AFFINE POISSON STRUCTURES IN ANALYTICAL MECHANICS

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Abstract

If the space-time is a product of the space and the time the Poisson structure on the phase bundle is used to describe dynamics of mechanical systems. Further it is shown that if the space-time is a fibration over the time, then the Poisson structure has to be replaced by an affine Poisson structure.

1 . TIME-DEPENDENT SYSTEMS

1.1. Time Independent Systems

In order to define a time-independent system the space-time has to be the product of space and time represented by the real line $\mathbb{R}$. For a time-independent system with the configuration manifold Q the infinitesimal dynamics is a submanifold D of TT^*Q. In particular cases D is the image of a vector field. The cotangent bundle T^*Q with the canonical 2-form ω_Q is a symplectic manifold. The tangent bundle TT^*Q of the cotangent bundle with the tangent 2-form $d_{\mathsf{T}}\omega_Q$ is a symplectic manifold as well[4, 5]. We say that the system is *Lagrangian* if the dynamics D is a Lagrange submanifold of $(\mathsf{TT}^*Q, d_{\mathsf{T}}\omega_Q)$.

Let us denote by τ_Q the canonical projection $\tau_Q\colon \mathsf{T}Q \to Q$ and by π_Q the canonical projection $\pi_Q\colon \mathsf{T}^*Q \to Q$. There are three, fundamental for the analytical mechanics, isomorphisms of vector bundles:

$$\kappa_Q\colon (\tau_{\mathsf{T}Q}\colon \mathsf{TT}Q \to \mathsf{T}Q) \longrightarrow (\mathsf{T}\tau_Q\colon \mathsf{TT}Q \to \mathsf{T}Q) \tag{1.1}$$

$$\alpha_Q\colon (\mathsf{T}\pi_Q\colon \mathsf{TT}^*Q \to \mathsf{T}Q) \longrightarrow (\pi_{\mathsf{T}Q}\colon \mathsf{T}^*\mathsf{T}Q \to \mathsf{T}Q) \tag{1.2}$$

$$\beta_Q\colon (\mathsf{T}\pi_Q\colon \mathsf{TT}^*Q \to \mathsf{T}Q) \longrightarrow (\pi_{\mathsf{T}^*Q}\mathsf{T}^*\mathsf{T}^*Q \to \mathsf{T}^*Q) \tag{1.3}$$

The mapping α_Q is also a symplectomorfism of $(\mathsf{TT}^*Q, \mathsf{T}\pi_Q)$ and $(\mathsf{T}^*\mathsf{T}Q, \pi_{\mathsf{T}Q})$. The mapping β_Q is a symplectomorphism of $(\mathsf{TT}^*Q, \mathsf{T}\pi_Q)$ and $(\mathsf{T}^*\mathsf{T}^*Q, \pi_{\mathsf{T}^*Q})$.

Let the dynamics D of a system be a Lagrangian submanifold of $(\mathsf{TT}^*Q, \mathsf{T}\pi_Q)$. It follows that $\alpha_Q(D)$ and $\beta_Q(D)$ are Lagrangian submanifolds of $(\mathsf{T}^*\mathsf{T}Q, \pi_{\mathsf{T}Q})$ and

$(\mathsf{T}^*\mathsf{T}^*Q, \pi_{\mathsf{T}^*Q})$ respectively. By a theorem of Hörmander $\alpha_Q(D)$ and $\beta_Q(D)$ can be generated (at least locally) by a function (or a Morse family of functions) on a submanifold of $\mathsf{T}Q$ or T^*Q respectively. The generating function on $\mathsf{T}Q$ (if it exists) is called the Lagrangian of the system. The generating function on T^*Q is called the Hamiltonian generating function of the system. In the following we shall consider systems with the dynamics generated by a Lagrangian defined on $\mathsf{T}Q$.

1.2. Time-Dependent Systems. Inhomogeneous formulation.

Let us assume that, as before, the space-time is the product of the space and the time. Let Q be the manifold of space configurations of a system. $M = Q \times \mathbb{R}$ is the manifold of the space-time configurations of the system. Let $\zeta\colon M \to Q$ be the canonical projection. An infinitesimal configuration is a vector $v \in \mathsf{T}M$ such that $\mathsf{T}\zeta(v) = (\zeta(\tau_M v), \partial_t)$. $\mathsf{T}_1 M$ will denote the bundle of infinitesimal configurations. The phase bundle is the product $P = \mathsf{T}^*Q \times \mathbb{R}$. Let $\eta\colon P \to \mathbb{R}$ be the canonical projection. For each $t \in \mathbb{R}$ the fiber $P_t = \eta^{-1}(t)$ is a symplectic manifold. An infinitesimal state is a vector $w \in \mathsf{T}P$ such that $\mathsf{T}\eta(w) = (\eta(\tau_P w), \partial_t)$. We denote by $\mathsf{T}_1 P$ the bundle of infinitesimal states. The dynamics of the system is a submanifold D of $\mathsf{T}_1 P$. Let D_t denote a subset of D defined by $D_t = \{D \ni w; \eta(\tau_P w) = t \in \mathbb{R}\}$. Since $P = \mathsf{T}^*Q \times \mathbb{R}$, we have also $\mathsf{T}P = \mathsf{T}\mathsf{T}^*Q \times \mathsf{T}\mathbb{R}$ and $D_t = \overline{D}_t \times (t, \partial_t)$. We say that the system is *Lagrangian* if $\overline{D}_t$ is a Lagrangian submanifold of $(\mathsf{T}\mathsf{T}^*Q, d_\mathsf{T}\omega_Q)$ for each $t \in \mathbb{R}$. It follows (see previous section) that $\overline{D}_t$ can be generated by the Lagrangian L_t on $\mathsf{T}Q$ or by the Hamiltonian generating function H_t on T^*Q. Families of functions L_t, H_t define functions L, H on $\mathsf{T}Q \times \mathbb{R}$ and $\mathsf{T}^*Q \times \mathbb{R}$ respectively. A procedure of generating the component $\overline{D}$ of the dynamics can be formulated in terms of the canonical Poisson structure on P. This formulation is equivalent to the described one.

1.3. Time-Dependent Systems. Homogeneous formulation.

In the homogeneous formulation the configuration manifold is the manifold M. Infinitesimal configurations are vectors tangent to M. $\mathsf{T}M$ is the manifold of infinitesimal configurations. If the system is Lagrangian, i. e., the dynamics is generated by the Lagrangian L on $\mathsf{T}Q \times \mathbb{R}$, we define a function $\widehat{L}\colon \mathsf{T}M \to \mathbb{R}$ by the formula

$$\widehat{L}(v) = sL_t(\overline{v})$$

where $\overline{v}$ is the component of v in $\mathsf{T}Q$ and $\mathsf{T}\zeta(v) = (t, s\partial_t)$.

The function $\widehat{L}$ generates a Lagrangian submanifold of $\mathsf{T}^*\mathsf{T}M$ and, consequently, of $\mathsf{T}\mathsf{T}^*M$ (Section 1.2). We denote by $\widehat{D}$ the generated by $\widehat{L}$ submanifold of $\mathsf{T}\mathsf{T}^*M$. It can also be generated also by a Hamiltonian.

Proposition 1 *The manifold $\widehat{D}$ is generated by a function equal to zero and defined on a submanifold $C \subset \mathsf{T}^*M$*

$$C = \{\mathsf{T}^*M \ni (p, t, \epsilon); -\epsilon = H_t(p)\}$$

We can also get D_t from $\widehat{D}$ directly. It is easy to verify that $\mathsf{T}\mathsf{T}^*Q$ is canonicaly identified with the reduction of $\mathsf{T}\mathsf{T}^*M$ with respect to the coisotropic submanifold $\mathsf{T}K_t$ where $K_t \subset \mathsf{T}^*M$ defined by

$$K_t = \{\mathsf{T}^*M \ni (p, t', \epsilon); t' = t\}.$$

Let us denote this reduction by $\mathsf{T}\varrho_t$.

Proposition 2

$$D_t = \mathsf{T}\varrho_t(D)$$

2. AFFINE SPACES AND AFFINE BUNDLES

2.1. Principal Affine Spaces

An *affine space* is a triple (A, V, α), where A is a set, V is a real vector space of finite dimension and α is a mapping $\alpha\colon A \times A \to V$ such that

1. $\alpha(a_3, a_2) + \alpha(a_2, a_1) + \alpha(a_1, a_3) = 0$;
2. the mapping $\alpha(\cdot, a)\colon A \to V$ is bijective for each $a \in A$.

We will write for brevity $a_2 - a_1$ instead of $\alpha(a_2, a_1)$. We will denote by $a + v$ the unique point $a' \in A$ such that $a' - a = v$.

We consider quadruples (A, V, α, v_0), where (A, V, α) is an affine space and v_0 is a distinguished nonzero vector in the *model space* V of the affine space (A, V, α). Such objects will be called *principal affine spaces.* A *principal affine mapping* from (A, V, α, v_0) to (B, W, β, w_0) is an affine mapping φ from (A, V, α) to (B, W, β) such that $\bar{\varphi}(v_0) = w_0$ ($\bar{\varphi}$ is the linear part of φ. A principal affine space (A, V, α, v_0) can be considered as a principal bundle with the structural group $\mathbb{R}$ and the action

$$(r, a) \mapsto a + sv_0.$$

The category of principal affine spaces has a distiguished object $I = (\mathbb{R}, \mathbb{R}, -, 1)$. The *affine dual* to a principal affine space (A, V, α, v_0) is a principal affine space $(A^\#, V^\#, \alpha^\#, f_0)$ where $A^\#$ is the space of all principal affine mappings from (A, V, α, v_0) to I, $V^\#$ is the vector space of affine functions on the quotient vector space $V/_{\{v_0\}}$, $\alpha^\#(\varphi - \varphi') = \varphi - \varphi'$ and f_0 is the equal 1 constant function.

2.2. Affine Bundles

Let $\xi\colon E \to N$ be a vector fibration. An *affine fibration modelled on* ξ is a differential fibration $\eta\colon A \to N$ and a differentiable mapping $\rho\colon A \times_N A \to E$ such that

1. $\xi \circ \rho = \eta \times_N \eta$,
2. $\rho(a_3, a_2) + \rho(a_2, a_1) = \rho(a_3, a_1)$ for each triple $(a_3, a_2, a_1) \in A \times_N A \times_N A$,
3. for each local section $\sigma\colon U \to A$ of η, the mapping $\rho_\sigma\colon \eta^{-1}(U) \to \xi^{-1}(U)$ defined by
$$\rho_\sigma(a) = \rho(a, \sigma(\eta(a)))$$
is a diffeomorphism.

A *principal affine fibration* is an affine fibration with e nowhere vanishing section of the model vector fibration. It follows that a fiber of a principal affine fibration is a principal affine space. The affine dual to a principal affine fibration we define in the obvious way.

An affine fibration modelled on the trivial vector fibration $pr_N: N \times \mathbb{R} \to N$ is usually interpreted as a principal fibration with structure group $\mathbf{R}$. We denote by $\mathbf{I}$ the trivial principal fibration $(pr_1: \{1\} \times \mathbb{R} \to \{1\})$.

Let $\mathbf{Z} = (\zeta: Z \to N, \rho: Z \times_N Z \to N \times \mathbb{R})$ be an affine fibration modelled on the trivial fibration $pr_N: N \times \mathbb{R} \to N$. We define an equivalence relation in the set of all pairs (m, σ), where m is a point in N and σ is a section of ζ. Two pairs (m, σ) and (m', σ') are equivalent if $m' = m$ and $\mathrm{d}(\sigma' - \sigma)(m) = 0$. We have identified the section $\sigma' - \sigma$ of pr_N with a function on N for the purpose of evaluating the differential $\mathrm{d}(\sigma' - \sigma)(m)$. We denote by $\mathsf{P}Z$ the set of equivalence classes. The class of (m, σ) will be denoted by $\mathrm{d}\sigma(m)$ and will be called the *differential* of σ at m. We define a mapping $\mathsf{P}\zeta: \mathsf{P}Z \to N$ by $\mathsf{P}\zeta(\mathrm{d}\sigma(m)) = m$. We define a mapping

$$\mathsf{P}\rho: \mathsf{P}Z \times_N \mathsf{P}Z \to \mathsf{T}^* N$$

by

$$\mathsf{P}\rho(\mathrm{d}\sigma_2(m), \mathrm{d}\sigma_1(m)) = \mathrm{d}(\sigma_2 - \sigma_1)(m).$$

The pair $\mathrm{P}\mathbf{Z} = (\mathsf{P}\zeta, \mathsf{P}\rho)$ is an affine fibration modelled on the fibration $\pi_N: \mathsf{T}^* N \to N$. This fibration is called the *phase fibration* of $\mathbf{Z} = (\zeta, \rho)$. Let φ be a section of $\mathsf{P}\zeta$ and let σ be a section of ζ. We define the *differential* $\mathrm{d}\varphi$ of φ by $\mathrm{d}\varphi = \mathrm{d}(\varphi - \mathrm{d}\sigma)$. Since for two sections σ, σ' of ζ we have $\mathrm{d}(\mathrm{d}\sigma - \mathrm{d}\sigma') = \mathrm{dd}(\sigma - \sigma') = 0$ it follows that the definition of the differential does not depend on the choice of σ.

For each $\mathbf{Z}$ the manifold $\mathsf{P}Z$ is a symplectic manifold[6].

3. AFFINE POISSON STRUCTURES

3.1. Homogeneous Formulation of the Dynamics

In the first section we have assumed that the space-time is the product of the space and the time (represented by the real line $\mathbb{R}$). This assumption implies that we have chosen a reference frame. In this section we formulate the dynamics of a nonrelativistic system in a frame-independent way. We represent the time by the real line. The space-time is a fibration over the time. It follows that the manifold of space-time configurations of a system is a fibration

$$\zeta: M \to \mathbb{R}.$$

Let us denote by Q_t the fiber over $t \in \mathbb{R}$ of the fibration. Infinitesimal configurations are vectors tangent to M. $\mathsf{T}M$ is the manifold of infinitesimal configurations. The phase bundle is the cotangent bundle $\mathsf{T}^* M$. Let $\widehat{\eta}: \mathsf{T}^* M \to \mathbb{R}$ be the canonical projection $\widehat{\eta} = \zeta \circ \pi_M$. The dynamics of a system is a submanifold $\widehat{D}$ of $\mathsf{TT}^* M$. We say that the system is *lagrangian* if D is the Lagrangian submanifold of $(\mathsf{TT}^* M, d_{\mathsf{T}}\omega_M)$. Let L be Lagrangian generating function of D. For a nonrelativistic system L is a homogeneous function on $\mathsf{T}M$. It follows that the Hamiltonian generating function is the zero function on a submanifold C of $\mathsf{T}^* M$.

3.2. Inhomogeneous Formulation of the Dynamics

In the formulation of the dynamics presented in Section 1.2 the existence of Lagrangian and Hamiltonian generating functions was possible because the space-time was assumed to be the product of the space and the time.

Let $\zeta\colon M \to \mathbb{R}$ be the configuraton manifold of a system, fibered over the time. By M_t we denote a fiber of the fibration ζ, $M_t = (\zeta)^{-1}(t)$. An infinitesimal configuration of the system is a vector $v \in \mathsf{T}M$ such that $\mathsf{T}\zeta(v) = (\zeta(\tau_M v), \partial_t)$. $\mathsf{T}_1 M$ will denote the bundle of infinitesimal configurations. For each $t \in \mathbb{R}$ a submanifold $\mathsf{T}_1 M_t$ of $\mathsf{T}_1 M$ is defined by

$$\mathsf{T}_1 M_t = \{\mathsf{T}_1 M \ni w;\ \eta(\tau_M w) = t\}.$$

The phase bundle is a fibration $\eta\colon P \to \mathbb{R}$ with $P_t = (\eta)^{-1}(t) = \mathsf{T}^* M_t$. For each $t \in \mathbb{R}$ the fiber P_t can be considered as the reduction of $\mathsf{T}^* M$ with respect to a coisotropic submanifold $K_t = \{\mathsf{T}^* \ni p;\ \zeta(\pi_M p) = t\}$. An infinitesimal state is a vector $w \in \mathsf{T}P$ such that $\mathsf{T}\eta(w) = (\eta(\tau_P w), \partial_t)$. We denote by $\mathsf{T}_1 P$ the bundle of infinitesimal states . For each $t \in \mathbb{R}$ a submanifold $\mathsf{T}_1 P_t$ of $\mathsf{T}_1 P$ is defined by

$$\mathsf{T}_1 P_t = \{\mathsf{T}_1 P \ni w;\ \eta(\tau_P w) = t\}.$$

Proposition 3 *A submanifold $\mathsf{T}_1 P_t$ is the reduction of $(\mathsf{T}\mathsf{T}^* M, d_{\mathsf{T}}\omega_M)$ with respect to a coisotropic submanifold $\mathsf{T}_1 K_t$ defined by*

$$\mathsf{T}_1 K_t = \{\mathsf{T}\mathsf{T}^* M \ni w;\ \mathsf{T}\pi_M(w) \in \mathsf{T}_1 M \textit{ and } \zeta(\tau_{\mathsf{T}^* M} \circ \pi_M(w)) = t\}.$$

It follows from this proposition that $\mathsf{T}_1 P_t$ is a symplectic manifold. The dynamics of the system is a submanifold D of $\mathsf{T}_1 P$. Let D_t denote a subset of D defined by

$$D_t = D \cap \mathsf{T}_1 P_t.$$

The system is *Lagrangian* if for each $t \in \mathbb{R}$ the dynamics D_t is a Lagrangian submanifold of $\mathsf{T}_1 P_t$. The existence of a Lagrangian generating function follows from the theorem

Theorem 1 *Let $\mathsf{T}_1\mathsf{T}M_t$ be a submanifold of $\mathsf{T}\mathsf{T}M$ defined by*

$$\mathsf{T}_1\mathsf{T}M_t = \{\mathsf{T}\mathsf{T}M \ni w; \mathsf{T}\tau_M w \in \mathsf{T}_1 M \textit{ and } \tau_{\mathsf{T}M} w \in \mathsf{T}M_t\}.$$

There are canonical isomorpisms of vector bundles

$$\widetilde{\kappa}_{M_t}\colon (\tau_{\mathsf{T}_1 M_t}\colon \mathsf{T}\mathsf{T}_1 M_t \to \mathsf{T}_1 M_t) \longrightarrow (\mathsf{T}\tau_M\colon \mathsf{T}_1\mathsf{T}M_t \to \mathsf{T}_1 M_t) \tag{3.1}$$

$$\widetilde{\alpha}_{M_t}\colon (\mathsf{T}\pi_M\colon \mathsf{T}_1\mathsf{T}^* M_t \to \mathsf{T}_1 M) \longrightarrow (\pi_{\mathsf{T}_1 M_t}\colon \mathsf{T}^*\mathsf{T}M \to \mathsf{T}M) \tag{3.2}$$

Proof. The bundle $\mathsf{T}\tau_M\colon \mathsf{T}_1\mathsf{T}M_t \to \mathsf{T}_1 M_t$ is defind as a subbundle of the bundle $\mathsf{T}\tau_M\colon \mathsf{T}\mathsf{T}M \to \mathsf{T}M$.

Also the bundle $\tau_{\mathsf{T}_1 M_t}\colon \mathsf{T}\mathsf{T}_1 M_t \to \mathsf{T}_1 M_t$ can be considered as a subbundle of the bundle $\tau_{\mathsf{T}M}\colon \mathsf{T}\mathsf{T}M \to \mathsf{T}M$. It is an easy exercise to verify that κ_M restricted to $\mathsf{T}\mathsf{T}_1 M_t$ gives the required isomorphism.

The isomorphism α_M is defined as the dual to κ_M. We define the isomorphism $\widetilde{\alpha}_{M_t}$ as the dual to $\widetilde{\kappa}_{M_t}$ as well. Since α_M is a symplectomorphism we conclude that also $\widetilde{\alpha}_{M_t}$ is a symplectomorphism.

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It follows that a Lagrangian system can be generated by a Lagrangian generating function defined on $\mathsf{T}_1 M$. The Hamiltonian formulation of a dynamics is more complicated and requires affine structures.

3.3. Affine Poisson Structures

This paragraph is based on the relation between Legendre transformation and the affine duality[7]. A *Lagrange bundle* is the trivial line bundle $\widehat{\mathsf{T}}_1 M = \mathsf{T}_1 M \times \mathbb{R}$ over $\mathsf{T}_1 M$. By ξ we denote the canonical projection

$$\xi \colon \widehat{\mathsf{T}}_1 M \to \mathsf{T}_1 M.$$

Lagrangians are sections of the fibration ξ. With respect to the projection

$$\tau_M \circ \xi \colon \widehat{\mathsf{T}}_1 M \to M$$

$\widehat{\mathsf{T}}_1 M$ is a special affine bundle. The affine dual $\widehat{\mathsf{T}}_1^{\#} M$ to this bundle is a *Hamiltonian bundle.*

Proposition 4 *The special affine bundle $\widehat{\mathsf{T}}_1^{\#} M$ is isomorphic to the cotangent bundle $\pi_M \colon \mathsf{T}^* M \to M$. The distinguished covector field ϑ is defined by $\langle v, \vartheta \rangle = 0$ for $\mathsf{T}\zeta v = 0$ and $\langle \partial_t, \vartheta \rangle = 1$.*

Proof. Let us fix $m \in M$. Elements of $\widehat{\mathsf{T}}_1^{\#} M$ over m are affine functions on $\mathsf{T}_{1,m} M$. An affine function on $\mathsf{T}_{1,m} M$ has the unique extension to a linear function on $\mathsf{T}_m M$, i. e., to an element of $\mathsf{T}^* M$. The distinguished element of $\widehat{\mathsf{T}}_1^{\#} M$ at m is the equal to 1 constant function. The liner extension of this function is a ζ-vertical covector, equal to 1 on vectors which project onto ∂_t.

■

With this isomorphism the line bundle structure of $\widehat{\mathsf{T}}_1^{\#} M$ is given by the canonical projection

$$\chi \colon \mathsf{T}^* M \to P.$$

Theorem 2 *There is a canonical isomorphism of affine bundles*

$$\widetilde{\beta}_{M_t} \colon (\tau_P \colon \mathsf{T}_1 P_t \to P_t) \longrightarrow (\mathsf{P}\chi \colon \mathsf{P}(\widehat{\mathsf{T}}_1^{\#} M_t) \to P_t),$$

which is also a symplectomorphism.

Proof. Let $\gamma \colon P_t \to \mathsf{T}^* M$ be a section of the fibration χ. We define a function $\widetilde{\gamma}$ on $\chi^{-1}(P_t)$ by the formula

$$\widetilde{\gamma}(\gamma(p) + s\vartheta) = -s.$$

We define a relation R from $C^\infty(\mathsf{T}^* M)$ to the space of sections of χ over P_t:

$\gamma \in R(f)$ if $f = \widetilde{\gamma}$ on $\chi^{-1}(P_t)$.

The relation R defines a relation

$$\mathrm{d}R \colon \mathsf{T}^* M \to \mathsf{P}(\widehat{\mathsf{T}}_1^{\#} M_t).$$

It is easy to verify that the composition $dR \circ \beta_M$ projects to an isomorphism $\widetilde{\beta}_{M_t}$ of affine bundles

$$\widetilde{\beta}_{M_t} \colon (\tau_P \colon \mathsf{T}_1 P_t \to P_t) \longrightarrow (\mathsf{P}\chi \colon \mathsf{P}(\widehat{\mathsf{T}}_1^{\#} M_t) \to P_t),$$

and that this isomorphism is a symplectomorphism.

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It follows that the dynamics of a Lagrangian system can be generated by a section of the fibration $\chi\colon \mathsf{T}_1 P \to P$. This section we call the Hamiltonian generating section. It is easy to verify that the image of the Hamiltonian generating section is the submanifold C of $\mathsf{T}^* M$ we mentioned in Section 3.1. The collection $(\widetilde{\beta}_{M_t})$ of isomorphisms defines a morphism Λ of affine bundles

$$\Lambda\colon (\mathsf{P}\chi\colon \mathsf{P}(\widehat{\mathsf{T}}_1^{\#} M) \to P) \longrightarrow (\tau_P \mathsf{T}_1 P\colon P).$$

Let $\Gamma(\chi)$ be the afine space of sections of the fibration χ. With the morphism Λ we define an *affine Poisson bracket* as a mapping

$$\{,\}\colon \Gamma(\chi) \times C^\infty(P) \to C^\infty(P)$$

defined by

$$\{\gamma, f\}(p) = \Lambda(d_p\gamma)(f).$$

The bracket $\{,\}$ has the following properties:

- it is affine with respect to the first and linear with respect to the second argument,
- the linear part is a linear Poisson bracket,
- for each section γ the mapping $C^\infty(P) \to C^\infty(P)\colon f \mapsto \{\gamma, f\}$ defines a canonical vector field on P.

A discussion on the concept of an affine Poisson structure will be given in a separate publication.

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