FUNDAMENTAL BOUNDS ON QUANTUM METROLOGY IN THE PRESENCE OF DECOHERENCE

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Parallelism: classical parameter estimation — quantum channel estimation

Quantum Metrology: Quantum Parameter Estimation Theory

Classical Parameter Estimation Game

- Input
- Parameter imprinting system
- Output
- Measurement
- Estimator

Quantum state

- Parameterised by $\varphi$

Output state

General set of positive operators

Estimate from probabilities

POVM

Estimator

- $\hat{\varphi}$

Parallelism: classical parameter estimation — quantum channel estimation
Setup for the optimal estimation strategy:

**A IM:**

- Find the optimal method of establishing $\tilde{\varphi}$ as close to $\varphi$, for all $\varphi \in \mathcal{S}_\varphi$.
- Minimise the average error: $\Delta \tilde{\varphi} = \sqrt{\langle (\tilde{\varphi} - \varphi)^2 \rangle}$
- Very hard!

We are given the set $\{\Lambda_\varphi : \varphi \in \mathcal{S}_\varphi\}$, for which we need to optimise over: the input state + the set of all POVMs + the estimator.
Can we ask any general questions?

N independent realisations of the estimated channel:

What is the scaling of the average error, $\Delta \tilde{\varphi}_N$, with the number of realisations $N$?

- **Classically**, as the realisations are independent, we cannot overcome the shot noise. Asymptotically ($N\to\infty$) the error can maximally scale as:
  $$\Delta \tilde{\varphi}_N = \frac{1}{\sqrt{N}} \Delta \tilde{\varphi}$$
  Shot Noise Limit (SNL)

- **Quantum mechanically**, input can be entangled and measurement can be non-local. Asymptotically ($N\to\infty$) the error can maximally scale as:
  $$\Delta \tilde{\varphi}_N = \frac{1}{N} \Delta \tilde{\varphi}$$
  Heisenberg limit (HL)

Channels considered that asymptotically achieve **Heisenberg Limit**,\[ \Delta \bar{\varphi}_N = \frac{1}{N} \Delta \bar{\varphi}, \] are **unitary**:\[ \Lambda_\varphi \left[ \rho_{in} \right] = U(\varphi) \rho_{in} U^\dagger(\varphi) \]

Do the **realistic physical channels**, which include losses/decoherence, also achieve **Heisenberg Limit**?\[ \Lambda_\varphi \left[ \rho_{in} \right] = \sum_{k=1}^{K} K_k(\varphi) \rho_{in} K_k^\dagger(\varphi) \]

**LET US INVESTIGATE SOME EXAMPLES ...**
**EXAMPLE 1A: OPTICAL INTERFEROMETER**

*N* independent **realisations** of the channel $\leftrightarrow$ *N* photon pure input state

$$\psi_{in}^N \xrightarrow{\varphi} \psi_{out}(\varphi) \xrightarrow{\text{POVM}} \tilde{\varphi}_N$$

- **Pure** *N* photon state
  $$|\psi_{in}^N\rangle = \sum_{n=0}^{N} \alpha_n |n\rangle_a \otimes |N-n\rangle_b$$
- **Output state**
  $$|\psi_{out}^N\rangle = \sum_{n=0}^{N} \alpha_n e^{in\varphi} |n\rangle_a \otimes |N-n\rangle_b$$
- **Set of POVM’s**

**Solutions in two scenarios:**

- **No knowledge** of the estimated $\varphi$.
  - **Flat "a priori" distribution**, $p(\varphi) = \frac{1}{2\pi}$.
  - **Optimal (entangled)** input state:
    $$\alpha_n = \sqrt{\frac{2}{N+2}} \sin \left( \frac{(n+1)\pi}{N+2} \right)$$
  - **Error scaling**:
    $$\Delta\tilde{\varphi}_N = \frac{\pi}{N+2}, \quad \Delta\tilde{\varphi}_N \rightarrow \infty = \frac{1}{N}$$

- **Highest sensitivity** to changes from $\varphi_0$.
  - **Delta "a priori" distribution**, $p(\varphi) \approx \delta(\varphi_0)$.
  - **Optimal (entangled)** input state:
    $$|\psi_{in}^N\rangle = \frac{1}{\sqrt{2}} (|N0\rangle + |0N\rangle)$$
  - **NOON state**:
  - **Error scaling**:
    $$\Delta\tilde{\varphi}_N|_{\varphi\approx\varphi_0} = \frac{1}{N}$$

[Berry and Wiseman, PRL 85, 5098 (2000)]

[Heisenberg Limit ! (unitary channel)]

Example 1B: **Optical Interferometer with Loss**

(photonic loss, imperfect detectors, non-optimal input)

- **Pure** $N$ photon state
- **Output** state
- Set of POVM's
- Estimator

- **Flat "a priori" distribution** of the estimated $\varphi$, $p(\varphi) = \frac{1}{2\pi}$.
- Channel is **no longer unitary** $\rightarrow$ mixed output $Q^N_{out}(\varphi)$.
- **For finite** $N$: optimal input state’s coefficients, $\alpha_n$, are found **numerically**:

<table>
<thead>
<tr>
<th>Loss in one arm: $\eta_a = \eta$, $\eta_b = 1$</th>
<th>Loss in both arms: $\eta_a = \eta_b = \eta$</th>
</tr>
</thead>
</table>

$$|\psi^N_{in}\rangle = \sum_{n=0}^N \alpha_n |n\rangle_a |N-n\rangle_b$$

**e.g.** $N = 100$
- $\eta = 0.8$
- $\eta = 0.5$

[Kolodyński and Demkowicz-Dobrzański, PRA 82,053804 (2010)]
EXAMPLE 1B: **OPTICAL INTERFEROMETER WITH LOSS**

Scaling of the error, $\Delta \tilde{\varphi}_N$, with the number of photons, $N$.

* e.g. Loss in both arms: $\eta_a = \eta_b = \eta$

- $\eta = 0.8$
- $\eta = 0.5$

- $\delta \varphi > \frac{1}{\sqrt{N}}$
- $\delta \varphi < \frac{1}{N}$

- **worse than SNL**
- **better than HL (disallowed)**

- For any type of loss we provide a lower bound on error with $\eta = \min(\eta_a, \eta_b)$

$$\Delta \tilde{\varphi}_N \geq \sqrt{\frac{1 - \eta}{\eta}} \frac{1}{\sqrt{N}} + O\left(\frac{1}{N}\right)$$

- **Shot Noise Limit !!!**

**ALREADY INFINITESMAL AMOUNT OF LOSS DESTROYS THE ASYMPTOTIC HEISENBERG SCALING**

[Kolodynski and Demkowicz-Dobrzański, PRA 82,053804 (2010)]
**Example 2A: Atomic Spectroscopy**

*N* two-level atoms (qubits) evolving for *fixed* time, *t*, oscillating *independently* with same estimated transition frequency *ω*.

\[
q_t^N (\omega) = U_{\omega t} \otimes N q_{in}^N U_{\omega t}^\dagger \otimes N
\]

**AIM:** BEST POSSIBLE ESTIMATE OF THE FREQUENCY, *ω*.

- An optimal (entangled) input state – GHZ state:
  \[
  q_{in}^N = |\psi_N\rangle \langle \psi_N|, \quad |\psi_N\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \otimes N + |1\rangle \otimes N \right)
  \]

- Error scaling:
  \[
  \Delta \tilde{\omega}_N = \frac{1}{\sqrt{t}} \cdot \frac{1}{N}
  \]

Heisenberg Limit !!!

[Huelga et al, PRL 79, 38653868 (1997)]
**Example 2B: Atomic Spectroscopy with Dephasing**

Bloch sphere picture:

\[ N \times \]

**What is now the asymptotic scaling of the error of estimated frequency, \( \omega \), with the number of atoms \( N \)?**


- Asymptotic error is given by

\[
\Delta \tilde{\omega}_{N \to \infty} = \sqrt{\frac{2\gamma}{t}} \cdot \frac{1}{\sqrt{N}}
\]

**Shot Noise Limit !!!**

**Already infinitesimal amount of dephasing destroys the asymptotic Heisenberg scaling**
Are there any channels that include *decoherence* and preserve the *Heisenberg scaling*?
Consider the **convex set of all quantum channels** mapping a general input density matrix onto an output one:

\[ \Lambda : \varrho_{in} \in B(\mathcal{H}_{d_{in}}) \rightarrow \varrho_{out} \in B(\mathcal{H}_{d_{out}}) \]

Consider the **unitary subset parameterised** by the estimated parameter: \( \Lambda \varphi \)

Consider the **family of subsets** for different strengths of the decoherence model: \( \Lambda \eta; \varphi \)

**Heisenberg Limit** for **extremal** (⊂ **unitary**) channels at the border.

**Shot Noise Limit** for ones inside the convex set.

**WHAT IS SO SPECIAL ABOUT THE NON-EXTREMAL CHANNELS?**

**WE COULD TRY TO MOVE THE ESTIMATED PARAMETER DEPENDENCE INTO THE "MIXING" PROBABILITY!**
For the **channel to possess SQL scaling** it is enough to show that the channel has the **following properties**:

1. **Non-extremal** – must **not** lie on the (non-flat) boundary of the convex set of all channels:
   \[
   \Lambda = \frac{1}{2} (\Lambda_1 + \Lambda_2) \quad \text{or in general} \quad \Lambda = \int d\mu \, p_\mu \Lambda_\mu
   \]

2. The **estimated parameter dependence** can be **moved** into the "mixing" probability distribution:
   \[
   \Lambda_\varphi = \int d\mu \, p_\mu (\varphi) \Lambda_\mu
   \]

3. The "mixing" **probability has to be regular** w.r.t. \( \varphi' \) at the estimated \( \varphi \):
   \[
   \left\langle \partial_{\varphi'} \ln p_\mu (\varphi') \right\rangle_{\varphi' = \varphi} = \int d\mu p_\mu (\varphi') \partial_{\varphi'} \ln p_\mu (\varphi') \bigg|_{\varphi' = \varphi} = 0
   \]
   e.g. not a Dirac delta function, \( p_\mu (\varphi) \neq \delta(\varphi) \)

Originally we had:

\[ N \times Q_{in}^{N} \rightarrow N \times CPTP \text{ channels} \rightarrow Q_{out}^{N} (\varphi) \rightarrow \text{POVM} \{M_r\}_{r=1}^{R} \rightarrow \tilde{\varphi} \]

*Markov chain of information about the parameter:*

\[ \varphi \rightarrow \Lambda \rightarrow Q_{out}^{N} \rightarrow \tilde{\varphi} \]

But NOW we have:

\[ N \times Q_{in}^{N} \rightarrow \int d\mu_1 p_{\mu_1}(\varphi) \Lambda_{\mu_1} \rightarrow \int d\mu_2 p_{\mu_2}(\varphi) \Lambda_{\mu_2} \rightarrow \int d\mu_N p_{\mu_N}(\varphi) \Lambda_{\mu_N} \rightarrow Q_{out}^{N} (\varphi) \rightarrow \text{POVM} \{M_r\}_{r=1}^{R} \rightarrow \tilde{\varphi} \]

*independent vars\{\mu_1, \mu_2, \ldots, \mu_N\}*

\[ \varphi \rightarrow \Lambda \rightarrow \tilde{\varphi} \]

SQL!!! Q.E.D.
\[ \Lambda_\varphi = \int d\mu \ p_\mu(\varphi) \Lambda_\mu \]

IN CASE OF OUR EXAMPLES...
EXAMPLE 2B: **ATOMIC SPECTROSCOPY WITH DEPHASING**

- one independent channel – one atom (qubit) evolving for time $t$.
- The channel, $\Lambda_{\gamma,t;\omega}$, is a **qubit-qubit** one. Due to dephasing it is **not unitary** and possesses two **Kraus operators**:

$$
\Lambda_{\gamma,t;\omega} : \left\{ \begin{array}{ll}
K_1 = \sqrt{\frac{1-e^{-\gamma t}}{2}} U_{\omega t} = \sqrt{\frac{1-e^{-\gamma t}}{2}} \begin{pmatrix} e^{i \frac{\omega t}{2}} & 0 \\ 0 & e^{-i \frac{\omega t}{2}} \end{pmatrix}, \\
K_2 = \sqrt{\frac{1-e^{-\gamma t}}{2}} \hat{\sigma}_z U_{\omega t} = \sqrt{\frac{1-e^{-\gamma t}}{2}} \begin{pmatrix} e^{i \frac{\omega t}{2}} & 0 \\ 0 & -e^{-i \frac{\omega t}{2}} \end{pmatrix} \end{array} \right. 
$$

We construct the required **”mixing” probability distribution**: $\Lambda_{\varphi} = \int d\mu \, p_{\mu}(\varphi) \Lambda_{\mu}$

$$
\Lambda_{\gamma,t;\omega} = \int d\mu \, p_{\mu}(\gamma,t;\omega) U_{\mu t} \quad \text{where} \quad U_{\mu t} [\rho] = U_{\mu t} \rho U_{\mu t}^\dagger 
$$

with the smearing probability, $p_{\mu}(\gamma,t;\omega)$, being a solution to the diffusion equation on the phase circle (group $U(1)$):

$$
\frac{\partial}{\partial t} p_{\mu}(\gamma,t;\omega) = \gamma \frac{\partial^2}{\partial \omega^2} p_{\mu}(\gamma,t;\omega) \quad \text{with} \quad p_{\mu}(\gamma,t=0;\omega) = \delta(\omega)
$$

$$
p_{\mu}(\gamma,t;\omega) = \frac{1}{2\pi} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \gamma t} \cos[n(\mu t - \omega t)] \right)
$$

**”mixing” probability distribution Constructed $\rightarrow$ Proved SQL Scaling**
Example 1B: Optical Interferometer with Loss

- one independent channel – one photon in the MZ interferometer.
- With loss the channel, $\Lambda_{\eta_a, \eta_b; \varphi}$, becomes a qubit-qutrit one, accounting for the possibility of losing the photon to the vacuum state at the output.
- Hence, the Kraus operators:

$$\Lambda_{\eta_a, \eta_b; \varphi} = \left\{ K_1 = \sqrt{1 - \eta_a} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \sqrt{1 - \eta_b} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} e^{-i\varphi} \sqrt{\eta_b} & 0 \\ 0 & \sqrt{\eta_a} \end{pmatrix} \right\}$$

We construct the required "mixing" probability distribution:

$$\Lambda_{\eta_a, \eta_b; \varphi} = (\eta_a + \eta_b - 1) \int d\mu p_\mu (\eta_a, \eta_b; \varphi) \Lambda_{\eta_a=1, \eta_b=1; \mu} + (1 - \eta_a) \Lambda_{\eta_a=0, \eta_b=1} + (1 - \eta_b) \Lambda_{\eta_a=1, \eta_b=0}$$

where the smearing probability, $p_\mu (\eta_a, \eta_b; \varphi)$, is a solution to the diffusion equation on the phase circle (group $U(1)$):

$$\frac{\partial}{\partial \gamma} p_\mu (\gamma; \varphi) = \frac{\partial^2}{\partial \varphi^2} p_\mu (\gamma; \varphi) \quad \text{with} \quad p_\mu (\gamma = 0; \varphi) = \delta (\varphi)$$

$$p_\mu (\gamma; \varphi) = \frac{1}{2\pi} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\gamma} \cos [n (\mu - \theta)] \right) \quad \text{with} \quad \gamma = \ln \frac{\eta_a + \eta_b - 1}{\sqrt{\eta_a \eta_b}}$$

"Mixing" probability distribution constructed $\Rightarrow$ Proved SQL scaling
CONCLUSIONS:

1. **Optical Interferometry with Loss**
   Asymptotically, we cannot do better than the **Shot Noise scaling** allows:
   \[
   \Delta \tilde{\varphi}_{N \rightarrow \infty} \geq \sqrt{\frac{1 - \eta}{\eta}} \cdot \frac{1}{\sqrt{N}}
   \]
   \[\text{[JK and RDD, PRA 82,053804 (2010)]}\]

2. **Atomic Spectroscopy with Dephasing**
   Asymptotically, we are following the **Shot Noise scaling**:
   \[
   \Delta \tilde{\omega}_{N \rightarrow \infty} = \sqrt{\frac{2\gamma}{t}} \cdot \frac{1}{\sqrt{N}}
   \]

3. **Generalisation:**
   \[
   \Lambda_{\eta_0; \varphi_0} = \int d\mu \, p_\mu(\eta_0; \varphi_0) \Lambda_{\eta_0; \mu}
   \]
   **If we can construct the "Mixing" Probability for a Non-Extremal Channel,** then we trivially prove asymptotic **Shot Noise Scaling.**

4. **Open Question:**
   How does this method overlaps and relates to other methods of finding asymptotic scaling?
   - Considering **purification** of the estimated channel – \[\text{[Escher et al, Nat. Phys. 7, 406 (2011)]}\]
   - Considering **extension** of the estimated channel – which proved **SNL** of all **full rank channels**. \[\text{[Fujiwara & Imai, J.Ph.A:Math.Theor.,41, 255304 (2008)]}\]