C}}$ is integral and $H^2(M^{\mathbb{C}},\mathbb{C}^*)$ enumerates its elements in such case. For $w \in W^a(M^{\mathbb{C}}, \gamma^{\mathbb{C}})$, (9) and (10) define a holomorphic bundle $L^{\mathbb{C}}$ over $LM^{\mathbb{C}}$ with holomorphic connection of curvature $\Omega^{\mathbb{C}}$, a holomorphic (2,0) form on $LM^{\mathbb{C}}$ defined by (13) with γ replaced by $\gamma^{\mathbb{C}}$. The isomorphism class of $L^{\mathbb{C}}$ is an element of $Q^a(LM^{\mathbb{C}},\Omega^{\mathbb{C}})$ which depends only on w. The amplitudes $A(\phi)$ defined by eq. (6) for $\phi : \Sigma \to M^{\mathbb{C}}$ are now in $\mathbb{C}^*$ for Σ without boundary and in $\bigotimes_i L^{\mathbb{C}}_{\phi\restriction(\partial\Sigma)_i}$ for $\partial\Sigma$ composed of loops $(\partial\Sigma)_i$. Holomorphisms D of $M^{\mathbb{C}}$ which preserve w, induce maps LD of $LM^{\mathbb{C}}$ lifting to isomorphisms of $L^{\mathbb{C}}$. If D is an antiholomorphism of $M^{\mathbb{C}}$ then D^*w is naturally an element of $\mathbb{H}^2(\overline{\mathcal{A}^2_{M^{\mathbb{C}}}})$, where $\overline{\mathcal{A}^2_{M^{\mathbb{C}}}}$ is the complex

$$\overline{\mathcal{O}^*_{M^{\mathbb{C}}}} \xrightarrow{\frac{1}{i}\, d\log} \overline{A^1_{M^{\mathbb{C}}}} \xrightarrow{d} A^2_{M^{\mathbb{C}}} \tag{16}$$

So is $-\overline{w}$ represented by $(\overline{g},-\overline{\eta},-\overline{\omega})$. If $D^*w = -\overline{w}$, LD lifts to an antisomorphism of $L^{\mathbb{C}}$.

4. BOSONIZED ACTIONS ON A GENERAL RIEMANN SURFACE.

As the first application of the abstract theory of the previous section, we shall consider the definition of (euclidean) probability amplitudes for (euclidean) bosonic fields defined on a general Riemann surface Σ and corresponding via bosonization[19,4,20] to fermionic fields b,c on Σ with spins $1-\lambda$ and λ respectively, $\lambda \in \frac{1}{2}\mathbb{Z}$. Σ is equipped with a metric g in local complex coordinates given by $g_{z\bar{z}}\, dz d\bar{z}$. The metric connection turns the bundle K^{-1} of holomorphic vectors $a\frac{\partial}{\partial z}$ over Σ into a line bundle with hermitian connection of curvature R satisfying

$$\int_\Sigma R = 2\pi(2\ \text{genus} - 2)\ . \tag{1}$$

Fermionic fields c are sections of a line bundle L over Σ with hermitian connection of curvature $-\lambda R$ and fields b are sections of $K \otimes L^{-1}$ (K is the bundle of holomorphic covectors on Σ). The metric g and (isomorphism class of) L encode the geometric information needed to define the free euclidean field theory of fields b and c (and their complex conjugates) with the action

$$\int_\Sigma b\ d\bar{z}\ \nabla_{\bar{z}} c + \text{c.c.} \tag{2}$$

This is the only information about the fermionic system that we shall use.

Bosonization reexpresses the functional integral of fermionic fields b, c and their complex conjugates by a functional integral of bosonic field φ on Σ with $\varphi \in \mathbb{R}/\frac{1}{2}\mathbb{Z}$. In order to represent correctly the conformal and fermion number anomalies, the euclidean action of φ should have besides the standard free field part $4\pi i \int_\Sigma (\partial\varphi)(\bar{\partial}\varphi)$ a term $(2\lambda-1)i \int_\Sigma R\varphi$. Unfortunately, the latter term is ill-defined on a Riemann surface of genus $g > 0$ as there are smooth field configurations with no smooth real valued φ . What is well defined, however, is the real 3-form

$$\gamma = (1-2\lambda)R\ d\varphi \tag{3}$$

on $M = \Sigma \times \mathbb{R}/\frac{1}{2}\mathbb{Z}$. The additional term in the action tries to integrate a globally non-existent 2-form ω s.t. $\gamma = d\omega$ over the 2-dimensional surface

$$\Sigma \ni \xi \mapsto \phi(\xi) = (\xi, \varphi(\xi)) \in M. \tag{4}$$

As γ is integral,

$$\int_M \gamma = \frac{1}{2}\,(1-2\lambda)2\pi(2g-2) \in 2\pi\,\mathbb{Z}\ , \tag{5}$$

the global action can still be defined up to an ambiguity in $2\pi\,\mathbb{Z}$, as we know from Section 3. For that, we shall have to choose $w \in W(M,\gamma)$ which we shall do in a special way.

Let us consider first a 2-form $(\frac{1}{2} -\lambda)R$ on Σ . This is a real closed integral form so that there exists a line bundle over Σ with hermitian connection of curvature $(\frac{1}{2} -\lambda)R$ (the isomorphism classes of such bundles form a $2g$-dimensional torus, the Jacobian). We shall construct a natural map ι which assigns to classes of bundles with curvature $(\frac{1}{2} -\lambda)R$ elements of $W(M,\gamma)$:

$$\iota : Q(\Sigma,(\tfrac{1}{2}-\lambda)R) \to W(M,\gamma) \quad . \tag{6}$$

It is in fact defined by a cup-product operation on the cohomologies of sheaf complexes, but we prefer to give a more down-to-ground definition.

Let $(g_{\alpha_0\alpha_1},\eta_\alpha)$ represent $q \in Q(\Sigma,(\frac{1}{2}-\lambda)R)$ on an open covering $\{O_\alpha\}$ of Σ. Let $\{O'_\beta\}$ be an open covering of the circle $\mathbb{R}/\frac{1}{2}\mathbb{Z}$ such that there exist smooth functions $\varphi_\beta : O'_\beta \to \mathbb{R}$ s.t. $\varphi = \varphi_\beta(\varphi) + \frac{1}{2}\mathbb{Z}$. Notice that $\varphi_{\beta_1}-\varphi_{\beta_0} \in \frac{1}{2}\mathbb{Z}$. Consider the covering $\{O_{(\alpha,\beta)} = O_\alpha \times O'_\beta\}$ of M. Order arbitrarily indices α. Set

$$\omega_{(\alpha,\beta)} = (1-2\lambda)R\,\varphi_\beta \quad , \tag{7}$$

$$\eta_{(\alpha_0,\beta_0)(\alpha_1,\beta_1)} = 2(\varphi_{\beta_1}-\varphi_{\beta_0})\eta_{\alpha_0} \quad \text{if} \quad \alpha_0 < \alpha_1 \quad , \tag{8}$$

$$g_{(\alpha_0,\beta_0)(\alpha_1,\beta_1)(\alpha_2,\beta_2)} = (g_{\alpha_0\alpha_1})^{2(\varphi_{\beta_2}-\varphi_{\beta_1})} \quad \text{if} \quad \alpha_0 < \alpha_1 < \alpha_2 . \tag{9}$$

It is easy to show that $[(g_{(\alpha_0,\beta_0)(\alpha_1,\beta_1)(\alpha_2,\beta_2)},\eta_{(\alpha_0,\beta_0)(\alpha_1,\beta_1)},\omega_{(\alpha,\beta)})]$ depends only on $q = [(g_{\alpha_0\alpha_1},\eta_\alpha)]$ and defines $\iota q \in W(M,\gamma)$.

Given $w = \iota q$ and a map (4), we may define the amplitude $A(\phi)$ by eq. (3.6). The class q may be chosen in a specific way as the isomorphism class of bundle $K^{-1/2} \otimes L$ where $K^{1/2}$ is a spin bundle, i.e. a line bundle with hermitian connection s.t. $K^{1/2} \otimes K^{1/2}$ is isomorphic to K. L is the line bundle carrying fermionic field c. Indeed, the curvature of $K^{1/2}$ is $-\frac{1}{2}R$ and that of L is $-\lambda R$ so that the curvature of $K^{-1/2}\otimes L$ is equal $(\frac{1}{2}-\lambda)R$, as required. Let us denote the amplitude of ϕ corresponding to this choice of q by $A_{K^{-1/2}\otimes L}(\phi)$. Its definition uses more geometric input that the fermionic theory which we bosonize : namely the isomorphism class of a spin bundle, i.e. a spin structure. We can get rid of this extra input by averaging the amplitude over the spin structures. Let us do this with the weight $\sigma(K^{1/2})$ which is the parity of the spin structure, i.e. -1 to the dimension of the space of sections of $K^{1/2}$ anihilated by covariant antiholomorphic derivative $d\bar{z}\,\nabla_{\bar{z}}$. The complete probability amplitude of the field $\xi \to \phi(\xi) = (\xi, \varphi(\xi))$, with the standard free field term included, is

$$e^{-S(\varphi)} = \exp[-4\pi i \int_\Sigma (\partial\varphi)(\bar{\partial}\varphi)] \sum_{[K^{1/2}]} \sigma(K^{1/2}) A_{K^{-1/2}\otimes L}(\phi) \quad . \tag{10}$$

It is clear that it uses as geometric input, besides the metric g, only the isomorphism class of line bundle L with hermitian connection, i.e. the input of the fermionic theory.

Let us cut the surface Σ along the homology basis a_i, b_i, $i = 1,\ldots,g$, meeting at one point, reducing the surface to the polygone Σ_c, see Fig. 3. For (smooth) field $\varphi : \Sigma \to \mathbb{R}/\frac{1}{2}\mathbb{Z}$, let us choose a smooth version $\tilde{\varphi} : \Sigma_c \to \mathbb{R}$ of it (i.e. $\varphi(\xi) = \tilde{\varphi}(\xi) + \frac{1}{2}\mathbb{Z}$). Denoting by $P_{K^{-1/2}\otimes L}$ the holonomy in $K^{-1/2} \otimes L$, it is easy to see using the definitions (3.6), (2.15) and (7) to (9) that

$$A_{K^{-1/2}\otimes L}(\phi) = \exp[(1-2\lambda)i \int_{\Sigma_c} R\tilde{\varphi}] \prod_{i=1}^{g} P_{K^{-1/2}\otimes L}(a_i)^{2\int_{b_i} d\varphi} \cdot P_{K^{-1/2}\otimes L}(b_i)^{-2\int_{a_i} d\varphi} . \tag{11}$$

This shows that (10) reproduces the bosonic field amplitudes defined by Alvarez-Gaumé et al. in[20]. From eq. (10), it is clear that $e^{-S(\varphi)}$ is defined intrinsically, i.e. if D is a diffeomorphism of Σ then

$$e^{-S(\varphi\circ D)} = e^{-S'(\varphi)} \tag{12}$$

where, if S corresponds to metric g and bundle L over Σ then S' does to D^*g and D^*L. In the work of Alvarez-Gaumé et al.[20] this required an explicite check of modular covariance of expressions defined with the use of a marking of the Riemann surface Σ .

5. WESS-ZUMINO-WITTEN QUANTUM FIELD THEORY.

Wess-Zumino-Witten (WZW) model[3] is another two-dimensional field theory where part of the action is defined by integrals of the local prime forms of a closed but not exact 3-form γ . Fields take values in a compact Lie group G and

$$\gamma = \frac{k}{12\pi} \operatorname{tr}(g^{-1}dg)^3 . \tag{1}$$

For concreteness, we shall first consider only the case of $G = SU(2)$ and its complexification $G^{\mathbb{C}} = SL(2,\mathbb{C})$ with (3,0) holomorphic form $\gamma^{\mathbb{C}}$ given by the same formula. γ and $\gamma^{\mathbb{C}}$ are integral if and only if $k \in \mathbb{Z}$. As the mapping $g \to g^{-1}$ changes the sign of k , we shall limit ourselves to $k \in \mathbb{N}$. In Appendix 3, we give an explicit local representative $(g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_\alpha)$ of class $w \in W^a(G^{\mathbb{C}},\gamma^{\mathbb{C}})$ which restricted to G defines an element of $W(G,\gamma)$. Since $H^2(G^{\mathbb{C}},\mathbb{C}^*) = H^2(G,U(1)) = 1$, these classes are unique.

In the euclidean Wess-Zumino-Witten model on a Riemann surface Σ ,

the field configuration is a map $g : \Sigma \to G$, or more generally $g : \Sigma \to G^{\mathbb{C}}$. 1
Eq. (3.6) allows now to define a part $A(g) \equiv A_\Sigma(g)$ of the probability amplitude of field configuration g coming from $\gamma^{\mathbb{C}}$. It is a non-zero complex number, of modulus one if g takes values in G . The other part of the amplitude comes from the standard action of the chiral σ-model. The complete amplitude is

$$e^{-S_\Sigma(g)} = \exp[\frac{ik}{4\pi} \int_\Sigma \mathrm{tr}(g^{-1}\partial g)(g^{-1}\bar{\partial} g)] A_\Sigma(g) \tag{2}$$

and defines global euclidean action $S_\Sigma(g)$ up to $2\pi i \mathbb{Z}$. In (2), we have chosen the special value of the coupling constant of the chiral model, which renders the combined model conformal invariant[3]. Since every mapping g from Σ to $SU(2)$ can be extended to a map $\tilde{g} : B \to SU(2)$ where B is a 3-dimensional compact oriented manifold with boundary $\partial\Sigma$ then due to eq. (3.7), we may rewrite (3) as

$$e^{-S_\Sigma(g)} = \exp[\frac{ik}{4\pi} \int_\Sigma \mathrm{tr}(g^{-1}\partial g)(g^{-1}\bar{\partial} g) + \frac{ik}{12\pi} \int_B \mathrm{tr}(\tilde{g}^{-1} d\tilde{g})^3] . \tag{3}$$

Let us start with some formal considerations. In quantum field theory, we shall have to calculate functional integrals given formally as

$$\int_{\Sigma G} F(g) e^{-S_\Sigma(g)} D_\Sigma g \tag{4}$$

where $F(g)$ are functionals of g , e.g. $F(g) = \prod_{j=1}^{n} g_{\alpha_j \beta_j}(\xi_j)$, and $D_\Sigma g$ is the Haar measure on the group ΣG of maps from Σ into G .

If $\Sigma = P\mathbb{C}^1$, the functional integral (4) may be given an interpretation in operator formalism. Consider the space of smooth sections ψ of line bundle L over LG associated to (a representative of) the class in $W(G,\gamma)$. $P\mathbb{C}^1 = \mathbb{C}^1 \cup \{\infty\} = D_0 \cup D_\infty$, where D_0 is the disc $|z| \leq 1$ and D_∞ is the opposite one. Formal examples of sections ψ are provided by functional integrals over fields defined on D_0 with boundary values fixed:

$$\psi_F(h) = \int_{D_0 G} F(g)\, e^{-S_{D_0}(g)} \delta(g\restriction_{\partial D_0} h^{-1}) D_{D_0} g \tag{5}$$

where $e^{-S_{D_0}(g)}$ is defined by (2) with Σ replaced by D_0 and takes values in $L_{g\restriction_{\partial D_0}}$. We may consider formal scalar product of states ψ defined as

$$(\psi_1,\psi_2) = \int_{LG} \overline{\psi_1(h)}\, \psi_2(h)\, D_{S^1} h \tag{6}$$

If we set $\theta g(z) = g(\frac{1}{\bar{z}})$ then it is easy to see that for $g : D_0 \to G$ (in any trivilization)

$$\overline{e^{-S_{D_o}(g)}} = e^{-S_{D_\infty}(\theta g)} \quad . \tag{7}$$

Let F_1, F_2 be functionals of $g : \Sigma = P\mathbb{C}^1 \to G$ depending only on fields on D_o. If we denote $(\theta F_1)(g) = \overline{F_1(\theta g)}$ then formally

$$\int_{\Sigma G} (\theta F_1)(g) F_2(g)\, e^{-S_\Sigma(g)} D_\Sigma g$$

$$= \int_{LG} D_{S^1} h\Big(\int_{D_\infty G} (\theta F_1)(g_\infty)\, e^{-S_{D_\infty}(g_\infty)} \delta(g_\infty\restriction_{\partial D_o} h^{-1}) D_{D_\infty} g\Big)$$

$$\cdot\Big(\int_{D_o G} F_2(g_o) e^{-S_{D_o}(g_o)} \delta(g_o\restriction_{\partial D_o} h^{-1}) D_{D_o} g_o\Big)$$

$$= \int_{LG} D_{S^1} h\, \overline{\Big(\int_{D_o G} F_1(g_o) e^{-S_{D_o}(g_o)} \delta(g_o\restriction_{\partial D_o} h^{-1}) D_{D_o} g_o\Big)}$$

$$\cdot\Big(\int_{D_o G} F_2(g_o) e^{-S_{D_o}(g_o)} \delta(g_o\restriction_{\partial D_o} h^{-1}) D_{D_o} g_o\Big)$$

$$= (\psi_{F_1}, \psi_{F_2}) \quad . \tag{8}$$

This shows that the left hand side should be positive for $F_1 = F_2$ which is the physical positivity condition for the WZW model on Riemann sphere.

Notice that the formal definition (5) may be rewritten as

$$\psi_F(h) = \int_{D_o G} F(gg_h)\, e^{-S_{D_o}(gg_h)} \delta(g|_{\partial D_o}) D_{D_o} g \tag{9}$$

where g_h is a fixed map from D_o into G such that $g_h\restriction_{\partial D_o} = h$. Independence of the integral of the choice of g_h follows from the right invariance of the Haar measure $D_{D_o} g$. If A is a finite set, then the the Haar measure $D_A g$ on AG (i.e. the set of maps from A to G) possesses a richer invariance : for F an analytic function on $AG^{\mathbb{C}}$ and $g' \in AG^{\mathbb{C}}$

$$\int_{AG} F(gg') D_A g = \int_{AG} F(g) D_A g \ . \tag{10}$$

If we assume this invariance still to hold formally in infinite dimensional case then for F analytic functionals on $D_o G^{\mathbb{C}}$, (9) still makes sense for $h \in LG^{\mathbb{C}}$ and defines formally a holomorphic section of bundle $L^{\mathbb{C}}$ over $LG^{\mathbb{C}}$.

The above formal considerations motivate the choice of space $\Gamma^a(L^{\mathbb{C}})$ of the holomorphic sections of $L^{\mathbb{C}}$ as the space of quantum states of the WZW model. Below we shall show that this space carries a natural representation of a pair of Kac-Moody algebras $\widehat{su}(2)$ with central charge k (arising by geometric quantization of the classical symmetries of the theory). We shall decompose this representation into irreducible components recovering the spectrum of the latter conjectured by Gepner and Witten in[5].

Classical WZW theory possesses an extraordinarily rich symmetry : its classical (euclidean) equations of motion are $\partial(g^{-1}\bar{\partial}g) = 0$ and if $g_1(g_2)$ is any (anti-)holomorphic map with values in $G^{\mathbb{C}}$ then the transformation $g \to g_1 g g_2^{-1}$ maps local classical solutions into new ones. On the quantum level, this becomes the Kac-Moody symmetry.

It will be important to study this symmetry on the group level rather than on the level of Lie algebras. To this end, we shall first introduce group $\widehat{LG}^{\mathbb{C}}$ which is the central extension of loop group $LG^{\mathbb{C}}$ by $\mathbb{C}^*$. It appears naturally when we lift the classical symmetries to the symmetries of bundle $L^{\mathbb{C}}$ following the rules of geometric quantization. Here, we shall construct directly the final product. As a space, $\widehat{LG}^{\mathbb{C}} = L^{\mathbb{C}^*} \equiv L^{\mathbb{C}} \setminus$ zero section. To describe its multiplication law, let us notice a basic property of the WZW amplitudes e^{-S_Σ}, known as Polyakov formula[21]. If Σ is a Riemann surface with boundary, $g,h : \Sigma \to G^{\mathbb{C}}$, $h\restriction_{\partial\Sigma} \equiv 1$, then

$$e^{-S_\Sigma(gh)} = e^{-S_\Sigma(g)}\, e^{-S_\Sigma(h)}\, e^{\Gamma_\Sigma(g,h)} \tag{11}$$

where

$$\Gamma_\Sigma(g,h) = \frac{ik}{2\pi}\int_\Sigma \mathrm{tr}(g^{-1}\bar{\partial}g)(h\partial h^{-1}) \quad . \tag{12}$$

Notice that $e^{-S_\Sigma(h)}$ is a complex number since the fibers of $L^{\mathbb{C}}$ over constant loops are canonically isomorphic to $\mathbb{C}^1$. Thus (11) is an equality of elements of $\bigotimes_i L^{\mathbb{C}}_{g\restriction(\partial\Sigma)_i}$. It is easily proven by comparing the t-derivatives of both sides along a homotopy h_t fixed at $\partial\Sigma$ between h and 1. All what is needed is the formula for the derivative of the global action $e^{-S_\Sigma(g_t)}$ over t valid in any trivialization if $\frac{\partial}{\partial t} g_t\restriction_{\partial\Sigma} = 0$:

$$\frac{d}{dt}\, S_\Sigma(g_t) = -\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\, \partial(g^{-1}\frac{\partial g_t}{\partial t})\, g_t^{-1}\,\bar{\partial}\, g_t \quad . \tag{13}$$

We proove a generalization of (13) admitting any $\frac{\partial g_t}{\partial t}\restriction_{\partial\Sigma}$ in Appendix 4. The multiplication law in $L^{\mathbb{C}}$ is defined by

$$(\lambda_1 e^{-S_{D_o}(g_1)})\cdot(\lambda_2 e^{-S_{D_o}(g_2)}) = \lambda_1\lambda_2\, e^{-S_{D_o}(g_1g_2)-\Gamma_{D_o}(g_1,g_2)} \tag{14}$$

where $g_i : D_o \to G^{\mathbb{C}}$, $i = 1,2$. That the definition is correct and defines group operation in $\widehat{LG}^{\mathbb{C}}$, follows easily from Polyakov formula (11) and the basic property of Γ_Σ

$$\Gamma_\Sigma(g_1,g_2) = \Gamma_\Sigma(g_1,g_2g_3) - \Gamma_\Sigma(g_1g_2,g_3) + \Gamma_\Sigma(g_2,g_3) \quad . \tag{15}$$

$\mathbb{C}^*$ can be embedded into the fiber of $L^{\mathbb{C}}$ over the constant loop 1 . This way we obtain the exact sequence of groups

$$1 \to \mathbb{C}^* \to \widehat{LG}^{\mathbb{C}} \to LG^{\mathbb{C}} \to 1 \quad . \tag{16}$$

Consider the antiholomorphic involution $j : g \mapsto g^\dagger$ on $G^{\mathbb{C}}$. For $w \in W^a(G^{\mathbb{C}},\gamma^{\mathbb{C}})$,

$$j^* w = -\bar{w} \tag{17}$$

since both sides have the same curvature. The involution Lj of $LG^{\mathbb{C}}$ lifts canonically (G is simply connected) to an antiholomorphic involution. We shall use symbole $\hat{g}^\dagger$ for the image of $\hat{g} \in L^{\mathbb{C}}$ under it. The involution transforms covariantly the probability amplitudes

$$(e^{-S_\Sigma(g)})^\dagger = e^{-S_\Sigma(g^\dagger)} \tag{18}$$

and is antimultiplicative, i.e.

$$(\hat{g}_1 \cdot \hat{g}_2)^\dagger = \hat{g}_2^\dagger \cdot \hat{g}_1^\dagger \quad . \tag{19}$$

The set of points $\hat{g}$ of $LG^{\mathbb{C}}$ s.t. $\hat{g}^\dagger = \hat{g}^{-1}$ is the set of vectors of length one in line bundle L over LG . We shall denote it by $\widehat{LG}$. Due to (19), the multiplication does not lead out of $\widehat{LG}$. This way we get a central extention of LG by $U(1)$:

$$1 \to U(1) \to \widehat{LG} \to LG \to 1 \quad . \tag{20}$$

Central extensions (16) and (20) depend on k . For $k = 1$, we obtain the so-called universal one . The extensions for higher k can be obtained from the universal one by division by $\mathbb{Z}_k$.

Space $\Gamma^a(L^{\mathbb{C}})$ of holomorphic sections of $L^{\mathbb{C}}$ carries two commuting representations ℓ and π of $\widehat{LG}^{\mathbb{C}}$ defined by means of left and right multiplication. If $\hat{g}_1,\hat{g}_2 \in \widehat{LG}^{\mathbb{C}}$ projecting to $g_1,g_2 \in LG^{\mathbb{C}}$ and if $\psi \in \Gamma^a(L^{\mathbb{C}})$ then

$$(\ell(\hat{g}_1)\pi(\hat{g}_2)\psi)(g) = \hat{g}_1 \cdot \psi(g_1^{-1} g g_2^{\dagger -1}) \cdot \hat{g}_2^\dagger \quad . \tag{21}$$

Denote by $L^\pm G^{\mathbb{C}} \equiv L^\pm$ the subgroups of $LG^{\mathbb{C}}$ composed of boundary values of holomorphic maps of $D_{\substack{o\\ \infty}}$ into $G^{\mathbb{C}}$. $L^\pm$ can be also considered as subgroups of $\widehat{LG}^{\mathbb{C}}$ composed of elements $e^{-S_{D_o}(g)}$, $g \in L^+$, and $e^{S_{D_\infty}(g)}$ i.e. elements dual to $e^{-S_{D_\infty}(g)}$, $g \in L^-$. Notice that for constant loops g_o forming $L^+ \cap L^-$, $e^{-S_{D_o}(g_o)} = e^{S_{D_\infty}(g_o)}$ since $e^{-S_\Sigma(g_o)} = 1$ so that they lift to the same element of $\widehat{LG}^{\mathbb{C}}$ via L^+ and L^- defining an embedding of $G^{\mathbb{C}}$ into $\widehat{LG}^{\mathbb{C}}$. The Lie algebras of L^+ and L^- are spanned by

$\frac{1}{2} i\sigma_j z^n$, $n \geq 0$ and $\frac{1}{2} i\sigma_j z^n$, $n \leq 0$ respectively where σ_j are the Pauli matrices. Define

$$J^j_n \psi = \frac{d}{d\varepsilon} \restriction_{\varepsilon=o} \ell(e^{\frac{1}{2} i\varepsilon\sigma_j z^n})\psi \tag{22}$$

and

$$\bar{J}^j_n \psi = \frac{d}{d\varepsilon} \restriction_{\varepsilon=o} r(e^{\frac{1}{2} i\varepsilon\sigma_j z^n})\psi \quad . \tag{23}$$

Straightforward computation based on eq. (14), see Appendix 5, shows that

$$[J^j_n, J^i_m] = \sum_k \varepsilon^{jik} J^k_{n+m} + \frac{k}{2} m\delta^{ji}\delta_{n+m,o} \quad . \tag{24}$$

The same relation holds for $\bar{J}$'s . This shows that our representation of $\widehat{LG}^{\mathbb{C}} \times \widehat{LG}^{\mathbb{C}}$ in $\Gamma^a(L^{\mathbb{C}})$ becomes on the level of Lie algebra a pair of commuting representations of the $\widehat{su}(2)$ Kac-Moody algebra with central charge equal k [23].

The representation of $\widehat{LG}^{\mathbb{C}} \times \widehat{LG}^{\mathbb{C}}$, defined somewhat ad hoc, becomes quite natural if we represent states in $\Gamma^a(L^{\mathbb{C}})$ by formal functional integrals (5). Indeed, it is easy to see that formally, for $g_1, g_2 \in L^+$

$$\ell(g_1) r(g_2)\psi_F = \psi_{F'} \tag{25}$$

where

$$F'(g) = F(g_1^{-1} g \, g_2^{\dagger -1}) \quad . \tag{26}$$

Besides, in the formal scalar product (8),

$$\ell(g_1)^\dagger = \ell(\theta g_1^\dagger) \ , \ r(g_2)^\dagger = r(\theta g_2^\dagger) \quad . \tag{27}$$

The action of $L^+ \times L^+$ together with rules (27) determine the representation of $\widehat{LG}^{\mathbb{C}} \times \widehat{LG}^{\mathbb{C}}$. On the algebraic level, eq. (27) becomes the unitarity rule[23]

$$J^{i\dagger}_n = -J^i_{-n} \quad , \quad \bar{J}^{i\dagger}_n = -\bar{J}^i_{-n} \quad .$$

We would like to reduce the representation of $\widehat{LG}^{\mathbb{C}} \times \widehat{LG}^{\mathbb{C}}$ in $\Gamma^a(L^{\mathbb{C}})$. To this end, we shall search for lowest weight (LW) vectors in $\Gamma^a(L^{\mathbb{C}})$, the building blocks of the LW subrepresentations[22]. By definition, the LW vectors are states ψ satisfying

$$\ell(g_1) r(g_2)\psi = a_1^{-2j_1} \bar{a}_2^{-2j_2} \psi \tag{29}$$

for $g_1, g_2 \in L^+$, $g_1(0) = \begin{pmatrix} a_1 & 0 \\ * & a_1^{-1} \end{pmatrix}$, $g_2(0) = \begin{pmatrix} a_2 & 0 \\ * & a_2^{-1} \end{pmatrix}$. j_1, j_2 are the left, right (iso-)spins of the LW vector ψ , $0 \leq j_1, j_2 \in \frac{1}{2}\mathbb{Z}$. On constant loops g_o

$$\psi(g_o) = \psi_o(g_o) e^{-S_{D_o}(g_o)} \tag{30}$$

where ψ_o is a holomorphic function on $G^{\mathbb{C}}$. Due to (18), eqs. (21) and (14) imply that for any $g_1, g_2 \in L^+$

$$\psi(g_1 g_2^\dagger) = [\ell(g_1 g_1(0)^{-1}) r(g_2 g_2(0)^{-1})\psi](g_1 g_2^\dagger)$$
$$= (g_1 g_1(0)^{-1}) \cdot \psi(g_1(0) g_2(0)^\dagger) \cdot (g_2(0)^{\dagger -1} g_2^\dagger)$$
$$= \psi_o(g_1(0) g_2(0)^\dagger)\, e^{-S_{D_o}(g_1 g_2^\dagger)} \quad . \tag{31}$$

On the other hand, using constant loops $g_{10} = \begin{pmatrix} a_1 & 0 \\ * & a_1^{-1} \end{pmatrix}$, $g_{20} = \begin{pmatrix} a_2 & 0 \\ * & a_2^{-1} \end{pmatrix}$, we obtain from (29)

$$\psi_o(g_{10}^{-1} g_0 g_{20}^{\dagger -1}) = a_1^{-2j_1}\, \bar{a}_2^{-2j_2}\, \psi_o(g_0) \quad . \tag{32}$$

Relation (32) means that ψ_o is the LW vector for the left and right regular representation of G in holomorphic functions on $G^{\mathbb{C}}$. Now, the standard result says that the left and right spins of ψ_o have to be equal and ψ_o is unique up to a factor. Indeed, (32) implies that

$$\psi_o(g_{10} g_{20}^\dagger) = a_1^{2j_1}\, \bar{a}_2^{2j_2}\, \psi_o(1) \quad . \tag{33}$$

Since for $a \neq 0$

$$\begin{pmatrix} a & b \\ c & \frac{1}{a} + \frac{bc}{a} \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\bar{b}}{\bar{a}} & 1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} \bar{a} & 0 \\ \bar{b} & \bar{a}^{-1} \end{pmatrix} , \tag{34}$$

we infer that $j_1 = j_2 = j$ and for $g_o = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\psi_o(g_o) = a^{2j} \psi_o(1) \; . \tag{35}$$

Summarizing, LW vectors ψ in $\Gamma^a(L^{\mathbb{C}})$, see eq. (29), have to have equal spins $j_1 = j_2 = j$ and then, for $g_1, g_2 \in L^+$,

$$\psi(g_1 g_2^\dagger) = \text{const}\; a^{2j}\, e^{-S_{D_o}(g_1 g_2^\dagger)} \quad . \tag{36}$$

where $g_1(0) g_2(0)^\dagger = \begin{pmatrix} a & * \\ * & * \end{pmatrix}$.

By the Birkhoff theorem [22], chapter 8, loops of the form $g_1 g_2^\dagger$, $g_1, g_2 \in L^+$ form an open dense set in $LG^{\mathbb{C}}$. Hence eq. (36), for each j, may have only one solution $\psi \in \Gamma^a(L^{\mathbb{C}})$, up to a multiplicative constant. We shall see that such a solution exists if and only if $j \leq \frac{k}{2}$. The proof of this fact is a variation of the proof of Prop. 11.3.1 of ref. [22].

Let us consider the action of $L^+ \times L^+$ on $LG^{\mathbb{C}}$ given by $g \mapsto g_1 g g_2^\dagger$. We want to describe the orbits of this action. First consider the orbits of $N^+ \times L^+$ in $LG^{\mathbb{C}}$ where $N^+ = \{g_1 \in L^+ \mid g_1(0) = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}\}$. As proven in [22], Section 8.4, these orbits are of the form

$$\Sigma_n = N^+ e_n L^- = N_n^+ e_n L^- \qquad (37)$$

where $e_n = \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \in LG^{\mathbb{C}}$,

$$N_n^+ = N^+ \cap e_n L_1^+ e_n^{-1} , \qquad (38)$$

and $L_1^+ = \{g \in L^+ : g(0) = 1\}$.

Relation (37) follows from the splitting

$$N^+ = N_n^+ (N^+ \cap e_n L^- e_n^{-1}) , \qquad (39)$$

elementary to prove. Another elementary splitting is

$$e_n L_1^+ e_n^{-1} = N_n^+ N_n^- \qquad (40)$$

where

$$N_n^- = N^- \cap e_n L_1^+ e_n^{-1} \qquad (41)$$

and $N^- = \{g \in L^- : g(\infty) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$. (42)

Since $U_n \equiv e_n L_1^+ L^-$ is an open (dense) set in $LG^{\mathbb{C}}$ containing e_n and

$$U_n = (e_n L_1^+ e_n^{-1}) e_n L^- = N_n^+ N_n^- e_n L^- , \qquad (43)$$

we see that the codimension of Σ_n is the dimension of $N_n^- = \begin{cases} 2n-1 , & n > 0 \\ -2n , & n < 0 \end{cases}$.

Another important information we shall need is that

$$U_n \setminus \Sigma_n \subset \begin{cases} \bigcup_{|m|<n} \Sigma_m , & n > 0 \\ \left(\bigcup_{|m|<n} \Sigma_m \right) \cup \Sigma_{-n} , & n < 0 \end{cases} . \qquad (44)$$

This is an easy consequence of the discussion in [22], Chapter 7.3. From the Hartogs theorem it follows that if ψ is a holomorphic section of $L^{\mathbb{C}}$ over $\Sigma_0 \cup \Sigma_1 = U_0 \cup U_1$ then it extends to a holomorphic section over $\Sigma_0 \cup \Sigma_1 \cup \Sigma_{-1} = U_0 \cup U_1 \cup U_{-1}$ since Σ_{-1} is a complex submanifold of the latter set of codimension > 1 . By induction, ψ can be continued to a global section of $L^{\mathbb{C}}$. To see when (36) defines a global section, it is then enough to study when it extends from Σ_0 to Σ_1 .

Consider loops $f_{cn} = \begin{pmatrix} 1 & 0 \\ cz^{-n} & 1 \end{pmatrix}$, $c \in \mathbb{C}^1$, $n = 1,2,\ldots$. $f_{cn} \in N_n^-$ and f_{c1} is a general element of N_1^- . If $c \neq 0$ then

$$f_{cn}\, e_n = \begin{pmatrix} z^n & 0 \\ c & z^{-n} \end{pmatrix} = \begin{pmatrix} 1 & \frac{z^n}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \bar{c} \\ -\frac{1}{c} & z^n \end{pmatrix}^{\dagger} \equiv g_{1cn} g_{2cn}^{\dagger} \tag{45}$$

where $g_{1cn} \in N^+$ and $g_{2cn} \in L^+$. We shall need to know the asymptotic behavior of $e^{-S_{D_o}(g_{1cn}g_{2cn}^{\dagger})}$ when $c \to 0$. Notice the singularity in

$$g_{1cn}g_{2cn}^{\dagger} = \begin{pmatrix} z^n & \frac{\bar{z}^n z^n - 1}{c} \\ c & \bar{z}^n \end{pmatrix} \tag{46}$$

where $c \to 0$ inside D_o. Using formula (A.4.6) for the derivative of action S_Σ (see Appendix 4), we infer that in any local trivialization

$$\frac{d}{dc} S_{D_o}(g_{1cn}g_{2cn}^{\dagger}) = -\frac{ik}{2\pi} \int_{D_o} \mathrm{tr}\; \partial\left(g_{2cn}^{\dagger -1} g_{1cn}^{-1} \frac{\partial(g_{1cn}g_{2cn}^{\dagger})}{\partial c}\right) g_{2cn}^{\dagger -1} g_{1cn}^{-1} \bar{\partial}(g_{1cn}g_{2cn}^{\dagger}) + \mathcal{O}(1) \tag{47}$$

where $\mathcal{O}(1)$ contains the boundary terms which are regular when $c \to 0$ as $g_{1c}g_{2c}^{\dagger}$ is regular on ∂D_o. Integrating by parts in (47) and using the fact that $\partial(g_{2cn}^{\dagger -1} g_{1cn}^{-1} \bar{\partial}(g_{1cn}g_{2cn}^{\dagger})) = 0$ ($g_{1c}g_{2c}^{\dagger}$ is a classical solution), we obtain

$$\frac{d}{dc} S_{D_o}(g_{1cn}g_{2cn}^{\dagger}) = \frac{ik}{2\pi} \int_{\partial D_o} \mathrm{tr}\; \frac{\partial(g_{2cn}^{\dagger -1} g_{1cn}^{-1})}{\partial c} \bar{\partial}(g_{1cn}g_{2cn}^{\dagger}) + \mathcal{O}(1)$$
$$= -\frac{kn}{c} + \mathcal{O}(1)\ . \tag{48}$$

Thus

$$e^{-S_{D_o}(g_{1cn}g_{2cn}^{\dagger})} = c^{kn}\, \mathcal{O}(1)\ . \tag{49}$$

In order to study the holomorphicity of LW state ψ satisfying (36) on U_1, we decompose each element of U_1 as $g_1 f_{c1}\, e_1 g_2^{\dagger}$, where $g_1 \in N_1^+$ and $g_2 \in L^+$. According to (43), this is a unique decomposition. Now for $c \neq 0$

$$\psi(g_1 f_{c1} e_1 g_2^{\dagger}) = \text{const.}\ a^{2j}\, e^{-S_{D_o}(g_1 g_{1c1} g_{2c1}^{\dagger} g_2^{\dagger})}$$
$$= \text{const.}\ a^{2j}\, g_1 \cdot e^{-S_{D_o}(g_{1c1}g_{2c1}^{\dagger})} \cdot g_2^{\dagger} \tag{50}$$

where a is defined by

$$(g_1 g_{1c1} g_{2c1} g_2^{\dagger})(0) = g_1(0) \begin{pmatrix} 0 & -\frac{1}{c} \\ c & 0 \end{pmatrix} g_2(0)^{\dagger} = \begin{pmatrix} a & * \\ * & * \end{pmatrix}.$$

Clearly (see (49)), the right hand side of (50) is analytic at $c = 0$ if and only if $j \leq \frac{k}{2}$. Thus in this and only this case ψ, as given by (36) extends by continuity to a unique holomorphic section of $L^{\mathbb{C}}$. From (36) it is obvious that ψ restricted to Σ_o satisfies (29). We still have to check this for the extension of ψ to the whole loop group. Any element $g \in LG^{\mathbb{C}} \setminus \Sigma_o$ may be written as $g_1 e_n g_2^{\dagger}$ for $g_1, g_2 \in L^+$ (orbits of $L^+ \times L^-$ on $LG^{\mathbb{C}}$ are of the form $\Sigma_n \cup \Sigma_{-n}$) . Hence g may be approximated by elements $g_1 f_{cn} e_n g_2^{\dagger} \in \Sigma_o$. Thus (29) follows by continuity. Notice that for $|n| > 2$,

$$\psi(g_1 e_n g_2^{\dagger}) = \lim_{c\to o} \psi(g_1 f_{cn} e_n g_2^{\dagger}) \tag{51}$$

$$= \text{const.} \lim_{c\to o} a^{2j} g_1 \cdot e^{-kS_{D_o}(g_{1cn} g_{2cn}^{\dagger})} \cdot g_2 = 0 \, , \tag{52}$$

so that LW state ψ vanishes on $\bigcup_{|n|\geq 2} \Sigma_n$. If the spin of the state is less than $\frac{k}{2}$ then ψ vanishes also on Σ_1 and Σ_{-1} .

The spectrum of the LW vectors found here : left-right spins equal taking values $0, \frac{1}{2}, 1 \ldots, \frac{k}{2}$ with multiplicity one, coincides with the spectrum conjectured by Gepner and Witten in[5] . From the general Peter-Weyl type theory for the representations of the Kac-Moody groups[24] it follows that the representation of $\widehat{LG}^{\mathbb{C}} \times \widehat{LG}^{\mathbb{C}}$ restricted to the subspace of the so-called strongly regular states decomposes into the direct sum of LW representations built on the vectors just described. We do not know if the strongly regular states are dense in $\Gamma^a(L^{\mathbb{C}})$. (This is however plausible as there are no highest weight vectors in $\Gamma^a(L^{\mathbb{C}})$) .

The formal candidates for our LW states are provided by functional integrals of type (5) :

$$\psi_j(h) = \int_{D_o G} N(g(0)^{\otimes_s 2j}) \, e^{-S_{D_o}(g)} \, \delta(g\restriction_{\partial D_o} h^{-1}) D_{D_o} g \tag{53}$$

where $N(g(0)^{\otimes_s 2j})$ stands for a normal ordering of the symmetric tensor power of matrix $g(0)$. Such normal powers are produced from point-splitted expressions by a limiting procedure with an appropriate multiplicative renormalization. How this works in details is described by the operator product expansions or fusion rules [21,5] which we discuss, from the present point of view, in [25].

6. SO(3) WESS-ZUMINO-WITTEN MODEL.

In the previous Section, we have studied the canoncially quantized WZW model with a simply connected compact group G taken to be $SU(2)$. Here, we shall examine the new topological aspect appearing if G is not simply connected : the presence of twisted sectors in the space of states.

To make the discussion free of group-theory complications, we shall take G to be $SO(3)$ and will denote by $\widetilde{G}$ its simply connected cover $SU(2)$ and by $G^{\mathbb{C}}$ its complexification equal $SL(2,\mathbb{C})/\mathbb{Z}_2$. 3-form γ $(\gamma^{\mathbb{C}})$ on G $(G^{\mathbb{C}})$ given by (5.1) is integral if and only if k is an even integer (the volume of G is half that of $\widetilde{G}$) . As before, we may take k positive. A representative of a single $w \in W^a(G^{\mathbb{C}},\gamma^{\mathbb{C}})$ whose restriction to G defines a single element of $W(G,\gamma)$ $(H^2(G^{\mathbb{C}},\mathbb{C}^*) = 1 = H^2(G,U(1)))$ is given in Appendix 3. The loop space $LG^{\mathbb{C}}$ has two components, $L_oG^{\mathbb{C}}$ of contractible loops and $L_tG^{\mathbb{C}}$ of twisted loops, which lift to curves in $\widetilde{G}$ whose ends differ by -1 . Line bundle $\mathcal{L}^{\mathbb{C}}\restriction_{L_oG^{\mathbb{C}}} \equiv \mathcal{L}_o^{\mathbb{C}}$ may be identified with $\widetilde{\mathcal{L}}^{\mathbb{C}}/\mathbb{Z}_2$ where $\widetilde{\mathcal{L}}^{\mathbb{C}}$ is the line bundle over $L\widetilde{G}^{\mathbb{C}}$ (recall that $\mathbb{Z}_2 \subset \widetilde{G}^{\mathbb{C}} \subset \widehat{L\widetilde{G}^{\mathbb{C}}} \subset \mathcal{L}^{\mathbb{C}}$) . Thus $\Gamma^a(\mathcal{L}_o^{\mathbb{C}})$ may be identified with the subspace $\Gamma^a_{even}(\widetilde{\mathcal{L}}^{\mathbb{C}})$. Since the action of $\widehat{L\widetilde{G}^{\mathbb{C}}} \times \widehat{L\widetilde{G}^{\mathbb{C}}}$ on $\Gamma^a(\mathcal{L}^{\mathbb{C}})$ commutes with -1 , we obtain its representation in $\Gamma^a(\mathcal{L}_o^{\mathbb{C}})$ whose LW states correspond to even LW states in $\Gamma^a(\widetilde{\mathcal{L}}^{\mathbb{C}})$, i.e. to integer spins $j_1 = j_2 \leq \frac{k}{2}$ appearing with multiplicity one.

We still have to study $\mathcal{L}^{\mathbb{C}}\restriction_{L_tG^{\mathbb{C}}} \equiv \mathcal{L}_t^{\mathbb{C}}$. We shall define left and right actions of $\widehat{L\widetilde{G}^{\mathbb{C}}}$ on $\mathcal{L}_t^{\mathbb{C}}$ (again fixed completely by the rules of geometric quantizations of the classical symmetries of the theory). This actions gives rise to a representation of $\widehat{L\widetilde{G}^{\mathbb{C}}} \times \widehat{L\widetilde{G}^{\mathbb{C}}}$ in $\Gamma^a(\mathcal{L}_t^{\mathbb{C}})$. Classifying the LW vectors for this representation, we shall recover the spectrum of the model derived originally by Gepner and Witten from the one for the $SU(2)$ case by use of the modular transformation properties of the characters of the $\widehat{su}(2)$ Kac-Moody algebra[5].

$$\text{Let} \quad e_{1/2} = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}\mathbb{Z}_2 \in L_tG^{\mathbb{C}} . \text{ Clearly, } L_tG^{\mathbb{C}} = L_oG^{\mathbb{C}}e_{1/2} .$$

We shall represent elements of $\mathcal{L}_t^{\mathbb{C}}$ by probality amplitudes of the WZW model. Let $D'_\infty = \{z \in D_\infty \mid |z| \leq 2\}$. $S_2 = \{z \in D_\infty \mid |z| = 2\} \cong S^1$. Let $g : D'_\infty \to G^{\mathbb{C}}$ be such that $g\restriction_{S_2} = e_{1/2}$. Let $\ell \in \mathcal{L}^{\mathbb{C}}_{e_{1/2}}$. Then $\ell e^{S_{D'_\infty}(g)}$ ($e^{S_{D'_\infty}(g)}$ is dual to $e^{-S_{D'_\infty}(g)}$) may be naturally considered as an element of $\mathcal{L}^{\mathbb{C}}_{g\restriction\partial D_o}$ and every element in $\mathcal{L}_t^{\mathbb{C}}$ can be described this way. If $g_1 : D'_\infty \to G^{\mathbb{C}}$, $g_1\restriction_{\partial D'_\infty} = g\restriction_{\partial D'_\infty}$, then $g_1 = gh$ where $h\restriction_{\partial D'_\infty} = 1$. We want to compare $\ell e^{S_{D'_\infty}(g_1)}$ and $\ell e^{S_{D'_\infty}(g)}$. Two cases may arise. If h lifts to $\widetilde{h} : D'_\infty \to \widetilde{G}^{\mathbb{C}}$, $\widetilde{h}\restriction_{\partial D'_\infty} = 1$, then there exists a homotopy h_t s.t. $h_o = 1$, $h_1 = h$ and $h_t\restriction_{\partial D'_\infty} = 1$. Polyakov's formula (5.11) applies and gives

$$\ell e^{S_{D'_\infty}(gh)} = e^{S_{D'_\infty}(h)-\Gamma_{D'_\infty}(g,h)} \ell e^{S_{D'_\infty}(g)} . \tag{1}$$

If h lifts to $\tilde{h} : D'_\infty \to \tilde{G}^{\mathbb{C}}$ s.t. $\tilde{h}\restriction_{\partial D_\infty} = 1$, and $\tilde{h}\restriction_{S_2} = -1$ then there exists a homotopy h_t, $h_o = 1$, $h_1 = h$, s.t. $h_t\restriction_{\partial D_\infty} = 1$ and $h_t\restriction_{S_2} = e_{1/2}(e^{2\pi it})$. In this case, using formula (A.4.6) (Appendix 4) we see that

$$\frac{d}{dt}[S_{D'_\infty}(gh_t)-S_{D'_\infty}(h_t)+\Gamma_{D'_\infty}(g,h_t)]$$
$$= \frac{ik}{4\pi}\int_{S_2} \operatorname{tr} h_t^{-1}\frac{\partial h_t}{\partial t} g^{-1}dg = -\frac{i\pi k}{2} . \tag{2}$$

(Notice that forms ω_α and $\eta_{\alpha_0\alpha_1}$ given in Appendix 3 do not contribute to (2)). Thus in this case

$$\ell e^{S_{D'_\infty}(gh)} = (-1)^{k/2} e^{S_{D'_\infty}(h)-\Gamma_{D'_\infty}(g,h)} \ell e^{S_{D'_\infty}(g)} . \tag{3}$$

Similarly,

$$\ell e^{S_{D'_\infty}(hg)} = ((-1)^{k/2}) e^{S_{D'_\infty}(h)-\Gamma_{D'_\infty}(h,g)} \ell e^{S_{D'_\infty}(g)} \tag{4}$$

where $(-1)^{k/2}$ appears if h lifts to $\tilde{h}$ with $\tilde{h}\restriction_{\partial D_\infty} = 1$ and $\tilde{h}\restriction_{S_2} = -1$.

Before describing the action of $\widehat{L\tilde{G}^{\mathbb{C}}}$ on $L_t^{\mathbb{C}}$, let us rewrite the one on $\mathcal{T}^{\mathbb{C}}$ in a different form. If $g_1 : D_o \to \tilde{G}^{\mathbb{C}}$, $g : D_\infty \to \tilde{G}^{\mathbb{C}}$ and $\tilde{g}_1, \tilde{g}$ are their extensions to $\Sigma = P\mathbb{C}^1$ then

$$(\lambda_1 e^{-S_{D_o}(g_1)})\cdot(\lambda_2 e^{S_{D_\infty}(g)}) = \lambda_1\lambda_2\, e^{S_\Sigma(\tilde{g})} e^{-S_{D_o}(g_1)}\cdot e^{-S_{D_o}(\tilde{g})}$$
$$= \lambda_1\lambda_2 e^{S_\Sigma(\tilde{g})} e^{-S_{D_o}(g_1\tilde{g})-\Gamma_{D_o}(g_1,\tilde{g})}$$
$$= \lambda_1\lambda_2\, e^{S_\Sigma(\tilde{g})-S_\Sigma(\tilde{g}_1\tilde{g})-\Gamma_{D_o}(g_1,\tilde{g})} e^{S_{D_\infty}(\tilde{g}_1 g)}$$
$$= \lambda_1\lambda_2\, e^{-S_\Sigma(\tilde{g}_1)+\Gamma_{D_\infty}(\tilde{g}_1,g)} e^{S_{D_\infty}(\tilde{g}_1 g)} . \tag{5}$$

Similarly,

$$(\lambda_2 e^{S_{D_\infty}(g)})\cdot(\lambda_1 e^{-S_{D_o}(g_1)}) = \lambda_1\lambda_2 e^{-S_\Sigma(\tilde{g}_1)+\Gamma_{D_\infty}(g,\tilde{g}_1)} e^{S_{D_\infty}(g\tilde{g}_1)} . \tag{6}$$

Eqs. (5) and (6) may serve as a guide-line in the definition of the action of $\widehat{L\tilde{G}^{\mathbb{C}}}$ on $L_t^{\mathbb{C}}$. Let $g_1, g_2 : D_o \to \tilde{G}^{\mathbb{C}}$, $\tilde{g}_1, \tilde{g}_2$ be their extensions to $D_o \cup D'_\infty$ s.t. $\tilde{g}_1, \tilde{g}_2\restriction_{S_2} = 1$ and $g : D'_\infty \to G^{\mathbb{C}}$, $g\restriction_{S_2} = e_{1/2}$. Let $\Sigma' = D_o \cup D'_\infty$. We shall define

$$(\lambda_1 e^{-S_{D_o}(g_1)})\cdot(\ell e^{S_{D'_\infty}(g)}) = \lambda_1 e^{-S_{\Sigma'}(\tilde{g}_1)+\Gamma_{D'_\infty}(\tilde{g}_1,g)}\ell e^{S_{D'_\infty}(\tilde{g}_1 g)} \tag{7}$$

$$(\ell e^{S_{D'_\infty}(g)})\cdot(\lambda_2 e^{-S_{D_o}(g_2)}) = \lambda_2 e^{-S_{\Sigma'}(\tilde{g}_2)+\Gamma_{D'}(g,\tilde{g}_2)}\ell e^{S_{D'_\infty}(g\tilde{g}_2)} . \tag{8}$$

A straightforward verification shows that this defines commuting left and right actions of $\widehat{L\tilde{G}^{\mathbb{C}}}$ on $L_t^{\mathbb{C}}$. Notice how in particular -1 acts on $L_t^{\mathbb{C}}$. If $\tilde{g}_1$ prolongs -1 on D_o to D'_∞ , $\tilde{g}_1\restriction_{S_2} = 1$, then, using (4) and (3), we obtain

$$\begin{aligned} e^{-S_{D_o}(-1)}\cdot(\ell e^{S_{D'_\infty}(g)}) &= e^{-S_{\Sigma'}(\tilde{g}_1)+\Gamma_{D'_\infty}(\tilde{g}_1,g)}\ell e^{S_{D'_\infty}(\tilde{g}_1 g)} \\ &= (-1)^{k/2} e^{-S_{\Sigma'}(\tilde{g}_1)+S_{D'_\infty}(\tilde{g}_1)+\Gamma_{D'_\infty}(\tilde{g}_1,g)-\Gamma_{D'_\infty}(\tilde{g}_1,g)}\ell e^{S_{D'_\infty}(g)} \\ &= (-1)^{k/2}\ell e^{S_{D'_\infty}(g)} = (\ell e^{S_{D'_\infty}(g)})\cdot e^{-S_{D_o}(-1)} . \end{aligned} \tag{9}$$

The actions of $\widehat{L\tilde{G}^{\mathbb{C}}}$ on $L_t^{\mathbb{C}}$ carry over to $\Gamma^a(L_t^{\mathbb{C}})$ inducing a representation $\ell\times r$ of $\widehat{L\tilde{G}^{\mathbb{C}}}\times\widehat{L\tilde{G}^{\mathbb{C}}}$ on the sector of twisted states :

$$(\ell(\hat{g}_1)r(\hat{g}_2)\psi)(g) = \hat{g}_1\cdot\psi(g_1^{-1}g g_2^{+^{-1}})\cdot\hat{g}_2^{+} . \tag{10}$$

This representation splits again into irreducible LW representations built over LW states satisfying condition (5.29). We shall find all LW states in $\Gamma^a(L_t^{\mathbb{C}})$. Notice, that due to (9), LW states can involve only integer spins if $k/2$ is even and only half-integer ones if $k/2$ is odd. In fact, the possible spins are more restricted.

As the LW condition of (5.29) fixes the states along orbits $N^+g(N^+)^+$ in $L_tG^{\mathbb{C}}$, we have to study the geometry of the latter. Under the mapping $L_tG^{\mathbb{C}}\ni g\mapsto g e_{1/2}\in L_oG^{\mathbb{C}}$, they become orbits N^+gM^- in $L_oG^{\mathbb{C}}$ where $M^- = \{g\in L^- : g(0) = \begin{pmatrix} * & 0\\ * & *\end{pmatrix}\}$. Their classification requires an easy refinement of the classification of orbits N^+gL^- in $L\tilde{G}^{\mathbb{C}}$ given in the previous sections. Namely, orbits N^+gM^- have the form

$$\Sigma_n^1 = N^+\varepsilon_n M^- \quad\text{and}\quad \Sigma_n^2 = N^+e_n M^- \tag{11}$$

where $\varepsilon_n = \begin{pmatrix} 0 & -z^n\\ z^{-n} & 0\end{pmatrix}\mathbb{Z}_2$, $e_n = \begin{pmatrix} z^n & 0\\ 0 & z^{-n}\end{pmatrix}\mathbb{Z}_2$ and n is integer (as is easy to see $\Sigma_n^1\cup\Sigma_n^2 = \Sigma_n/\mathbb{Z}_2$, where Σ_n are the orbits discussed in Section 5). Let $N_o^{+1} = \{g\in N^+ | g(0) = \begin{pmatrix} 1 & 0\\ * & 1\end{pmatrix}\}$, $N_n^{+1} = N^+\cap e_n N_o^{+1} e_n^{-1}$, $N_n^{+2} = N^+\cap\varepsilon_n N_o^{+1}\varepsilon_n^{-1}$. Then $\Sigma_n^1 = N_n^{+1}\varepsilon_n M^-$ and $\Sigma_n^2 = N_n^{+2}e_n M^-$ and these identities lead to unique decompositions of the elements of the orbits. Moreover, we have splittings :

$$e_n N_o^{+1} e_n^{-1} = N_n^{+1} N_n^{-1} \ , \ \varepsilon_n N_o^{+1} \varepsilon_n^{-1} = N_n^{+2} N_n^{-2} \tag{12}$$

where $N_n^{-1} = N^- \cap e_n N_o^{+1} e_n^{-1}$ and $N_n^{-2} = N^- \cap \varepsilon_n N_o^{+1} \varepsilon_n^{-1}$. Σ_o^1 is an open dense orbit. Since

$$U_n^1 = e_n \Sigma_o^1 = e_n N_o^{+1} \varepsilon_o M^- = N_n^{+1} N_n^{-1} \varepsilon_n M^- \ , \tag{13}$$

$$U_n^2 = \varepsilon_n \Sigma_o^1 = N_n^{+2} N_n^{-2} e_n M^- \ , \tag{14}$$

the codimension of Σ_n^1 is equal $\dim N_n^{-1} = \begin{cases} 2n \ , & n > 0 \\ -2n \ , & n \le 0 \end{cases}$ and the codimension of Σ_n^2 equals $\dim N_n^{-2} = \begin{cases} 2n-1 \ , & n > 0 \\ -2n+1 \ , & n \le 0 \end{cases}$. Besides

$$U_n^1 \setminus \Sigma_n^1 \subset \begin{cases} \bigcup\limits_{|m|<n} (\Sigma_m^1 \cup \Sigma_m^2) \cup \Sigma_n^2 \ , & n > 0 \ , \\ \bigcup\limits_{n<m\le -n} (\Sigma_m^1 \cup \Sigma_m^2) \ , & n \le 0 \ , \end{cases} \tag{15}$$

and

$$U_n^2 \setminus \Sigma_n^2 \subset \begin{cases} \bigcup\limits_{|m|<n} (\Sigma_m^1 \cup \Sigma_m^2) \ , & n > 0 \ , \\ \bigcup\limits_{n<m\le -n} (\Sigma_m^1 \cup \Sigma_m^2) \cup \Sigma_n^1 \ , & n < 0 \ , \end{cases} \tag{16}$$

which establishes the hierarchy of the orbits. Proofs of the facts listed above are elementary.

Let us use relation (5.29) to compute a LW state with spins j_1 and j_2 on the open dense orbit $\Sigma_o^1 e_{1/2}^{-1} = N^+ \varepsilon_{1/2} (N^+)^\dagger$ of $N^+ \times N^+$ in $LG_t^{\mathbb{C}}$. Let $\tilde{\varepsilon}_{1/2} : D'_\infty \to G^{\mathbb{C}}$ be such that $\tilde{\varepsilon}_{1/2}\restriction_{\partial D_o} = \varepsilon_{1/2}$, $\tilde{\varepsilon}_{1/2}\restriction_{S_2} = e_{1/2}$. For a LW state ψ , represent

$$\psi(\varepsilon_{1/2}) = \ell e^{S_{D'_\infty}(\tilde{\varepsilon}_{1/2})} \tag{17}$$

for $\ell \in L_{e_{1/2}}$. Then for $g_1, g_2 \in N^+$, $g_1(0) = \begin{pmatrix} a_1 & 0 \\ * & a_1^{-1} \end{pmatrix}$, $g_2(0) = \begin{pmatrix} a_2 & 0 \\ * & a_2^{-1} \end{pmatrix}$

$$\begin{aligned} \psi(g_1 \varepsilon_{1/2} g_2^\dagger) &= a_1^{2j_1} \bar{a}^{2j_2} (\ell(g_1) r(g_2)\psi)(g_1 \varepsilon_{1/2} g_2^\dagger) \\ &= a_1^{2j_1} \bar{a}_2^{2j_2} g_1 \cdot (\ell e^{S_{D'_\infty}(\tilde{\varepsilon}_{1/2})}) \cdot g_2^\dagger \\ &= a_1^{2j_1} \bar{a}_2^{2j_2} e^{-S_{\Sigma'}(\tilde{g}_1) + \Gamma_{D'_\infty}(\tilde{g}_1, \tilde{\varepsilon}_{1/2})} (\ell e^{S_{D'_\infty}(\tilde{g}_1 \tilde{\varepsilon}_{1/2})}) \cdot g_2^\dagger \\ &= a_1^{2j_1} \bar{a}^{2j_2} e^{-S_{\Sigma'}(\tilde{g}_1) - S_{\Sigma'}(\widetilde{g_2^\dagger}) + \Gamma_{D'_\infty}(\tilde{g}_1, \tilde{\varepsilon}_{1/2}) + \Gamma_{D'_\infty}(\tilde{g}_1 \tilde{\varepsilon}_{1/2}, \widetilde{g_2^\dagger})} \\ &\qquad \ell e^{S_{D'_\infty}(\tilde{g}_1 \tilde{\varepsilon}_{1/2} g_2^\dagger)} \end{aligned} \tag{18}$$

where $\tilde{g}_i$ extends g_i to Σ' so that $\tilde{g}_i\restriction_{S_2} = 1$, $i = 1,2$. For (18) to determine ψ on the dense open orbit $\Sigma_o^1 e_{1/2}^{-1}$ it is necessary and sufficient that the right hand side does not change when $g_1 \mapsto g_1 \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $g_2 \mapsto \pm g_2 \begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}$. As the action of -1 on the sections was already studied, we can consider only the + sign case. Using equation (5.13), we obtain after an easy algebra

$$\frac{d}{da} [S_{D'_\infty}(\tilde{g}_1\tilde{\varepsilon}_{1/2}\tilde{g}_2^{\dagger}) - S_{\Sigma'}(\tilde{g}_1) - S_{\Sigma'}(\tilde{g}_2^{\dagger}) + \Gamma_{D'_\infty}(\tilde{g}_1, \tilde{\varepsilon}_{1/2}) + \Gamma_{D'_\infty}(\tilde{g}_1\tilde{\varepsilon}_{1/2}, \tilde{g}_2^{\dagger})]$$

$$= -\frac{ik}{2\pi} \int_{\partial D_o} \mathrm{tr}\, g_1^{-1} \frac{dg_1}{da}\, \varepsilon_{1/2} d\varepsilon_{1/2}^{-1} = -\frac{k}{a}. \tag{19}$$

On the other hand,

$$\frac{d}{da} \log a_1^{2j_1} \bar{a}_2^{2j_2} = 2(j_1+j_2)\frac{1}{a} \tag{20}$$

so that the right hand side of (18) is a-independent if and only if

$$j_1+j_2 = \frac{k}{2}. \tag{21}$$

For a pair of spins satisfying this condition, integer for even $\frac{k}{2}$ and half-integer for odd $\frac{k}{2}$, eq. (18) defines a holomorphic section of $L^{\mathbb{C}}$ over $\Sigma_o^1 e_{1/2}^{-1}$ up to a constant factor. Remains to be seen if we can extend this section to a holomorphic section over entire $L_t G^{\mathbb{C}}$. Obstructions can arise only when extending the section to orbits of codimension 1, i.e. to $\Sigma_o^2 e_{1/2}^{-1}$ and $\Sigma_1^2 e_{1/2}^{-1}$.

By eq. (14), any element of $U_o^2 e_{1/2}^{-1}$ may be uniquely written as $g_1' \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} e_{1/2}^{-1} g_2'^{\dagger}$ where $g_1' \in N_o^{+2}$, $g_2' \in N^+$. It belongs to $\Sigma_o^2 e_{1/2}^{-1} \subset U_o^2 e_{1/2}^{-1}$ if and only if $b = 0$, for $b \neq 0$ being an element of $\Sigma_o^1 e_{1/2}^{-1}$. Indeed,

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} e_{1/2}^{-1} = \begin{pmatrix} 1 & 0 \\ b^{-1} & 1 \end{pmatrix} \varepsilon_{1/2} \begin{pmatrix} -\bar{b}^{-1} & -z \\ 0 & -\bar{b} \end{pmatrix}^{\dagger} \equiv g_{1b}\varepsilon_{1/2} g_{2b}^{\dagger}. \tag{22}$$

Now, $\psi(g_1' g_{1b} \varepsilon_{1/2} g_{2b}^{\dagger} g_2'^{\dagger}) \equiv \psi(g_b)$ is given by the right hand side of (18) with $g_1' g_{1b} \mapsto g_1$, $g_2' g_{2b} \mapsto g_2$. Writing it in a somewhat more convenient way, we obtain

$$\psi(g_b) = a_1^{2j_1} \bar{a}_2^{2j_2}\, e^{-S_{\Sigma'}(\tilde{g}_{1b}) - S_{\Sigma'}(\tilde{g}_{2b}^{\dagger}) + \Gamma_{D'_\infty}(\tilde{g}_{1b}, \tilde{\varepsilon}_{1/2}) + \Gamma_{D'_\infty}(\tilde{g}_{1b}\tilde{\varepsilon}_{1/2}, \tilde{g}_{2b}^{\dagger})}$$
$$g_1' \cdot (\ell e^{S_{D'_\infty}(\tilde{g}_{1b}\tilde{\varepsilon}_{1/2}\tilde{g}_{2b}^{\dagger})}) \cdot g_2'^{\dagger} \tag{23}$$

where $g_1'(0) = \begin{pmatrix} a_1 & 0 \\ * & a_1^{-1} \end{pmatrix}$ and $g_2'(0) \begin{pmatrix} -\bar{b}^{-1} & 0 \\ 0 & -\bar{b} \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ * & a_2^{-1} \end{pmatrix}$. Hence

$a_1^{2j_1} \bar{a}_2^{2j_2} = O(b^{-2j_2})$ when $b \to 0$. To study the asymptotics of the exponential expression on the right hand side of (23), we apply formula (A.4.6) to infer that

$$\frac{d}{db}\,[S_{D'_\infty}(\tilde{g}_{1b}\tilde{\varepsilon}_{1b}\tilde{g}^{+}_{2b})-S_{\Sigma'}(\tilde{g}_{1b})-S_{\Sigma'}(\tilde{g}^{+}_{2b})+\Gamma_{D'_\infty}(\tilde{g}_{1b},\tilde{\varepsilon}_{1b})+\Gamma_{D'_\infty}(\tilde{g}_{1b}\tilde{\varepsilon}_{1/2},\tilde{g}^{+}_{2b})]$$

$$= -\frac{ik}{2\pi}\int_{\partial D_o} \mathrm{tr}\,[\,\frac{\partial \tilde{g}^{+-1}_{2b}}{\partial b}\,\bar{\partial}\tilde{g}^{+}_{2b}+\tilde{g}^{-1}_{1b}\,\frac{\partial\tilde{g}_{1b}}{\partial b}\,\tilde{\varepsilon}_{1/2}\tilde{g}^{+}_{2b}d(g^{+-1}_{2b}\,\varepsilon^{-1}_{1/2})]$$

$$+\, O(1) = \frac{k}{b} + O(1) \qquad (24)$$

so that

$$\psi(g_b) = O(b^{k-2j_2}) \qquad (25)$$

and $\psi(g_b)$ is analytic at $b = 0$ since $j_2 \le \frac{k}{2}$.

Similarly, every element of $U_1^2 e_{1/2}^{-1}$ may be uniquely written as $g'_1\begin{pmatrix}1 & 0\\ cz^{-1} & 1\end{pmatrix} e_{1/2}g'^{+}_2$ where $g'_1 \in N_1^{+2}$ and $g'_2 \in N^+$. It belongs to $\Sigma_1^2 e_{1/2}^{-1}$ if and only if $c = 0$, for $c \neq 0$ being an element of $\Sigma_o^1 e_{1/2}^{-1}$. Since

$$\begin{pmatrix}1 & 0\\ cz^{-1} & 1\end{pmatrix} e_{1/2} = \begin{pmatrix}1 & \frac{z}{c}\\ 0 & 1\end{pmatrix}\varepsilon_{1/2}\begin{pmatrix}\bar{c} & 0\\ 1 & \frac{1}{c}\end{pmatrix}^{+} \equiv g_{1c}\varepsilon_{1/2}g^{+}_{2c}\ , \qquad (26)$$

$\psi(g'_1 g_{1c}\varepsilon_{1/2}g^{+}_{2c}g'^{+}_2) \equiv \psi(g_c)$ is given by the right hand side of (23) with $c \to b$ and a_1, a_2 defined by relations $g'_1(0) = \begin{pmatrix}a_1 & 0\\ * & a_1^{-1}\end{pmatrix}$, $g'_2(0)\begin{pmatrix}\bar{c} & 0\\ 1 & \frac{1}{\bar{c}}\end{pmatrix} = \begin{pmatrix}a_2 & 0\\ * & a_2^{-1}\end{pmatrix}$. Thus $a_1^{2j_1}\bar{a}_2^{2j_2} = O(c^{2j_2})$. On the other hand, the left equality of (24) still holds (with $c \mapsto b$) and produces a regular contribution. Hence

$$\psi(g_c) = O(c^{2j_2}) \qquad (27)$$

and ψ is analytic at $c = 0$. By induction based on the Hartogs theorem, ψ extends now to a global holomorphic section, for which (5.29) follows by continuity.

Summarizing, in the twisted sector the LW states correspond with multiplicity 1, to spins j_1, j_2 , $j_1+j_2 = \frac{k}{2}$, j_1, j_2 integer for $\frac{k}{2}$ even and half-integers if $\frac{k}{2}$ is odd. This coincides with the Gepner-Witten result[5].

For both SU(2) and SO(3) case, the spectrum of the LW states found here gives immediately modular invariant partition functions of the theory on the torus. These are the two infinite series from all possible modular invariants built from bilinears in $\widehat{su}(2)$ Kac-Moody algebra characters, see[26,27].

APPENDIX 1.

We shall describe explicitly the magnetic-monopole line bundle. The monopole curvature 2-form

$$\omega = \frac{1}{2} e\mu \sum_{i,j,k=1}^{3} \varepsilon^{ijk} \frac{x^i}{|\vec{x}|^3} dx^j dx^k , \tag{1}$$

see (1.3), extends to a (2,0)-form $\omega^{\mathbb{C}}$ on $M^{\mathbb{C}} = \{\vec{y} \in \mathbb{C}^3 | \vec{y}^{\,2} \notin]-\infty,0]\}$ given by (1) with $\vec{y} \mapsto \vec{x}$ and $|\vec{y}| \equiv \sqrt{\vec{y}^{\,2}}$ (with the square root cut along the negative axis). Cover $M^{\mathbb{C}}$ by $\{\mathcal{O}_{-1}, \mathcal{O}_o, \mathcal{O}_1\}$,

$$\begin{aligned} \mathcal{O}_{-1} &= \{\vec{y} \mid \mathrm{Re}\, \frac{y_1}{|\vec{y}|} > 0\} , \\ \mathcal{O}_{0} &= \{\vec{y} \mid \frac{y_2^2+y_3^2}{\vec{y}^{\,2}} \notin]-\infty,0]\} , \\ \mathcal{O}_{1} &= \{\vec{y} \mid \mathrm{Re}\, \frac{y_1}{|\vec{y}|} < 0\} . \end{aligned} \tag{2}$$

For $e\mu \in \frac{1}{2}\mathbb{Z}$, put

$$\begin{aligned} \eta_{-1} &= e\mu\left(\frac{y_1}{|\vec{y}|} - 1\right) \frac{y_3 dy_2 - y_2 dy_3}{y_2^2+y_3^2} = \eta_o , \\ \eta_1 &= e\mu\left(\frac{y_1}{|\vec{y}|} + 1\right) \frac{y_3 dy_2 - y_2 dy_3}{y_2^2+y_3^2} , \\ g_{-10} &= 1 , \; g_{01} = \left(\frac{iy_2+y_3}{\sqrt{\frac{y_2^2+y_3^2}{\vec{y}^{\,2}}}\, |\vec{y}|} \right)^{2e\mu} . \end{aligned} \tag{3}$$

$(g_{\alpha_0\alpha_1}, \eta_\alpha)$ defines the unique element of $Q^a(M^{\mathbb{C}}, \omega^{\mathbb{C}})$ and, by restriction to $\mathbb{R}^3 \setminus \{0\}$, the unique element of $Q(M,\omega)$, i.e. the monopole line bundle.

APPENDIX 2.

The group of orientation preserving reparametrizations of S^1, Diff_+S^1, acts on the loop space LM by $\phi \mapsto \phi\circ\tau^{-1}$, $\tau \in \mathrm{Diff}_+S^1$. The canonical isomorphism of the fibers of the line bundle L over loops ϕ and $\phi\circ\tau^{-1}$ may be used to lift this action to the action of Diff_+S^1 on L preserving the hermitian structure and the connection given by (3.10). Here we shall show that the connection projects from L to the line bundle L/Diff_+S^1 over LM/Diff_+S^1. One has to check that if (τ_t) is a curve in Diff_+S^1, $\tau_o = \mathrm{id}$, then its lift $\ell(t) = \tau(t)\ell_o$ to L, $\ell_o \in L$, is horizontal.

Let in local presentation

$$\ell_o = (\mathcal{U}_A, \phi, z) \tag{1}$$

where $\mathcal{U}_A$ is given by (3.8), $\phi \in \mathcal{U}_A \subset LM$ and $z \in \mathbb{C}$. Then

$$\ell(t) = (\mathcal{U}_{A_t}, \phi\circ\tau_t^{-1}, z) \tag{2}$$

where if A corresponds to the triangulation of S^1 by intervals b and vertices v and to assignment α_b, α_v then A_t is related to the triangulation by $\tau_t(b)$ and $\tau_t(v)$ with the unchanged assignment of α's. Changing the trivialization by the transition function G_{A_tA} given by (3.9), we obtain

$$\ell(t) = (\mathcal{U}_A, \phi\circ\tau_t^{-1}, G_{A_tA}(\phi\circ\tau_t^{-1})z) \ . \tag{3}$$

We have to show that the covariant derivative

$$\frac{D}{dt}\,\ell(t) = [G_{A_tA}(\phi\circ\tau_t^{-1})^{-1}\,\frac{d}{dt}\,G_{A_tA}(\phi\circ\tau_t^{-1}) - i\langle X_t, E_A\rangle]\,\ell(t) = 0 \tag{4}$$

where

$$X_t = \frac{\partial}{\partial t}\,(\phi\circ\tau_t^{-1}) \ , \tag{5}$$

see (3.10). It is enough to show (4) for $t = 0$. For simplicity, let us assume that $\tau_t(v) > v$ for each vertex v of the triangulation of S^1 (in the natural order on the circle). The general case proceeds the same way. Denote by b_v^+ and b_v^- two intervals b bordering on v, b_v^+ later than b_v^-, see Fig. 4. From eq. (3.9), we compute

$$G_{A_tA}(\phi\circ\tau_t^{-1}) = \exp[i \sum_v \int_v^{\tau_t(v)} (\phi\circ\tau_t^{-1})^* \eta_{\alpha_{b_v^-}\alpha_{b_v^+}}]$$

$$\cdot \prod_v \frac{g_{\alpha_v\alpha_{b_v^-}\alpha_{b_v^+}}(\phi\circ\tau_t^{-1}(v))}{g_{\alpha_v\alpha_{b_v^-}\alpha_{b_v^+}}(\phi(v))} \ . \tag{6}$$

On the other hand, by definition (3.10),

$$\langle X_o, E_A\rangle = -\sum_b \int_b \frac{\partial\tau_o}{\partial t} \rfloor \phi^*\omega_{\alpha_b} - \sum_{\substack{v,b\\ v\in\partial b}} \langle \frac{\partial\tau_o}{\partial t}(v), \phi^*\eta_{\alpha_v\alpha_b}\rangle$$

$$= \sum_v \langle \frac{\partial\tau_o}{\partial t}(v), \phi^*(\eta_{\alpha_v\alpha_{b_v^+}} - \eta_{\alpha_v\alpha_{b_v^-}})\rangle \quad . \tag{7}$$

Thus

$$G_{A_tA}(\phi)^{-1} \frac{d}{dt}\, G_{A_oA}(\phi\circ\tau_o^{-1}) - i\langle X_o, E_A\rangle$$

$$= i\sum_v \langle \frac{\partial\tau_o}{\partial t}(v), \phi^*(\eta_{\alpha_{b_v^-}\alpha_{b_v^+}} - \frac{1}{i}\, g^{-1}_{\alpha_v\alpha_{b_v^-}\alpha_{b_v^+}}\, dg_{\alpha_v\alpha_{b_v^-}\alpha_{b_v^+}}$$

$$- \eta_{\alpha_v\alpha_{b_v^+}} + \eta_{\alpha_v\alpha_{b_v^-}})\rangle = 0 \tag{8}$$

by (3.3).

APPENDIX 3.

In order to describe a representative of $w \in W^a(SL(2,\mathbb{C})\ ,\ \frac{k}{12\pi}\,\mathrm{tr}(g^{-1}dg)^3 \equiv \gamma^{\mathbb{C}})$ let us parametrice

$$SL(2,\mathbb{C}) = \{z_o + i\sum_{i=1}^{3} z_i\sigma_i \mid z_o, z_i \in \mathbb{C}\ ,\ z_o^2 + z^2 = 1\} \ . \tag{1}$$

We shall use the (stereographic) coordinates $\vec{\xi}, \vec{\zeta}$ on the subsets $z_o \notin\ ]-\infty,-1]$ and $z_o \notin [1,\infty[$ respectively, $\vec{\xi},\vec{\zeta} \in \mathbb{C}^3\ ,\ \vec{\xi}^{\,2}, \vec{\zeta}^{\,2} \notin\ ]-\infty,-1]$,

$$\frac{1-\vec{\xi}^{\,2}}{1+\vec{\xi}^{\,2}} = z_o = \frac{\vec{\zeta}^{\,2}-1}{\vec{\zeta}^{\,2}+1} \quad , \tag{2}$$

$$\frac{2\xi_i}{1+\vec{\xi}^{\,2}} = z_i = \frac{2\zeta_i}{\vec{\zeta}^{\,2}+1} \quad . \tag{3}$$

An easy computation gives

$$\gamma^{\mathbb{C}} = \frac{8k}{\pi}\, \frac{1}{(1+\vec{\xi}^{\,2})^3}\, d\xi^1 d\xi^2 d\xi^3 = -\frac{8k}{\pi}\, \frac{1}{(\vec{\zeta}^{\,2}+1)^3}\, d\zeta^1 d\zeta^2 d\zeta^3 \ . \tag{4}$$

Cover $SL(2,\mathbb{C})$ by

$$O_{-2} = \{|\vec{\xi}^{\,2}| < 1\}\ ,\quad O_{-1} = \{\mathrm{Re}\,\frac{\xi_1}{|\vec{\xi}|} > 0\}\ ,$$

$$O_0 = \{\frac{\xi_2^2+\xi_3^2}{\vec{\xi}^2} \notin\ ]-\infty,0]\}\ , \tag{5}$$

$$O_1 = \{\mathrm{Re}\,\frac{\xi_1}{|\vec{\xi}|} < 0\}\ ,\quad O_{-2} = \{|\vec{\zeta}^{\,2}| < 1\}$$

where $|\vec{\xi}| = \sqrt{\vec{\xi}^{\,2}}$. Let

$$\omega = \int_0^1 \frac{4kt^2}{\pi(t^2\,\vec{\xi}^2+1)}\,dt \sum_{i,j,k} \varepsilon^{ijk}\xi^i d\xi^j d\xi^k \tag{6}$$

and ω' be given by the same formula with $\vec{\zeta} \to \vec{\xi}$. Notice that they are well defined for $\vec{\xi}^2, \vec{\zeta}^2 \notin\ ]-\infty,-1]$ and that $d\omega = \gamma = -d\omega'$ there. Define for integer k

$$\omega_{-2} = \omega\ ,\quad \omega_{-1} = \omega\ ,\quad \omega_0 = \omega\ ,\quad \omega_1 = \omega\ ,\quad \omega_2 = -\omega'\ ,$$

$$\eta_{-2,-1} = 0\ ,\ \eta_{-2,0} = 0\ ,\ \eta_{-2,1} = 0\ ,\ \eta_{-1,0} = 0\ ,\ \eta_{0,1} = 0\ ,$$

$$\eta_{-1,2} = \frac{k}{2}\left(\frac{\xi_1}{|\vec{\xi}|} - 1\right)\frac{\xi_3 d\xi_2 - \xi_2 d\xi_3}{\xi_2^2+\xi_3^2} = \eta_{0,2}\ ,$$

$$\eta_{1,2} = \frac{k}{2}\left(\frac{\xi_1}{|\vec{\xi}|} + 1\right)\frac{\xi_3 d\xi_2 - \xi_2 d\xi_3}{\xi_2^2+\xi_3^2}\ , \tag{7}$$

$$g_{-2,-1,0} = 1\ ,\quad g_{-2,0,1} = 1\ ,\quad g_{-1,0,2} = 1\ ,$$

$$g_{0,1,2} = \left(\frac{i\xi_2+\xi_3}{\sqrt{\frac{\xi_2^2+\xi_3^2}{\vec{\xi}^2}}\ |\vec{\xi}|}\right)^k .$$

It is easy to check that $(g_{\alpha_0\alpha_1\alpha_2}, \eta_{\alpha_0\alpha_1}, \omega_\alpha)$ determines the unique element of $W^a(SL(2,\mathbb{C}),\gamma^{\mathbb{C}})$ and, by restriction to $SU(2)$ (i.e. to $\vec{\xi}$ and $\vec{\zeta}$ real), the unique element of $W(SU(2),\gamma)$. Notice the relation between (7) and the local data of the monopole bundle of Appendix 1.

For even k , define another representative of $w \in W^a(SL(2,\mathbb{C}),\gamma^{\mathbb{C}})$ by putting

$$\tilde{\omega}_{-2} = \omega\ ,\quad \tilde{\omega}_2 = -\omega'\ ,$$

$$\tilde{\omega}_{-1} = \omega + \frac{k}{8|\vec{\xi}|^3}\sum_{i,j,k} \varepsilon^{ijk}\xi^i d\xi^j d\xi^k = \tilde{\omega}_1 = \tilde{\omega}_0\ ,$$

$$\tilde{\eta}_{-2,-1} = \frac{k}{4}\left(\frac{\xi_1}{|\vec{\xi}|} - 1\right)\frac{\xi_3 d\xi_2 - \xi_2 d\xi_3}{\xi_2^2+\xi_3^2} = \tilde{\eta}_{-2,0} = \tilde{\eta}_{-1,2}\ ,$$

$$\tilde{\eta}_{-2,1} = \frac{k}{4}\left(\frac{\xi_1}{|\vec{\xi}|}+1\right)\frac{\xi_3 d\xi_2 - \xi_2 d\xi_3}{\xi_2^2+\xi_3^2} = \tilde{\eta}_{0,2} \ ,$$
$$\| \qquad \tilde{\eta}_{1,2}$$

$$\tilde{\eta}_{-1,0} = 0 = \tilde{\eta}_{0,1} \ ,$$

$$\tilde{g}_{-2,-1,0} = 1 = \tilde{g}_{0,1,2} \ ,$$

$$\tilde{g}_{-2,0,1} = \left(\frac{i\xi_2-\xi_3}{\sqrt{\frac{\xi_2^2+\xi_3^2}{\vec{\xi}^2}}\ |\vec{\xi}|}\right)^{k/2} \ ,$$

$$\tilde{g}_{-1,0,2} = \left(\frac{i\xi_2+\xi_3}{\sqrt{\frac{\xi_2^2+\xi_3^2}{\vec{\xi}^2}}\ |\vec{\xi}|}\right)^{k/2} \ .$$

The multiplication by -1 in $SL(2,\mathbb{C})$ corresponds to $\vec{\xi} \to -\frac{\vec{\xi}}{\vec{\xi}^2}$, $\vec{\zeta} \to -\frac{\vec{\zeta}}{\vec{\zeta}^2}$. A straightforward check shows that $(\tilde{g},\tilde{\eta},\tilde{\omega})$ is invariant under this transformation so that it defines the unique element in $W^a(SL(2,\mathbb{C})/\mathbb{Z}_2,\gamma^{\mathbb{C}})$ and in $W(SO(3),\gamma)$.

APPENDIX 4.

We shall compute here $\frac{d}{dt} S_\Sigma(g_t)$ for a t-dependent map $g_t : \Sigma \to G^{\mathbb{C}}$ and Σ any oriented 2-dimensional compact surface with boundary. First, we shall differentiate the $A_\Sigma(g_t)$ part of the amplitude, see (5.2), as given by formula (3.6)

$$-\frac{d}{dt}\log A_\Sigma(g_t) = \frac{d}{dt}\Big[-i\sum_c \int_c g_t^*\omega_{\alpha_c} + i\sum_{\substack{b,c\\ b\subset\partial c}} \int_b g_t^*\eta_{\alpha_b\alpha_c}\Big] - \sum_{\substack{v,b,c\\ v\in\partial b\\ b\subset\partial c}} g_{\alpha_v\alpha_b\alpha_c}(g_t(v))^{-1}\frac{d}{dt} g_{\alpha_v\alpha_b\alpha_c}(g_t(v)) \ . \tag{1}$$

Let $\tilde{g}_t : \Sigma \to \Sigma \times G^{\mathbb{C}}$ be obtained from g_t by putting $\tilde{g}_t(\xi) = (\xi, g_t(\xi))$. Denote by $\tilde{X}_t$ any vector field tangent to $\Sigma \times G^{\mathbb{C}}$ s.t. $X_t(\tilde{g}_t(\xi)) = \frac{d}{dt}\tilde{g}_t(\xi)$. For a form χ on $G^{\mathbb{C}}$ denote by $\tilde{\chi}$ its pull-back to $\Sigma \times G^{\mathbb{C}}$ by the projection on the second factor. We have

$$\frac{d}{dt}\int g_t^*\chi = \frac{d}{dt}\int \tilde{g}_t^*\tilde{\chi} = \int g_t^* L_{\tilde{X}_t}\tilde{\chi} \ . \tag{2}$$

where $L_{\tilde{X}_t} \tilde{\chi} = \tilde{X}_t \rfloor d\tilde{\chi} + d(\tilde{X}_t \rfloor \tilde{\chi})$ is the Lie derivative of $\tilde{\chi}$ with respect to $\tilde{X}_t$. Hence

$$-\frac{d}{dt} \log A_\Sigma(g_t) = -i\sum_c \int_c \tilde{g}_t^* L_{\tilde{X}_t} \tilde{\omega}_{\alpha_c} + i \sum_{b,c} \int_b \tilde{g}_t^* L_{\tilde{X}_t} \tilde{\eta}_{\alpha_b \alpha_c}$$

$$- \sum_{v,b,c} \langle \tilde{X}_t(\tilde{g}_t(v)), \tilde{g}^{-1}_{\alpha_v \alpha_b \alpha_c} \, d\tilde{g}_{\alpha_v \alpha_b \alpha_c} \rangle$$

$$= -i\sum_c \int_c \tilde{g}_t^*(\tilde{X}_t \rfloor \tilde{\gamma}^{\mathbb{C}}) - i\sum_c \int_c \tilde{g}_t^* d(\tilde{X}_t \rfloor \tilde{\omega}_{\alpha_c})$$

$$+ i \sum_{b,c} \int_b \tilde{g}_t^*(\tilde{X}_t \rfloor d\tilde{\eta}_{\alpha_b \alpha_c}) + i \sum_{b,c} \int_b \tilde{g}_t^* d(\tilde{X}_t \rfloor \tilde{\eta}_{\alpha_b \alpha_c})$$

$$- \sum_{v,b,c} \langle \tilde{X}_t(\tilde{g}_t(v)), \tilde{g}^{-1}_{\alpha_v \alpha_b \alpha_c} \, dg_{\alpha_v \alpha_b \alpha_c} \rangle$$

$$= -i \int_\Sigma \tilde{g}_t^*(X_t \rfloor \tilde{\gamma}^{\mathbb{C}}) - i \sum_{b,c} \int_b \tilde{g}_t^*(\tilde{X}_t \rfloor (\tilde{\omega}_{\alpha_c} - d\tilde{\eta}_{\alpha_b \alpha_c}))$$

$$+ i \sum_{v,b,c} \langle \tilde{X}_t(\tilde{g}_t(v)), \tilde{\eta}_{\alpha_b \alpha_c} - \frac{1}{i} g^{-1}_{\alpha_v \alpha_b \alpha_c} \, dg_{\alpha_v \alpha_b \alpha_c} \rangle$$

$$= -i \int_\Sigma \tilde{g}_t^*(\tilde{X}_t \rfloor \tilde{\gamma}^{\mathbb{C}}) - i \sum_{b,c} \int_b \tilde{g}_t^*(\tilde{X}_t \rfloor (\tilde{\omega}_{\alpha_c} - \tilde{\omega}_{\alpha_b} - d\tilde{\eta}_{\alpha_b \alpha_c}))$$

$$- i \sum_{b \subset \partial\Sigma} \int_b \tilde{g}_t^*(\tilde{X}_t \rfloor \tilde{\omega}_{\alpha_b})$$

$$+ i \sum_{v,b,c} \langle \tilde{X}_t(\tilde{g}_t(v)), \tilde{\eta}_{\alpha_b \alpha_c} - \tilde{\eta}_{\alpha_v \alpha_c} + \tilde{\eta}_{\alpha_v \alpha_b} - \frac{1}{i} g^{-1}_{\alpha_v \alpha_b \alpha_c} \, dg_{\alpha_v \alpha_b \alpha_c} \rangle$$

$$- i \sum_{\substack{v,b \\ v \in \partial b \\ b \subset \partial\Sigma}} \langle \tilde{X}_t(\tilde{g}_t(v)), \tilde{\eta}_{\alpha_v \alpha_b} \rangle$$

$$= -i \int_\Sigma \tilde{g}_t^*(\tilde{X}_t \rfloor \tilde{\gamma}^{\mathbb{C}}) - i \sum_{b \subset \partial\Sigma} \int_b \tilde{g}_t^*(\tilde{X}_t \rfloor \tilde{\omega}_{\alpha_b})$$

$$- i \sum_{\substack{v,b \\ v \in \partial b, b \subset \partial\Sigma}} \langle \tilde{X}_t(\tilde{g}_t(v)), \tilde{\eta}_{\alpha_v \alpha_b} \rangle \tag{3}$$

where we have used the defining properties of $(g_{\alpha_0 \alpha_1 \alpha_2}, \eta_{\alpha_0 \alpha_1}, \omega_\alpha)$. Now

$$-i \int_\Sigma \tilde{g}_t^*(\tilde{X}_t \rfloor \tilde{\gamma}^{\mathbb{C}}) = -\frac{ik}{4\pi} \int \mathrm{tr}(g_t^{-1} \frac{\partial g_t}{\partial t})(g_t^{-1} dg_t)^2$$

$$= \frac{ik}{4\pi} \int \mathrm{tr}(g_t^{-1} \frac{\partial g_t}{\partial t}) d(g_t^{-1} dg_t)$$

$$= \frac{ik}{4\pi} \int_{\partial\Sigma} \mathrm{tr}(g_t^{-1} \frac{\partial g_t}{\partial t})(g_t^{-1} dg_t) - \frac{ik}{4\pi} \int_\Sigma \mathrm{tr}\, \partial(g_t^{-1} \frac{\partial g_t}{\partial t})(g_t^{-1} \bar{\partial} g_t)$$

$$- \frac{ik}{4\pi} \int \mathrm{tr}\, \bar{\partial}(g_t^{-1} \frac{\partial g_t}{\partial t})(g_t^{-1} \partial g_t) \ . \tag{4}$$

The contribution of the sigma model action $S_\Sigma(g_t)$ (see (5.2)) gives

$$
\begin{aligned}
&- \frac{ik}{4\pi} \frac{d}{dt} \int_\Sigma tr(g_t^{-1}\partial g_t)(g_t^{-1}\bar\partial g_t) \\
&= \frac{ik}{4\pi} \int_\Sigma tr(g_t^{-1} \frac{\partial g_t}{\partial t})(g_t^{-1}\partial g_t)(g_t^{-1}\bar\partial g_t) \\
&- \frac{ik}{4\pi} \int_\Sigma tr(g_t^{-1}\partial(\frac{\partial g_t}{\partial t}))(g_t^{-1}\bar\partial g_t) \\
&+ \frac{ik}{4\pi} \int_\Sigma tr(g_t^{-1}\partial g_t)(g_t^{-1}\frac{\partial g_t}{\partial t})(g_t^{-1}\bar\partial g_t) \\
&- \frac{ik}{4\pi} \int_\Sigma tr(g_t^{-1}\partial g_t)(g_t^{-1}\bar\partial(\frac{\partial g_t}{\partial t})) \\
&= - \frac{ik}{4\pi} \int_\Sigma tr\ \partial(g_t^{-1}\frac{\partial g_t}{\partial t})(g_t^{-1}\bar\partial g_t) \\
&+ \frac{ik}{4\pi} \int_\Sigma tr\ \bar\partial(g_t^{-1}\frac{\partial g_t}{\partial t})(g_t^{-1}\partial g_t) \ . \qquad (5)
\end{aligned}
$$

Eqs. (3), (4) and (5) together yield

$$
\begin{aligned}
\frac{d}{dt} S_\Sigma(g_t) &= - \frac{ik}{2\pi} \int_\Sigma tr\ \partial(g_t^{-1}\frac{\partial g_t}{\partial t})(g_t^{-1}\bar\partial g_t) \\
&+ \frac{ik}{4\pi} \int_{\partial\Sigma} tr(g_t^{-1}\frac{\partial g_t}{\partial t})(g_t^{-1}dg_t) - i \sum_{b\subset\partial\Sigma} \int_b \tilde{g}_t^{*}(\tilde{X}_t \rfloor \tilde{\omega}_{\alpha_b}) \\
&-i \sum_{\substack{v,b \\ v\in\partial b,\, b\subset\partial\Sigma}} \langle \tilde{X}_t(\tilde{g}_t(v)), \tilde{\eta}_{\alpha_v \alpha_b} \rangle \ . \qquad (6)
\end{aligned}
$$

Of course (6) implies (5.13) but it provides also the boundary terms occurring when $\frac{\partial g_t}{\partial t}$ does not vanish on $\partial\Sigma$.

Notice that due to (3.10), eq. (6) may be rewritten as

$$
\begin{aligned}
\frac{D}{dt} \exp[-S\ (g_t)] = \{&\frac{ik}{2\pi} \int_\Sigma tr\ \partial(g_t^{-1}\frac{\partial g_t}{\partial t})(g_t^{-1}\bar\partial g_t) \\
&- \frac{ik}{4\pi} \int_{\partial\Sigma} tr(g_t^{-1}\frac{\partial g_t}{\partial t})(g_t^{-1}dg_t)\} \exp[-S_\Sigma(g_t)] \qquad (7)
\end{aligned}
$$

where $\frac{D}{dt}$ is the covariant derivative.

APPENDIX 5.

In order to prove (5.24), it is enough to check the commutation relations in the Lie algebra of $\widehat{LG}^{\mathbb{C}}$ as ℓ and r are two commuting representations of $\widehat{LG}^{\mathbb{C}}$. The only non-trivial relation is the commutator

$$\left[\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{S_{D_\infty}\left(e^{\frac{1}{2} i\varepsilon\sigma_j z^n}\right)}, \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{-S_{D_0}\left(e^{\frac{1}{2} i\varepsilon\sigma_\ell z^m}\right)}\right] \tag{1}$$

with $n < 0$ and $m \geq 0$. This is equal to

$$\frac{d}{d\varepsilon^2}\Big|_{\varepsilon=0} e^{S_{D_\infty}(g_1^{-1})} \cdot e^{-S_{D_0}(g_2^{-1})} \cdot e^{S_{D_\infty}(g_1)} \cdot e^{-S_{D_0}(g_2)} \tag{2}$$

where

$$g_1 = e^{\frac{1}{2} i\varepsilon\sigma_j z^n} \quad , \quad g_2 = e^{\frac{1}{2} i\varepsilon\sigma_\ell z^m} \quad . \tag{3}$$

Setting $\tilde{g}_1 = e^{\frac{1}{2} i\varepsilon\sigma_j f}$, where f extends z^n on D_∞ to D_0, we may rewrite expression (2) :

$$(2) = \frac{d}{d\varepsilon^2}\Big|_{\varepsilon=0} e^{S_\Sigma(\widetilde{g_1^{-1}})+S_\Sigma(\tilde{g}_1)} e^{-S_{D_0}(\widetilde{g_1^{-1}})} \cdot e^{-S_{D_0}(g_2^{-1})} \cdot e^{-S_{D_0}(\tilde{g}_1)} \cdot e^{-S_{D_0}(g_2)}$$

$$= \frac{d}{d\varepsilon^2}\Big|_{\varepsilon=0} \exp[\Gamma_\Sigma(\widetilde{g_1^{-1}},\tilde{g}_1)-\Gamma_{D_0}(\widetilde{g_1^{-1}},g_2^{-1})-\Gamma_{D_0}(\tilde{g}_1,g_2)$$

$$e^{S_{D_0}(\widetilde{g_1^{-1}}g_2^{-1})} \cdot e^{S_{D_0}(\tilde{g}_1 g_2)}$$

$$= \frac{d}{d\varepsilon^2}\Big|_{\varepsilon=0} \exp[\Gamma_{D_0}(\widetilde{g_1^{-1}},\tilde{g}_1)-\Gamma_{D_0}(\widetilde{g_1^{-1}},g_2^{-1})-\Gamma_{D_0}(\tilde{g}_1,g_2)$$

$$- \Gamma_{D_0}(\widetilde{g_1^{-1}}g_2^{-1},\tilde{g}_1 g_2)] e^{-S_{D_0}(\widetilde{g_1^{-1}}g_2^{-1}\tilde{g}_1 g_2)} \tag{4}$$

where we have used (5.11) and (5.14). The expression in brackets equals

$$- \frac{ik\varepsilon^2}{8\pi} \int_{D_0} tr(\sigma_j\sigma_\ell)(\bar{\partial} f)(\partial z^m) + O(\varepsilon^2)$$

$$= - \frac{ik\varepsilon^2}{4\pi} m\delta^{j\ell} \int_{\partial D_0} z^{n+m} \frac{dz}{z} + O(\varepsilon^2)$$

$$= \varepsilon^2 \frac{k}{2} m \, \delta^{j\ell} \, \delta_{n+m,0} + O(\varepsilon^2) \quad . \tag{5}$$

On the other hand,

$$\tilde{g}_1^{-1} g_2^{-1} \tilde{g}_1 g_2 = e^{\frac{1}{2} i\varepsilon^2 \sum_k \varepsilon^{j\ell k} \sigma_k f z^m} (1+O(\varepsilon^2)) \quad .$$

Now it follows from (5.11) and (5.13) $(S(e^{O(\varepsilon^2)}) = O(\varepsilon^4))$ that if $n+m \geq 0$ then

$$\frac{d}{d\varepsilon^2}\Big|_{\varepsilon=0} e^{-S_{D_0}(\tilde{g}_1^{-1} g_2^{-1} \tilde{g}_1 g_2)} = \frac{d}{d\varepsilon^2}\Big|_{\varepsilon=0} e^{-S_{D_0}\left(e^{\frac{1}{2} i\varepsilon^2 \sum_k \varepsilon^{j\ell k} \sigma_k z^{n+m}}\right)} \tag{6}$$

and if $n+m < 0$ then

$$\frac{d}{d\varepsilon^2}\Big|_{\varepsilon=0} e^{-S_{D_0}(\tilde{g}_1^{-1} g_2^{-1} \tilde{g}_1 g_2)} = \frac{d}{d\varepsilon^2}\Big|_{\varepsilon=0} e^{S_{D_\infty}\left(\frac{1}{2} i\varepsilon^2 \sum_k \varepsilon^{j\ell k} \sigma_k z^{n+m}\right)} \quad . \tag{7}$$

Gathering (5), (6) and (7), we obtain for $n+m \gtrless 0$

$$(1) = \frac{d}{d\varepsilon^2}\Big|_{\varepsilon=0} e^{\begin{cases} -S_{D_0} \\ S_{D_\infty} \end{cases} \left(\frac{1}{2} i\varepsilon^2 \sum_k \varepsilon^{j\ell k} \sigma_k z^{n+m}\right)} + \frac{k}{2} m\delta^{j\ell} \delta_{n+m,0} \tag{8}$$

what was to be shown.

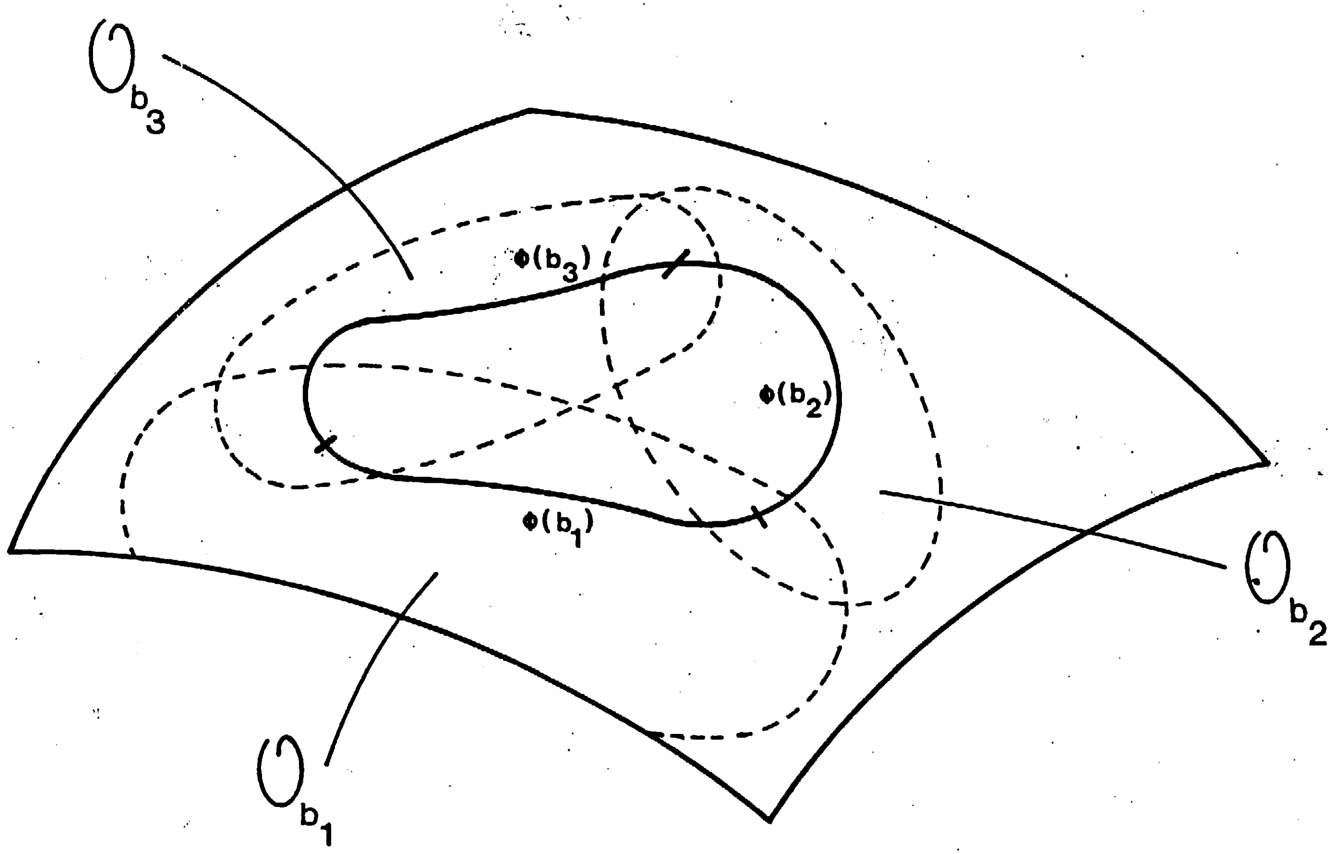

Fig. 1

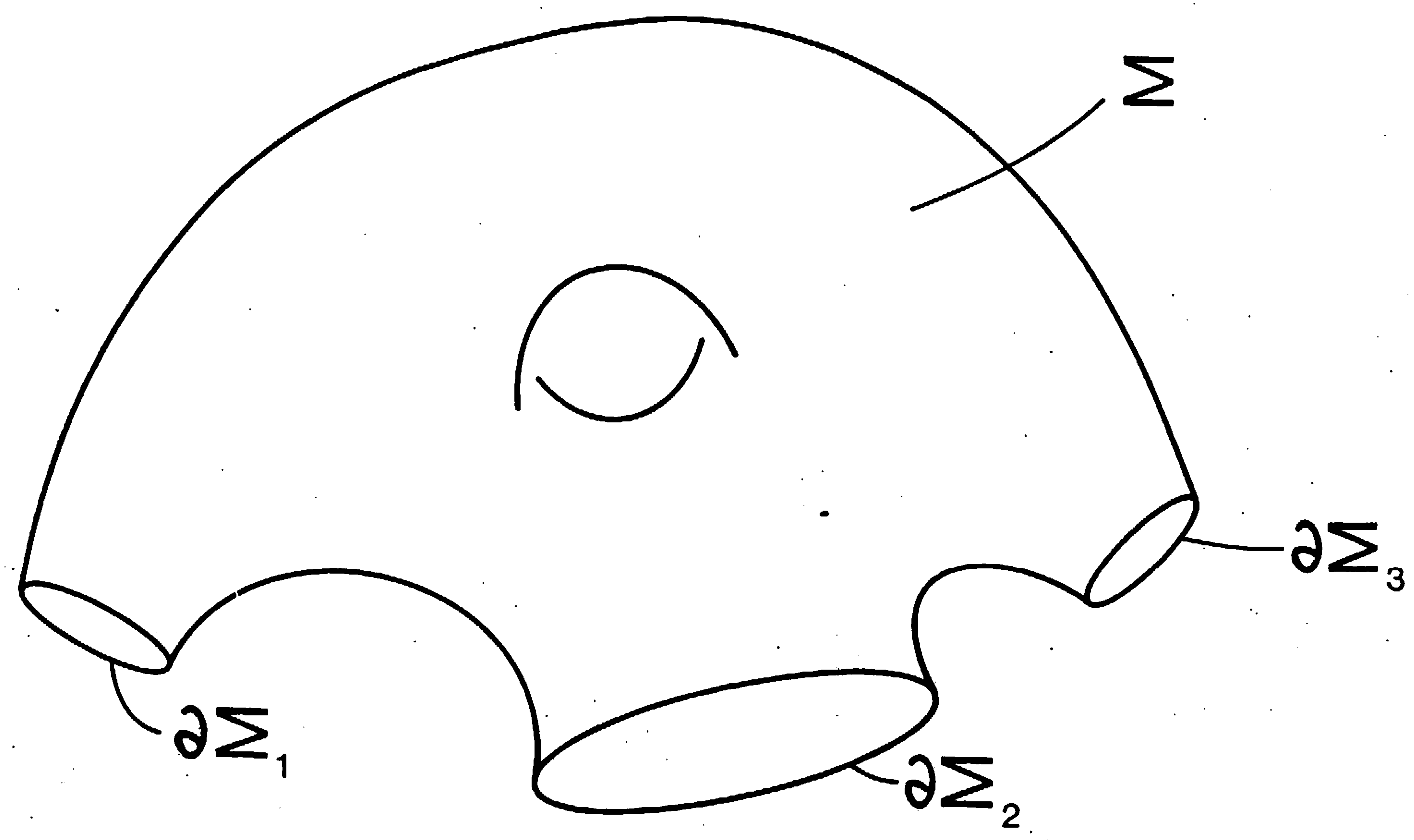

Fig. 2

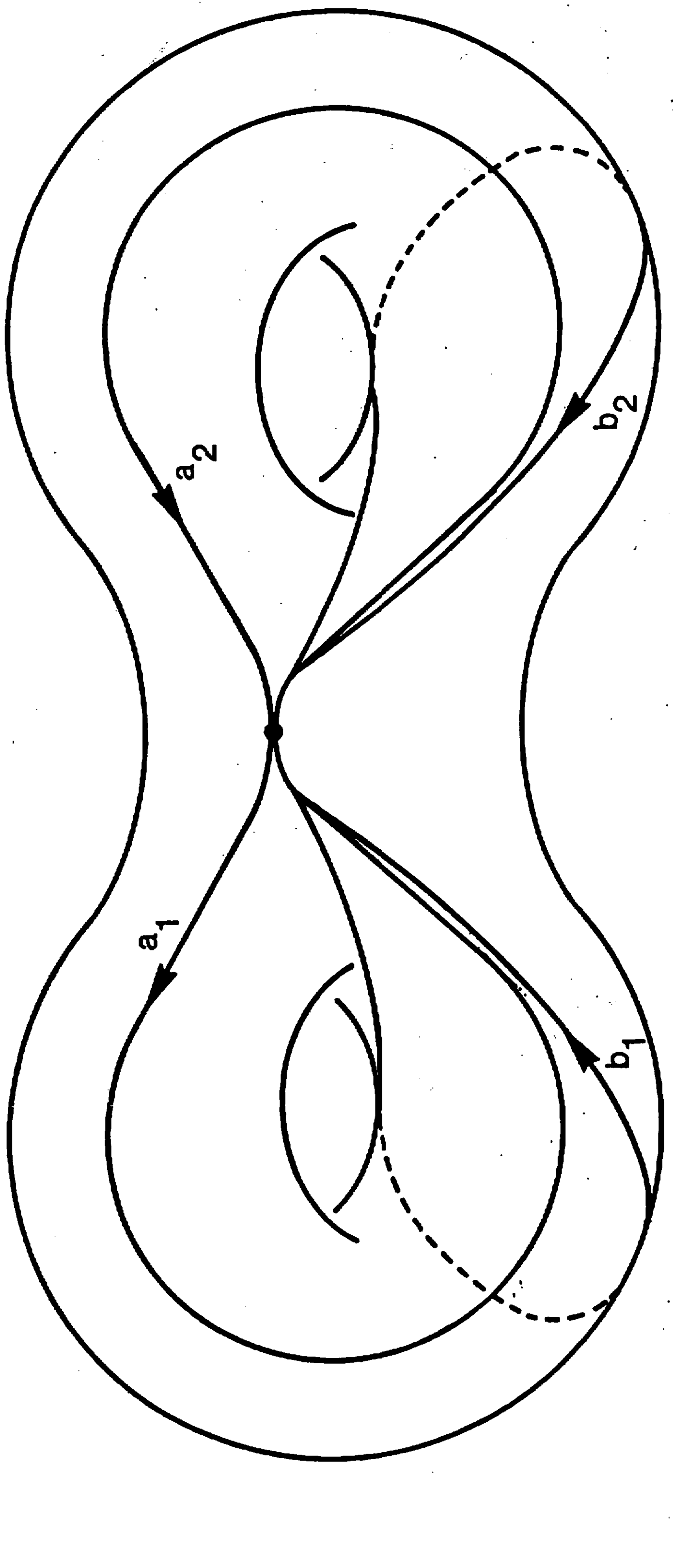

Fig. 3

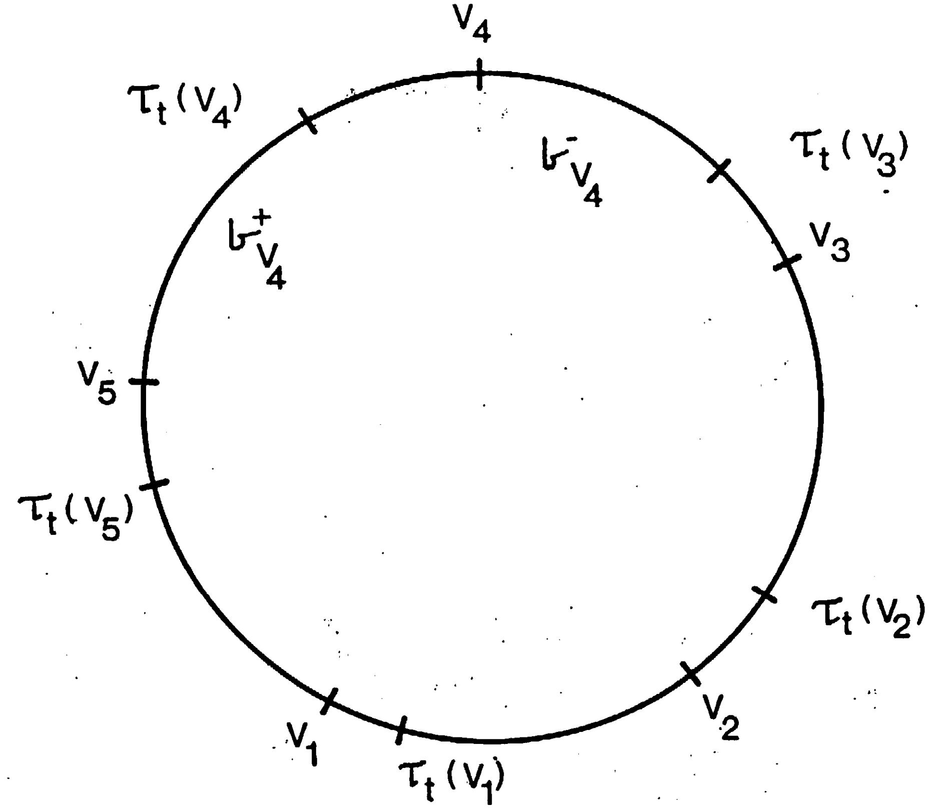

Fig. 4

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