Variational principles for dissipative systems

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Credo

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  - virtual work function $\sigma \colon C^1 \rightarrow \mathbb{R}$,

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- Ingredients:
  - configuration manifold Q,
  - constrained set $C^1 \subset \mathsf{T}Q$,
  - virtual work function $\sigma\colon C^1 \longrightarrow \mathbb{R}$,
- with the properties:
  - for each $q \in C^0 = \tau_Q(C^1)$ the set $C^1_q = \mathsf{T}_q Q \cap C^1$ is a cone, i.e. $\lambda v \in C^1_q$ for each $v \in C^1_q,\ \lambda \geqslant 0$,
  - virtual work function is positive homogeneous, i.e. $\sigma\colon (\lambda v) = \lambda\sigma(v)$ for $\lambda \geqslant 0$.

The principle of virtual work

is incorporated in the definition of the constitutive set

$$S = \{f \in \mathsf{T}^*Q; q = \pi_Q(f) \in C^0,\ \forall_{v \in C^1_q}\ \sigma(v) - \langle f, v \rangle \geqslant 0\}.$$

The main reference

Włodzimierz M. TULCZYJEW

"The Origin of Variational Principles"

Banach Center Publications 59, "Classical and Quantum Integrability", Warszawa 2003
also math-ph/0405041

Legendre-Fenchel transformation

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The constitutive set S derived from the work function σ is obtained by applying the Legendre-Fenchel transformation to functions

$$\sigma_q\colon C^1_q \longrightarrow \mathbb{R}.$$

The Legendre-Fenchel transforms S_q are then combined

$$S = \bigcup_{q\in C^0} S_q$$

The inverse Legendre-Fenchel transformation

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and the function

$$\sigma : C \longrightarrow \sup_{f \in S} \langle f, v \rangle$$

Properties of the L-F transformation

- The L-F transform S of a positive homogeneous function $\sigma : C \to \mathbb{R}$ is convex and closed.
- The inverse L-F transform $\sigma : C \to \mathbb{R}$ of a subset $S \subset V^*$ is convex and closed (the overgraph of σ is closed).
- The L-F transformation and the inverse L-F transformation establish a one to one correspondence between positive homogeneous closed convex functions defined on cones in V and non empty closed convex subsets of V^*.

It follows that the constitutive set provides a complete characterization of a convex static system

Partially controlled systems

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$$\begin{array}{ccc} \overline{Q} & \xrightarrow{\overline{U}} & \mathbb{R}\,, \\ {\scriptstyle \eta}\downarrow & & \\ Q & & \end{array} \tag{3}$$

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The constitutive set S derived from the potential $\overline{U}$:

$$S = \{f \in \mathsf{T}^*Q; \exists_{\overline{q} \in \overline{Q}}\ \eta(\overline{q}) = \pi_Q(f), \forall_{\overline{v} \in \mathsf{T}_{\overline{q}}\overline{Q}}\ \langle \mathrm{d}\overline{U}, \overline{v} \rangle = \langle f, \mathsf{T}\eta(\overline{v}) \rangle \}$$

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A point $\overline{q} \in \overline{Q}$ 'contributes' to S if and only if $\langle \mathrm{d}\overline{U}, \overline{v} \rangle = 0$ for each vertical $\overline{v} \in \mathsf{T}_{\overline{q}}\overline{Q}$.

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The critical set of the family $(\overline{U}, \eta)$:

$$Cr(\overline{U}, \eta) = \{\overline{q} \in \overline{Q}; \forall_{\overline{v} \in \mathsf{T}_{\overline{q}}\overline{Q}} \quad \text{if} \quad \mathsf{T}\eta(\overline{v}) = 0 \quad \text{then} \quad \langle \mathrm{d}\overline{U}, \overline{v} \rangle = 0\}$$

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\Big\downarrow{\scriptstyle \eta} & & & & \\
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\tag{7}
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For each $\overline{q}$ the function $\overline{\sigma}_{\overline{q}}\colon \mathsf{T}_{\overline{q}}\overline{Q} \longrightarrow \mathbb{R}$ is convex.

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$$S = \{ f \in \mathsf{T}^*Q ; \exists_{\overline{q} \in \overline{Q}}\ \eta(\overline{q}) = \pi_Q(f), \forall_{\overline{v} \in \mathsf{T}_{\overline{q}}\overline{Q}}\ \langle \overline{\sigma}, \overline{v} \rangle \geqslant \langle f, \mathsf{T}\eta(\overline{v}) \rangle \}$$

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The critical set

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Convex generating families

'Contribution' of $\overline{q} \in Cr(\overline{\sigma}, \eta)$ to S

$$S_{\overline{q}} = \{f \in \mathsf{T}^*Q; \pi_Q(f) = \eta(\overline{q}),\ \forall_{\overline{v} \in \mathsf{T}_{\overline{q}}\overline{Q}}\ \langle \overline{\sigma}, \overline{v} \rangle \geqslant \langle f, \mathsf{T}\eta(\overline{v}) \rangle\}$$

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Proposition.

$$S_{\overline{q}} = \{f \in \mathsf{T}^*Q; \pi_Q(f) = \eta(\overline{q}) = q,\ \forall_{\overline{v} \in \mathsf{T}_q Q}\ \langle \sigma_{\overline{q}}, v\rangle \geqslant \langle f, v\rangle\}$$

where

$$\sigma_{\overline{q}} \colon \mathsf{T}_q Q \to \mathbb{R} \colon v \mapsto \inf_{\overline{v}} \overline{\sigma}(\overline{v}),\ \ \overline{v} \in \mathsf{T}_{\overline{q}}\overline{Q},\ \mathsf{T}\eta(\overline{v}) = v.$$

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$\sigma_{\overline{q}}$ is well defined and convex.

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$\sigma_{\overline{q}}$ is well defined and convex.

If $Cr(\overline{\sigma}, \eta)$ is a section of η, the family $(\overline{\sigma}, \eta)$ can be reduced to a function $\sigma \colon \mathsf{T}Q \to \mathbb{R}$

Example 1

A point with configuration q_1 is tied to a fixed point q_0 with a spring of spring constant k_1. Points q_1 and q_2 are tied with a spring of spring constant k_2. The point q_1 is subject to friction and left free.

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The work form of the system is

$$\vartheta(q_1, q_2, v_1, v_2) = k_1(q_1 - q_0|v_1) + \mu\|v_1\| + k_2(q_2 - q_1|v_2 - v_1)$$

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A point (q_1, q_2) is critical if $\|k_1(q_1 - q_0) + k_2(q_1 - q_2)\| \leqslant \mu$

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For a critical point (q_1, q_2)

$$\inf_{v_1} \vartheta(q_1, q_2, v_1, v_2) = k_2(q_2 - q_1|v_2)$$

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$$S_{(q_1,q_2)} = \{f = k_2(q_2 - q_1)\}.$$

Example 2

A point with configuration q_1 is tied to a fixed point q_0 with a spring of spring constant k_1. Points q_1 and q_2 are tied with a spring of spring constant k_2. The point q_1 is left free and point q_2 is subject to friction.

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There is one critical point in a fibre

$$q_1 = q_0 + \frac{k_2}{k_1 + k_2}(q_2 - q_0).$$

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For a critical point (q_1, q_2)

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The family can be reduced to the function

$$(q_2, v_2) \mapsto \mu\|v_2\| + \frac{k_1 k_2}{k_1 + k_2}(q_2 - q_1|v_2).$$

Convex relation

Let a relation $\mathcal{R}\colon \mathsf{T}^*Q_1 \longrightarrow \mathsf{T}^*Q_2$ be generated by a convex function $G\colon \mathsf{T}K \longrightarrow \mathbb{R}$, $K \subset Q_1 \times Q_2$, i.e.

$$b \in \mathcal{R}(a) \text{ if } \langle b, v_2 \rangle - \langle a, v_1 \rangle \leqslant G(v_1, v_2)$$

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Let $D \subset \mathsf{T}^*Q_1$ be generated by a function $\sigma\colon \mathsf{T}C_1 \longrightarrow \mathbb{R}$.

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Let $D \subset \mathsf{T}^*Q_1$ be generated by a function $\sigma\colon \mathsf{T}C_1 \to \mathbb{R}$.

Theorem. *Let K and $C_1 \times Q_2$ have clean intersection and let $Y = K \cap (C_1 \times Q_2)$ then the family*

$$\rho\colon \mathsf{T}Y \longrightarrow \mathbb{R}\colon (q_1, q_2, v_1, v_2) \mapsto G(q_1, q_2, v_1, v_2) + \sigma(v_1)$$

is a generating family of $\mathcal{R}(D)$.

Application

Legendre transformation for dissipative systems.