Dirac Algebroids

Janusz Grabowski*, Katarzyna Grabowska†

*Polish Academy of Sciences
† University of Warsaw

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Dirac structures

- There is a symmetric pairing on the bundle $\mathcal{T}N = TN \oplus_N T^*N$:

$$
(X_1 + \alpha_1 \mid X_2 + \alpha_2) = \frac{1}{2} (\alpha_1(X_2) + \alpha_2(X_1)).
$$

- Courant-Dorfman bracket on the space of $\text{Sec}(\mathcal{T}N)$:

$$
[[X_1 + \alpha_1, X_2 + \alpha_2]] = [X_1, X_2] + \mathcal{L}_{X_1} \alpha_2 - \iota_{X_2} d\alpha_1.
$$

**Definition**

An almost Dirac structure on the smooth manifold $N$ is a subbundle $D$ of $\mathcal{T}N$ which is maximally isotropic with respect to the symmetric pairing $(\cdot | \cdot)$. If additionally the space of sections of $D$ is closed under the Courant-Dorfman bracket, we speak about a Dirac structure.

Note that here a subbundle $D$ may be supported on a submanifold $N_0 \subset N$. 
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Note that here a subbundle $D$ may be supported on a submanifold $N_0 \subset N$. 
The first integrability condition for the almost Dirac structure says that

\[ pr_{TN}(D) \subset TN_0, \]

so the Courant-Dorfman bracket reduces to a well-defined bracket \([\cdot, \cdot]_D\) on sections of \(D\).

The second integrability condition says that \([\cdot, \cdot]_D\) takes values in \(\text{Sec}(D)\):

\[ [\cdot, \cdot]_D : \text{Sec}(D) \times \text{Sec}(D) \to \text{Sec}(D) \subset \text{Sec}((TN)|_{N_0}). \]

By definition, an almost Dirac structure is a Dirac structure if and only if it satisfies both the integrability conditions.
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• For $\Pi \in \text{Sec}(\bigwedge^2 TN)$, $\tilde{\Pi} : T^*N \ni \alpha \mapsto _{\alpha}\Pi \in TN$,

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\text{graph}(\tilde{\Pi}) \subset TN \quad \text{is an almost Dirac structure.}
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If $\Pi$ is a Poisson tensor, then it is a Dirac structure.

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If $\omega$ is a closed 2-form, then it is a Dirac structure.

• For a distribution $\Delta$ on $N$,

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If $\Delta$ is integrable, then it is a Dirac structure.
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Double vector bundles

Definition

A **double vector bundle** is a manifold with two compatible vector bundle structures. Compatibility means that the Euler vector fields associated with the two structures commute.

- \( \tau_1, \tau_2, \tau'_1, \tau'_2 \) are v.b.
- The core

\[
C = \{ k \in K : \tau_1(k) = 0, \tau_2(k) = 0 \},
\]

\( \tau_0 \) is a v.b.

- \((\tau_1 \cdot \tau'_1), (\tau_2 \cdot \tau'_2)\) are v.b. morphisms

- There is one more (affine) bundle

\[
\tau_1 \times \tau_2 : K \rightarrow K_1 \times_M K_2
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modeled on the pull-back of the core

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K_1 \times_M K_2 \times_M C \rightarrow K_1 \times_M K_2
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First example is usually $TE...$

\[\tau : E \longrightarrow M\]
\[(x^a, y^i) \longmapsto (x^a)\]

\[\tau_M : TM \longrightarrow M\]
\[(x^a, \dot{x}^b) \longmapsto (x^a)\]

\[\pi : E^* \longrightarrow M\]
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\[\nabla_1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{\xi}_i \partial_{\dot{\xi}_i}\]

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\[
\nabla_{TM} : TE^* \longrightarrow TM
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\]
We can add vectors $v, w$ such that

$$T\pi(v) = T\pi(w).$$

For $v, w$ take curves $\gamma_v, \gamma_w$:

$$\pi \circ \gamma_v = \pi \circ \gamma_w.$$

$v + w$ is tangent to

$$t \mapsto \gamma_v(t) + \gamma_w(t).$$
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Second example: $T^*E^*$.

\[ \pi_{E^*} : T^*E^* \rightarrow E^* \]
\[ (x^a, \xi_i, p_b, y^j) \mapsto (x^a, \xi_i) \]

\[ \zeta : T^*E^* \rightarrow E \]
\[ (x^a, \xi_i, p_b, y^j) \mapsto (x^a, y^j) \]

\[ \nabla_1 = p_a \partial_{p_a} + y^i \partial_{y^i} \]

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Second example: $T^*E^*$.

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Second example: $T^*E^*$.

$$0_{E^*} \sim E^* \sim E^* \sim TM$$

$$E^* \xrightarrow{\sim} E^* \xrightarrow{\sim} E^*$$

$$T_0E^* \simeq E^*_x \oplus T_xM$$

$$T^*E^* \simeq E^*_x \oplus T^*_xM$$

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Second example: $T^*E^*$.

$T^*E^*$ is isomorphic to $T^*E$. The graph of the canonical isomorphism $\mathcal{R}$ is the lagrangian submanifold generated in

$$T^*(E^* \times E) \cong T^*E^* \times T^*E \quad \text{by} \quad E^* \times_M E \ni (\xi, y) \longmapsto \xi(y) \in \mathbb{R}.$$
Second example: $T^* E^*$.

\[ T^* E^* \xrightarrow{\pi_{E^*}} E^* \quad \xrightarrow{\pi_M \tau} T^* M \quad \xrightarrow{\zeta} E \]

\[ E^* \xrightarrow{\pi} M \]

\[ T^* (E^* \times E) \cong T^* E^* \times T^* E \quad \text{by} \quad E^* \times_M E \ni (\xi, y) \mapsto \xi(y) \in \mathbb{R}. \]

\[ \mathcal{R} : (x^a, y^i, p_b, \xi_j) \longmapsto (x^a, \xi_i, -p_b, y^j) \]
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\[
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T^*E^* & \xrightarrow{\pi_{E^*}} & E^* \\
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\end{array}
\quad
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Definition

A vector subbundle of a vector bundle $\tau : E \to M$ is a submanifold $F \subset E$ such that it is invariant with respect to the family of homotheties defined by the vector bundle structure $\tau$.

- Euler vector field is tangent to $F$;
- $F$ can be supported on a submanifold $M_0 \subset M$.

Definition

A double vector subbundle of a double vector bundle $K$ is a submanifold $D \subset K$ such that it is invariant with respect to both families of homotheties defined by the vector bundle structures $\tau_1$ and $\tau_2$.

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A double vector subbundle of a double vector bundle $K$ is a submanifold $D \subset K$ such that it is invariant with respect to both families of homotheties defined by the vector bundle structures $\tau_1$ and $\tau_2$.

- Both Euler vector fields are tangent to $D$;
- $D$ defines subbundles $F_1 \subset K_1$, $F_2 \subset K_2$ supported on $M_0 \subset M$. 
Dirac algebroids

*Linearity* of different geometrical structures is connected with double vector bundles.

- A connection $\Gamma$ on a vector bundle $F \to M$ is *linear* if the map

$$\tilde{\Gamma} : TF \longrightarrow VF \oplus_F (F \times_M TM) = F \times_M F \times_M TM$$

is a double vector bundle morphism.

- A Poisson tensor $\Pi$ on a vector bundle $F$ is linear if the corresponding map

$$\tilde{\Pi} : T^*F \longrightarrow TF$$

is a double vector bundle morphism.
Linearity of different geometrical structures is connected with double vector bundles.

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- \ldots
A general algebroid is a double vector bundle morphism covering the identity on $E^*$:

$$
\begin{align*}
\varepsilon(x^a, y^i, p_b, \xi_j) &= (x^a, \xi_i, p_b(x)y^k, c^{k}_{ij}(x)y^i\xi_k + \sigma^{a}_{j}(x)\rho_a) \\
\Pi_{\varepsilon} &= c^{k}_{ij}(x)\xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \rho^{b}_{i}(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma^{a}_{j}(x)\partial_{x^a} \otimes \partial_{\xi_j}.
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\[ \tau_1 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \xi_i), \]
\[ \tau_2 : (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) \mapsto (x^a, \dot{x}^b, y^k), \]

\[ \nabla_1 = p_a \partial_{p_b} + \dot{\xi}_j \partial_{\dot{\xi}_j} + y^i \partial_{y^i} + \dot{x}^b \partial_{\dot{x}^b}, \]
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**Definition**

A Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle \( E \) is an almost Dirac (resp., Dirac) subbundle \( D \) of \( TE^* \) being a double vector subbundle, i.e., \( D \) is not only a subbundle of \( \tau_1 : TE^* \to E^* \) but also a vector subbundle of the vector bundle \( \tau_2 : TE^* \to TM \oplus_M E \).
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\nabla_1 &= p_a \partial_{p_b} + \dot{\xi}_j \partial_{\dot{\xi}_j} + y^i \partial_{y^i} + \dot{x}^b \partial_{\dot{x}^b}, \\
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\[ \nabla_2 = p_a \partial_{p_b} + \xi_i \partial_{\xi_i} + \dot{\xi}_j \partial_{\dot{\xi}_j} \]
- $Ph_D = \tau_1(D)$ - phase bundle
- $Vel_D = \tau_2(D)$ - velocity bundle (anchor relation)
- $C_D \subset E^* \oplus_M T^* M$ - core bundle for $D$
The graph of any linear bivector field is a Dirac algebroid,

\[ \Pi = \frac{1}{2} c_{ij}^k(x) \xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + \rho_i^b(x) \partial_{\xi_i} \wedge \partial_x^b, \quad c_{ij}^k(x) = -c_{ji}^k(x), \]

\[ D\Pi = \{(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) : \dot{x}^b = \rho_i^b(x)y^k, \quad \dot{\xi}_j = c_{ij}^k(x)y^i \xi_k - \rho_j^a(x)p_a\}. \]

The phase bundle is \( E^* \), the velocity bundle is the graph of \( \rho : E \to TM \), the core bundle is the graph of \( -\rho^* \).

The graph of any linear two-form is a Dirac algebroid,

\[ \omega = \frac{1}{2} c_{ab}^k(x) \xi_k dx^a \wedge dx^b + \rho_i^b(x)d\xi_i \wedge dx^b, \quad c_{ab}^k(x) = -c_{ba}^k(x), \]

\[ D\omega = \{(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j, p_c, y^k) : y^i = \rho_i^a(x)\dot{x}^a, \quad p_a = c_{ab}^k(x)\xi_k\dot{x}^b - \rho_i^a(x)\dot{\xi}_i\}. \]

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The phase bundle is \(E^*\), the velocity bundle is the graph of \(\rho : TM \to E\), the core bundle is the graph of \(-\rho^*\).
Due to isotropy condition for $D$ we have the following.

**Theorem**

The core bundle of a Dirac algebroid $D$ is the annihilator subbundle $\text{Vel}_D^0 \subset T^* M \oplus M E^*$ of the anchor relation $\text{Vel}_D$:

$$\mathcal{C}_D = \text{Vel}_D^0.$$  

For Dirac-Lie algebroids we have the following.

**Theorem**

If $D$ is a Dirac-Lie algebroid, then it induces a Lie algebroid structure on the bundle $\text{Vel}_D$.  

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Dirac Algebroids  

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**Theorem**

*If $D$ is a Dirac-Lie algebroid, then it induces a Lie algebroid structure on the bundle $\text{Vel}_D$.***
\[ \begin{align*}
TE^* \oplus_{E^*} T^*E^* & \\
\tau_1 & \quad \tau_2 \\
E^* \oplus_M T^*M & \quad TM \oplus_M E \\
\pi & \\
M & \\
\end{align*} \]

\[ \begin{align*}
TE^* \oplus_{E^*} T^*E^* \\
\downarrow \\
E^* \times_M (E \oplus_M TM) \\
\downarrow \\
E^* \times_M (E \oplus_M TM) \\
\end{align*} \]

\[ \begin{align*}
\tau_1 & \quad \tau_2 \\
\pi & \\
E^* \oplus_M T^*M & \quad TM \oplus_M E \\
\downarrow \\
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\downarrow \\
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\[ \begin{align*}
(x, \xi, \dot{x}, \dot{\xi}, p, y) \\
\downarrow \\
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\[ TE^* \oplus_{E^*} T^*E^* \]
\[ \tau_1 \]
\[ \tau_2 \]
\[ E^* \]
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\[ TM \oplus_M E \]
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\[ M \]

\[ E^* \times_M (E \oplus_M TM) \times_M (T^*M \oplus_M E^*) \]

\[ (x, \xi, \dot{x}, \dot{\xi}, p, y) \]

\[ E^* \times_M (E \oplus_M TM) \]

\[ (x, \xi, \dot{x}, y) \]
\[
E^* \times_M (E \oplus_M TM) \times_M (T^* M \oplus_M E^*) \\
\downarrow \\
E^* \times_M (E \oplus_M TM) \\
\]

\[(x, \xi, \dot{x}, \dot{\xi}, p, y) \longleftrightarrow (x, \xi, \eta, \hat{\eta}, \zeta, \hat{\zeta})\]

- Coordinates in \( E \oplus_M TM = \text{Vel}_D \oplus V \),
  \[(x, \eta, \hat{\eta}) : \quad \text{Vel}_D = \{\hat{\eta} = 0\} \, .\]

- Dual coordinates in \( T^* M \oplus_M E = V^0 \oplus \text{Vel}_D^0 \),
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- For \( D \) we have now
  \[(x, \xi, \eta, \hat{\eta}, \zeta, \hat{\zeta}) : \quad \hat{\eta} = 0, \quad \hat{\zeta} \text{ arbitrary}, \ldots \]
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For $D$ we have now

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More conditions

$$\zeta_k = c^i_{jk}(x)\eta^i\xi_i, \quad \text{isotropy gives} \quad c^i_{jk}(x) = -c^i_{kj}(x).$$

If $Ph_D \subsetneq E^*$

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**Theorem**

*In the introduced coordinates we have*

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Example with application

We start with a Dirac algebroid $D$ and a vector subbundle $V \subset Vel_D$.

- $V \subset Vel_D \subset E \oplus_M TM$
- $\tilde{V} = (\tau_2^D)^{-1}(V)$;
- $V^0 \subset T^*M \oplus_M E^*$, $V^0 \supset C_D$
- $D^V = \tilde{V} + V^0$

**Definition**

The Dirac algebroid $D^V$ is called induced from $D$ by the subbundle $V$. 
Example with application

We start with a Dirac algebroid $D$ and a vector subbundle $V \subset Vel_D$.

Definition

The Dirac algebroid $D^V$ is called induced from $D$ by the subbundle $V$. 

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\begin{itemize}
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  \item $V \subset Vel_D \subset E \oplus_M TM$
  \item $\tilde{V} = (\tau_2^D)^{-1}(V)$;
  \item $V^0 \subset T^*M \oplus_M E^*$, $V^0 \supset C_D$
  \item $D^V = \tilde{V} + V^0$
\end{itemize}

Definition

The Dirac algebroid $D^V$ is called induced from $D$ by the subbundle $V$. 

\begin{itemize}
  \item $\tau_1^D$
  \item $\tau_2^D$
  \item $\pi^D$
  \item $\tau_1^D$
  \item $\tau_2^D$
  \item $\pi^D$
\end{itemize}
Example with application
We start with a Dirac algebroid $D$ and a vector subbundle $V \subset \text{Vel}_D$.

\[ \begin{align*}
& \text{Definition} \\
& \text{The Dirac algebroid } D^V \text{ is called } \text{induced} \text{ from } D \text{ by the subbundle } V. 
\end{align*} \]
Dirac algebroids in mechanics

How to obtain phase equations from a Lagrangian (or Hamiltonian):

- Bundle of configurations: $T^*M$, phase bundle: $T^*T^*M$.

$$D_L = \alpha^{-1}_M(dL(TM)), \quad D_H = \tilde{\omega}^{-1}_M(dH(T^*M)).$$
Dirac algebroids in mechanics

How to obtain phase equations from a Lagrangian (or Hamiltonian):

- Bundle of configurations: $TM$, phase bundle: $T^*M$.

\[
D_L = \alpha_M^{-1}(dL(TM)), \quad D_H = \tilde{\omega}_M^{-1}(dH(T^*M)).
\]
Bundle of configurations: $E$ (skew-algebroid), phase bundle: $E^*$. 

\[ DL = \varepsilon(dL(E)), \quad DH = \tilde{\Pi}(dH(E^*)). \]
Bundle of configurations: $E$ (skew-algebroid), constraints $W \subset E$.

- $W$ defines $V = \{y + v \in E \oplus_M TM : y \in W, \ v = \rho(y)\} \subset Vel_{D\pi}$.
- We induce $D^V_{\pi}$.
- $D^V_{\pi}$ gives the relations

$$\varepsilon_V : T^*E \longrightarrow TE^*, \quad \beta_V : T^*E^* \longrightarrow TE^*.$$


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