Liouville structures

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DARBOUX's THEOREM

Darboux, G., Sur le problème de Pfaff, *Bull. Sci. Math* **6** (1882)

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Let (M, ω) be a symplectic manifold of dimension $2n$. Every point a of M has an open neighbourhood U, which is the domain of a chart (U, φ) with local coordinates $x^1, \ldots, x^{2n}$, such that the 2-form ω has the local expression

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^{n+i} \wedge \mathrm{d}x^i$$

on U.

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For reasons of interpretation this structure can not be replaced by the corresponding cotangent fibration.

Definition

A Liouville structure is a vector fibration isomorphism

$$
\begin{array}{ccc}
P & \xrightarrow{\alpha} & \mathsf{T}^*Q \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_Q} \\
Q & = & Q
\end{array}
$$

This is a preliminary definition.

Example 1

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There is a canonical symplectomorphism

$$\mathsf{T}^*E \longleftrightarrow \mathsf{T}^*E^*$$

There are TWO different Liouville structures on T^*E.

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We have a vector bundle isomorphism, which is also a symplectomorphism

$$\begin{array}{ccc} \mathsf{T}P & \xrightarrow{\widetilde{\omega}} & \mathsf{T}^*P \\ {\scriptstyle \tau_P}\downarrow & & \downarrow{\scriptstyle \pi_P} \\ P & = & P \end{array}$$

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A form μ on E is linear if $\mathcal{L}_Z\mu = \mu$, where Z is the Euler vector field on E.
A 1-form ϑ is linear if it is a linear mapping of vector fibrations

$$\begin{array}{ccc} E & \xrightarrow{\ \vartheta\ } & \mathsf{T}^*E \\ {\scriptstyle \eta}\Big\downarrow & & \Big\downarrow \\ Q & \longrightarrow & E^* \end{array}$$

A 2-form μ is linear if the associated linear map $\widetilde{\mu}$

$$\begin{array}{ccc} \mathsf{T}E & \xrightarrow{\widetilde{\mu}} & \mathsf{T}^*E \\ {\scriptstyle \tau_E}\downarrow & & \downarrow{\scriptstyle \pi_E} \\ E & = & E \end{array}$$

is also a linear mapping of vector fibrations

$$\begin{array}{ccc} \mathsf{T}E & \xrightarrow{\widetilde{\mu}} & \mathsf{T}^*E \\ {\scriptstyle \mathsf{T}\eta}\downarrow & & \downarrow \\ \mathsf{T}Q & \longrightarrow & E^* \end{array}$$

Alternative definitions of a Liouville structure

- A bilinear non degenerate pairing

$$\langle\ ,\ \rangle : P \times_{(\pi,\tau_Q)} \mathsf{T}Q \longrightarrow \mathbb{R}.$$

$$\langle \alpha(p), v\rangle_Q = \langle p, v\rangle.$$

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- Linear symplectic form ω on P.

$$\langle\ ,\ \rangle : P \times_{(\pi,\tau_Q)} \mathsf{T}Q \longrightarrow : (p, v) \mapsto \omega(\chi_\pi(O_\pi(\pi(p)), p), \mathsf{T}O_{\tau_Q}(v)).$$

Here $\alpha^*\omega_Q = \omega$.

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- Linear and vertical 1-form ϑ (a Liouville form) with non degenerate

$$\omega = \mathrm{d}\vartheta.$$

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$$\vartheta - \alpha^* \vartheta_Q = \mathrm{d} f.$$

f is linear, $\mathrm{d} f$ is vertical, hence $f = 0$.

Relations

A vector fibration relation is differential relation of fibrations

$$\begin{array}{ccc} P & \xrightarrow{\rho} & P' \\ \pi \downarrow & & \downarrow \pi' \\ Q & \xrightarrow{\sigma} & Q' \end{array}$$

such that for each $(q', q) \in \mathrm{graph}(\sigma)$ the set $\mathrm{graph}(\rho) \cap (P'_{q'} \times P_q)$ is a vector subspace of $P'_{q'} \times P_q$.

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such that for each $(q', q) \in \mathrm{graph}(\sigma)$ the set $\mathrm{graph}(\rho) \cap (P'_{q'} \times P_q)$ is a vector subspace of $P'_{q'} \times P_q$. A Liouville structure morphism is a

vector fibration relation such that one of the following conditions is satisfied:

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- If the Liouville structures are characterized by .....

Propositions

Proposition. *Let* $\pi : P \to Q$ *be a vector fibration. If* K *is a closed submanifold of* P *such that for each* $q \in C = \pi(K)$ *the intersection* $K_q = K \cap P_q$ *of* K *with* $P_q = \pi^{-1}(q)$ *is a vector subspace of* P_q*, then* C *is a submanifold of* Q *and the dimension of* K_q *is locally constant and the mapping*

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Proposition. *If* K *is a submanifold of the cotangent bundle* T^*Q *with the properties*

1. *the dimension of* K *is equal to the dimension of* Q*,*
2. *for each* $q \in C = \pi_Q(K)$ *the intersection* $K_q = K \cap \mathsf{T}^*_q Q$ *of* K *with the fibre* $\mathsf{T}^*_q Q = \pi_Q^{-1}(q)$ *is a vector subspace of the fibre,*
3. *the Liouville form* ϑ_Q *vanishes on* $\mathsf{T}K$*,*

then $C \subset Q$ *is a submanifold and* $K = \mathsf{T}^\circ C$*.*

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A morphism of Liouville structures is the 'phase lift' of a differential relation between base manifolds.

The tangent functor

The *tangent functor* T associates the structure

$$\begin{array}{c} (\mathsf{T}P, \mathrm{d}_T\vartheta) \\ \mathsf{T}\pi \downarrow \\ \mathsf{T}Q \end{array}$$

with a Liouville structure

$$\begin{array}{c} (P, \vartheta) \\ \pi \downarrow \\ Q \end{array}$$

and the morphism

$$\begin{array}{ccc} (\mathsf{T}P, \mathrm{d}_T\vartheta) & \xrightarrow{\ \mathsf{T}\rho\ } & (\mathsf{T}P', \mathrm{d}_T\vartheta') \\ {\scriptstyle \mathsf{T}\pi}\downarrow & & \downarrow{\scriptstyle \mathsf{T}\pi'} \\ \mathsf{T}Q & \xrightarrow{\ \mathsf{T}\sigma\ } & \mathsf{T}Q' \end{array}$$

with a Liouville structure morphism

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The Hamilton functor

The *Hamilton functor* H is a covariant functor from the category of symplectic manifolds to the category of Liouville structures.

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It associates the Liouville structure

$$\begin{array}{c} (\mathsf{T}P, \mathrm{i}_T\omega) \\ \Big\downarrow \tau_P \\ P \end{array}$$

with a symplectic manifold (P, ω).

and the morphism

$$\begin{array}{ccc} (\mathsf{T}P, \mathrm{i}_T\omega) & \xrightarrow{\mathsf{T}\varphi} & (\mathsf{T}P', \mathrm{i}_T\omega') \\ \Big\downarrow{\scriptstyle \mathsf{T}\pi} & & \Big\downarrow{\scriptstyle \tau'_P} \\ P & \xrightarrow{\varphi} & P' \end{array} \qquad (237)$$

with a symplectomorphism $\varphi : (P, \omega) \longrightarrow (P', \omega')$.

Generating functions

A Liouville structure offers the possibility of generating from generating objects subsets of a symplectic manifold (P, ω) for which the Liouville structure is established. Such subsets are usually Lagrangian submanifolds.

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An example of a generating object is a constrained generating function

$$U : C \to \mathbb{R},$$

defined on a submanifold of $C \subset Q$. The set

$$S = \Big\{ f \in P \ ; \ \pi_Q(f) \in C \ \text{ and } \ \forall_{\delta q \in \mathsf{T}C} \ \text{ if } \ \tau_Q(\delta q) = \pi(f),$$

$$\text{then } \ \langle f, \delta q \rangle = \langle \mathrm{d}U, \delta q \rangle_C \Big\}$$

is the Lagrangian submanifold of (P, ω) generated by the constrained function U. This submanifold is an affine bundle over C, modelled on the vector bundle $\mathsf{T}^{\circ}C$.

Example 3

Let

$$\begin{array}{c} (\mathsf{T}P, \mathrm{d}_T\vartheta) \\ \Big\downarrow {\scriptstyle \mathsf{T}\pi} \\ \mathsf{T}Q \end{array} \qquad\qquad \begin{array}{c} (\mathsf{T}P, \mathrm{i}_T\mathrm{d}\vartheta) \\ \Big\downarrow {\scriptstyle \tau_P} \\ P \end{array}$$

be Liouville structures derived from a Liouville structure

$$\begin{array}{c} (P, \vartheta) \\ \Big\downarrow {\scriptstyle \pi} \\ Q \end{array}$$

The pairing

$$\langle\ ,\ \rangle : P \times_{(\pi,\tau_Q)} \mathsf{T}Q \to \mathbb{R}$$

is a differentiable function defined on the submanifold $P \times_{(\pi,\tau_Q)} \mathsf{T}Q \subset P \times \mathsf{T}Q$ and the diagonal Δ of $\mathsf{T}P \times \mathsf{T}P$ is the graph of the identity symplectomorphism in $(\mathsf{T}P, \mathrm{d}_T\omega)$.

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The diagonal Δ is generated by the function $-\langle\ ,\ \rangle$.