CASIMIR FORCES INDUCED BY BOSE-EINSTEIN CONDENSATION

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MOTTO

”Tis much better to do a little with certainty

& leave the rest for others that come after,

than to explain all things by conjecture

without making sure of any thing.”

Isaak Newton

HENDRIK BRUGT GERHARD CASIMIR (1909 - 2000)

- H. B. G. Casimir
"On the attraction between two perfectly conducting plates"
Proc. K. Ned. Akad. Wet. **51**, 793-795 (1948).

$$-\frac{\hbar c}{r^4}\frac{\pi^2}{240}$$

- H. B. G. Casimir, D. Polder
"The influence of retardation on the London-van der Waals forces"
Phys. Rev. **73**, 360-372 (1948).

$$U(r) = -\frac{\alpha_1\alpha_2}{r^7}\frac{23\hbar c}{4\pi}$$

TYPES OF THE CASIMIR EFFECT

- electromagnetic
- in quantum field theory
- in particle physics
- in cosmology
- in critical phenomena
- dynamical Casimir effect

STATING THE PROBLEM:

Derive the Casimir effect in an imperfect (interacting) Bose gas filling the volume contained between two infinite parallel plane walls.
Hamiltonian of the imperfect Bose gas:

$$H = H_0 + \frac{a}{V}\frac{N^2}{2}$$

$H_0 =$ kinetic energy (perfect gas Hamiltonian)
$a/V > 0$ =repulsive mean-field interaction per pair of bosons
$V =$ volume occupied by the system.
H is superstable!

METHOD OF ANALYSIS

Bose gas occupies volume $V = L^2 D$ of a rectangular box with linear dimensions $L \times L \times D$.
D denotes the distance between two $L \times L$ square walls.
The excess grand canonical free energy per unit wall area is defined by

$$\omega_s(T, D, \mu) = \lim_{L \to \infty} \left[\frac{\Omega(T, L, D, \mu)}{L^2} \right] - D\, \omega_b(T, \mu)$$

where $\omega_b(T, \mu)$ denotes the grand canonical potential per unit volume evaluated in the thermodynamic limit.
The Casimir force equals

$$F(T, D, \mu) = -\frac{\partial \omega_s(T, D, \mu)}{\partial D}$$

BOUNDARY CONDITIONS

One-particle kinetic energy $\epsilon(\mathbf{k}) = (k_x^2 + k_y^2 + k_z^2)\hbar^2/2m$
z-axis perpendicular to $L \times L$ walls

- periodic

$$k_z = \frac{2\pi}{D} n_z, \quad n_z = 0, \pm 1, \pm 2, ...$$

- Dirichlet

$$k_z = \frac{\pi}{D} n_z, \quad n_z = 1, 2, ...$$

- Neumann

$$k_z = \frac{\pi}{D} n_z, \quad n_z = 0, 1, 2, ...$$

$k_x, \ k_y$ -periodic b.c.

IMPORTANT RELATION

Grand canonical potential

$$\Omega(T, L, D, \mu) = -k_B T \ln \Xi(T, L, D, \mu)$$

$\Xi(T, L, D, \mu)$ is related to the analytic continuation of the perfect gas partition function Ξ_0 by

$$\Xi(T, L, D, \mu) = \exp\left[\frac{\beta L^2 D}{2a}\,\mu^2\right]\sqrt{\frac{L^2 D\beta}{2\pi a}}$$

$$\times(-i)\int_{\alpha-i\infty}^{\alpha+i\infty} dt\,\exp\left[\frac{\beta L^2 D}{a}\,\left(\frac{t^2}{2} - t\mu\right)\right]\Xi_0(T, L, D, t)$$

$$(\alpha < 0)$$

BULK PROPERTIES OF THE IMPERFECT BOSE GAS

The bulk grand canonical free-energy density

$$\omega_b(T,\mu) = -\lim_{L\to\infty} \frac{1}{L^3}\, k_B T\, \ln \Xi(T,L,L,\mu) \; = -p(T,\mu)$$

can be calculated with the use of the steepest descent method.
If $\mu < \mu_c = a n_{0,c}$

$$p(T,\mu) = \frac{1}{2} a n^2(T,\mu) + p_0(T,\mu - a n(T,\mu))$$

where $n(T,\mu)$ is the unique solution of the equation

$$n = n_0(T,\mu - a n)$$

BULK PROPERTIES OF THE IMPERFECT BOSE GAS

If $\mu > \mu_c = an_{0,c}$

$$p(T,\mu) = \frac{\mu^2}{2a} + p_0(T,0)$$

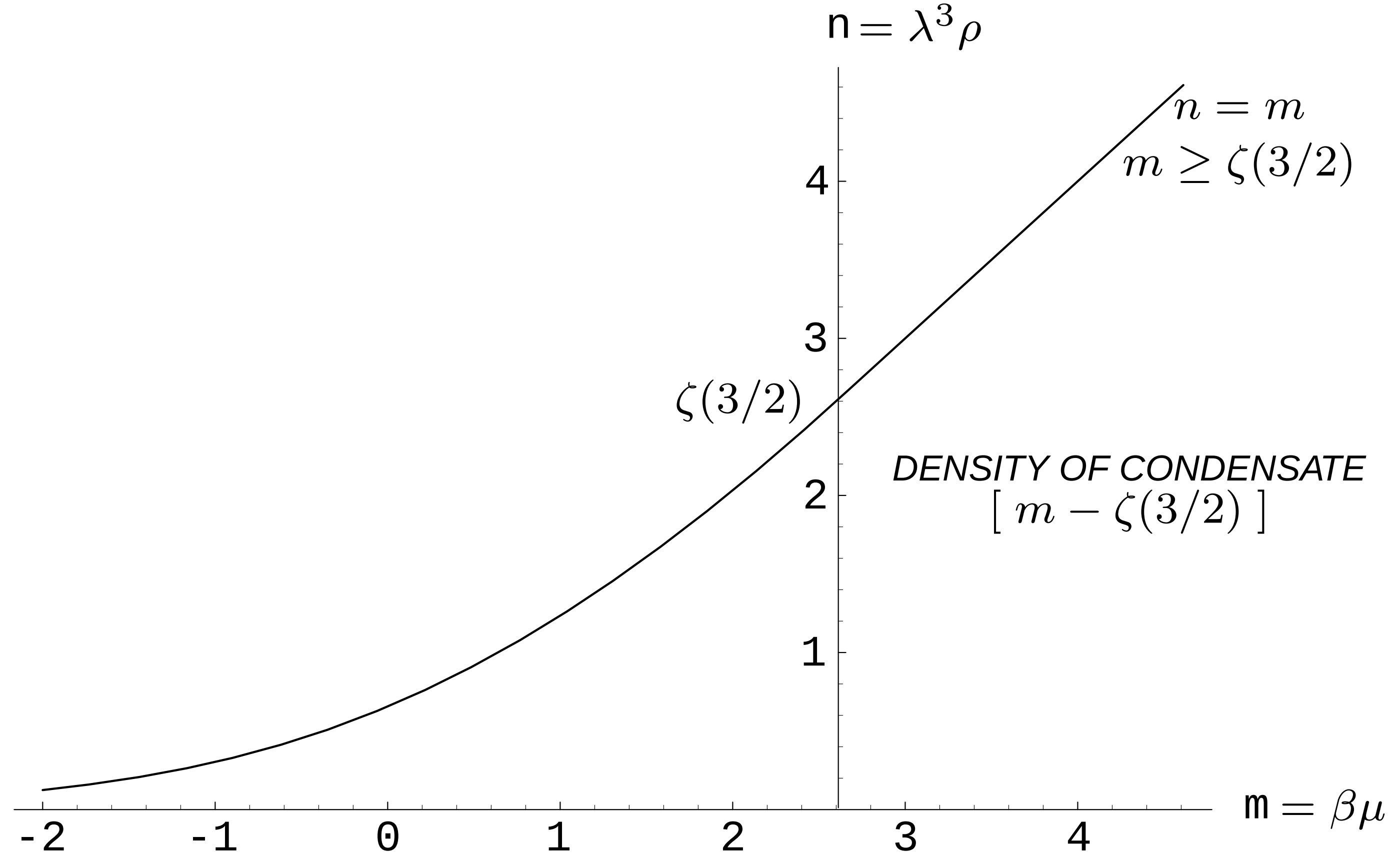
In the two-phase region

$$n = \frac{\mu}{a}$$

and the density of condensate is equal to

$$\left(\frac{\mu}{a} - n_{0,c} \right)$$

IMPERFECT BOSE GAS: CONDENSATION



$$\beta a/\lambda^3 = 1$$

CASIMIR FORCE: PERIODIC BOUNDARY CONDITIONS

The steepest descent method yields the asymptotic form of the excess free energy density. The Casimir force in the one-phase region near the condensation point equals

$$\frac{F(T,D,\mu)}{k_B T} = -\frac{1}{\pi D^3}\left[2\,\Psi(x) - x\,\Psi'(x)\right]$$

with

$$\Psi(x) = \sum_{n=1}^{\infty} \frac{1 + 2nx}{n^3} \exp(-2nx)$$

$$x = \frac{D}{\kappa_{per}}, \quad \kappa_{per} == \lambda \frac{an_c}{(an_c - \mu)} \frac{2\pi^{1/2}}{\zeta(3/2)}$$

CASIMIR FORCE IN THE PRESENCE OF CONDENSATE

In the two-phase region (in the presence of condensate) one observes a power-law decay

$$\frac{F(T,D,\mu)}{k_B T} = -\frac{2\zeta(3)}{\pi}\frac{1}{D^3}, \quad \mu > an_c$$

exactly the same, and with the same amplitude as in the perfect Bose gas.

IMPERFECT AND PERFECT GAS: COMPARING CRITICAL BEHAVIOR

Divergence of the range of exponential forces at the approach to condensation:

imperfect (mean-field) Bose gas

$$\kappa \sim (an_c - \mu)^{-1}$$

perfect Bose gas

$$\kappa_0 \sim (-\mu)^{-1/2}$$

ONE-PARTICLE DENSITY MATRIX
FOR $\alpha = -\mu/K_B T > 0$

THE CASE OF A PERFECT GAS:

$$< \mathbf{x}_2 | \hat{\rho}_1 | \mathbf{x}_1 >= F(|\mathbf{x}_2 - \mathbf{x}_1|)$$

$$\lambda^3 F(x) = \sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \exp\left[-\alpha\, j - \frac{\pi x^2}{j\lambda^2}\right]$$

$$= \frac{\lambda}{x} \exp\left(-2\frac{\sqrt{\pi\alpha}\, x}{\lambda}\right) + \sum_{s=1}^{\infty} \frac{\lambda}{x} \exp\left[-A^+(s)\frac{x}{\lambda}\right] 2\cos\left[-A^-(s)\frac{x}{\lambda}\right]$$

with

$$A^{\pm}(s) = \sqrt{2\pi}(\alpha^2 + 4\pi^2 s^2)^{1/4}\left[1 \pm \frac{\alpha}{(\alpha^2 + 4\pi^2 s^2)^{1/2}}\right] \quad .$$

BULK CORRELATION LENGTH AND RANGE OF CASIMIR FORCES

Correlation function of a perfect Bose gas

$$\lambda^6[n_2(r;\mu,T) - n^2] = \left[\sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \exp\left[-\alpha\, j - \frac{\pi r^2}{j\lambda^2}\right]\right]^2$$

$$\alpha = -\mu/k_B T, \qquad \lambda = h/\sqrt{2\pi m k_B T}$$

Large distance ($r \gg \lambda$) asymptotics

$$\lambda^6[n_2(r;\mu,T) - n^2] \cong \left(\frac{\lambda}{r}\right)^2 \exp\left(-\frac{r}{\xi_0(\mu)}\right) \qquad .$$

$$\xi_0(\mu) = \frac{h}{4\pi\sqrt{2m}} \frac{1}{\sqrt{-\mu}} = \text{range of Casimir force!}$$

PAIR CORRELATIONS IN AN IMPERFECT BOSE GAS

The hierarchy equations for the thermodynamic Green functions in the one-phase region $\mu < an_c$ imply the equality between the imperfect gas correlation function and the perfect gas correlation function calculated for the shifted chemical potential $[\mu - an(T,\mu)]$.
The range of exponentially decaying correlations equals

$$\xi_{imp} = \frac{\lambda}{4}\left(-\frac{k_B T}{\pi[\mu - an(T,\mu)]}\right)^{1/2}$$

$$\text{and diverges} \quad \sim \frac{\lambda}{2\zeta(3/2)}\left(1 - \frac{\mu}{an_{0,c}}\right)^{-1} \qquad ,$$

when μ approaches its critical value $\mu_c = an_{0,c}$ from below.

ELEMENTS OF CALCULATION

Knowledge of the asymptotic behavior of Bose functions

$$g_r(\alpha) = \sum_{q=1}^{\infty} \frac{\exp(-\alpha q)}{q^r}$$

when $\alpha \to 0$

$$g_{1/2}(\alpha) \cong \sqrt{\frac{\pi}{\alpha}}\,, \quad g_{-1/2}(\alpha) \cong \frac{1}{\alpha}\sqrt{\frac{\pi}{\alpha}}$$

permits to evaluate derivatives of the density with respect to the chemical potential at the condensation point $\mu = \mu_{imp,c} = n_c a$.

BULK CORRELATIONS AND CASIMIR FORCES

$\kappa =$ range of Casimir forces.
$\xi =$ range of bulk correlations

- perfect gas

  $\kappa_{0,periodic} = 2\kappa_{0,Dirichlet} = 2\kappa_{0,Neumann} = \xi_0$

  critical exponent $\nu = 1/2$

- imperfect (mean field) gas

  $\kappa_{periodic} = 2\kappa_{Dirichlet} = 2\kappa_{Neumann} = 2\sqrt{\pi}\xi$

  critical exponent $\nu = 1$

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