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Boundary Value Problems for Static Maxwell's Equations

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A generalized version of magnetostatics in differentiable manifolds is formulated. Different boundary value problems are treated as different representations of the same object as graphs of self-adjoint mappings. The Hodge theorem for a domain with the local segment property is proved.

Introduction

This paper adapts to magnetostatics the general approach to linear field theories outlined in References 12 and 13. The main idea of this approach is to treat different boundary value problems as different representations of the same object as graphs of self-adjoint mappings from Hilbertizable spaces to their duals. A generalized version of magnetostatics in differential manifolds instead of affine spaces is formulated. The potential can be a form of any degree. Only continuous operators appear in this formulation of magnetostatics. Consequently, difficulties typical for formulations based on unbounded operators are avoided.[6,7,10,14] In contrast to standard approaches, no use of Riemannian metrics is made. This results in a clearer conceptual structure and simpler proofs of fundamental theorems. One of the theorems establishes the relation between the dimensions of spaces of harmonic forms and the dimensions of the de Rham cohomology spaces for domains with the local segment property. A similar theorem for domains with the global segment property was obtained by Picard.[7,8]

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1. A geometric formulation of electro- and magneto-statics

Let us consider the de Rham complex on a smooth, real manifold M of dimension m:

$$0 \to \mathbb{R} \to C^0(M) \xrightarrow{d_0} C^1(M) \xrightarrow{d_1} C^2(M) \xrightarrow{d_2} \ldots C^m(M) \to 0.$$

Let Ω be a compact m-dimensional submanifold of M with a boundary. For each $0 \leqslant k \leqslant m$ the de Rham complex induces a sequence of linear operators of Hilbertizable spaces

$$X^{k,1} \xrightarrow{d_k} X^{k+1,0} \xrightarrow{d_{k+1}} X^{k+2,-1},$$

where $X^{q,s} = H^s(\Omega; \Lambda^q T^* M)$ are Sobolev spaces of sections of corresponding bundles. The sequence of conjugate operators will be denoted by

$$X_{k,1} \xleftarrow{d_k^*} X_{k+1,0} \xleftarrow{d_{k+1}^*} X_{k+2,-1},$$

where $X_{q,s}$ is the space dual to $X^{q,s}$. Elements of $X^{1,1}(X^{0,1})$ are interpreted as magnetic potentials (electrostatic potentials) and elements of $X^{2,0}$ $(X^{1,0})$ are magnetic inductions (electrostatic fields). Elements of $X_{1,1}$ $(X_{0,1})$ and $X_{2,0}$ $(X_{1,0})$ are electric currents (electric charges) and magnetic fields (displacement density), respectively. It is known, that $X_{k,0}$ can be identified with the space of L^2-sections of the bundle of k-vector densities on M (if M is oriented this bundle can be replaced by the bundle of $(m-k)$ forms). The magneto- and electro-statics of a material in the region Ω is a relation between potentials and sources expressed as a linear subspace Σ of $X^{k,1} \oplus X_{k,1}$. We list a number of properties of this subspace. The subspace Σ is the graph of a linear, continuous and self-adjoint mapping

$$\Lambda: X^{k,1} \to X_{k,1}$$

and this mapping has the form

$$\Lambda = d_k^* \hat{\Lambda} d_k,$$

where

$$\hat{\Lambda}: X^{k+1,0} \to X_{k+1,0}$$

is a linear, continuous, self-adjoint and strictly positive mapping. As a consequence of self-adjointness both Λ and $\hat{\Lambda}$ are differentials of positive quadratic functions. Let μ_k denote a fibre preserving quadratic mapping of $\Lambda^{k+1} T^* M$ into the bundle of scalar densities. This mapping represents the magnetic (electric) permeability of the material. The mapping $\hat{\Lambda}$ is the differential of the action function

$$\hat{L}: X^{k+1,0} \to \mathbb{R}: \phi \mapsto \int_\Omega \mu \circ \phi.$$

Consequently Λ is the differential of the function $L = \hat{L} \circ d_k$.

Theorem 1.1. *For each k* (i) $\ker \Lambda = \ker d_k$, (ii) $\operatorname{im} \Lambda = (\ker d_k)^\circ$ *if and only if $d_k X^{k,1}$ is closed in $X^{k+1,0}$.*

Proof. Since $\hat{\Lambda}$ is positive, $\ker \Lambda = \ker d_k$. Since Λ is self-adjoint $(\ker d_k)^\circ$ is the closure of $\operatorname{im}\Lambda$. But $\operatorname{im}\Lambda$ is closed if and only if the norm induced by L on $X^{k,1}/\ker d_k$ is equivalent to the norm of a quotient space. Because $\hat{\Lambda}$ is strictly positive (defines a norm equivalent to that of $X^{k+1,0}$) and

$$d_k\colon X^{k,1}/\ker d_k \to X^{k+1,0}$$

is an injection, these norms are equivalent if and only if $d_k X^{k,1}$ is closed in $X^{k+1,0}$.

Q.E.D.

The equality $\operatorname{im}\Lambda = (\ker\Lambda)^\circ$ means that the induced mapping

$$\Lambda\colon X^{k,1}/\ker\Lambda \to (\ker\Lambda)^\circ \subset X_{k,1}$$

is an isomorphism. It is reasonable to say that a theory with this property is elliptic, even if the corresponding Euler–Lagrange operator is not elliptic. The question arises whether ellipticity is a general property of static (linear) systems. In the case of generalized magnetostatics a partial answer is given by the following proposition.

Proposition 1.1. *Let Ω have a smooth boundary. Then $d_k X^{k,1}$ is closed in $X^{k+1,0}$.*

This proposition is a simple consequence of the Kodaira-Morrey decomposition theorem.[5] It is well known[14] that in the case of a non-smooth boundary the situation is much more complicated and that there is no use of the Morrey theory. The question of the ellipticity of the generalized magnetostatics remains open even in simplest situations (e.g. domains with edges).

2. Reductions

Let X be a closed subspace of $X^{k,1}$. The mapping

$$\Lambda_X := i^*\Lambda i\colon X \to X^*,$$

where $i\colon X \to X^{k,1}$ is the canonical injection, will be called the reduced mapping.

We study conditions under which the image of Λ_X is closed. Reasoning used in the proof of Theorem 1.1. shows that

Proposition 2.1. $\operatorname{im}\Lambda_X$ *is closed if and only if $d_k(X)$ is closed in $X^{k+1,0}$.*

In the case of a smooth boundary $d_k X^{k,1}$ is closed in $X^{k+1,0}$ and the statement '$d_k(X)$ is closed in $X^{k+1,0}$' is equivalent to the statement 'image of X in $X^{k,1}/\ker\Lambda$ is closed'. This proves the following.

Proposition 2.2. *Let Ω have a smooth boundary. Then* $\operatorname{im}\Lambda_X$ *is closed if and only if $X + \ker\Lambda$ is closed in $X^{k,1}$.*

There are two cases in which $X + \ker\Lambda$ is obviously closed:

(i) $X \subset \ker\Lambda$ or, equivalently, $(X)^\circ \supset \operatorname{im}\Lambda$,

(i) $X \supset \ker\Lambda$ or, equivalently, $(X)^\circ \subset \operatorname{im}\Lambda$.

In the following section we discuss special cases of reductions with respect to a closed subspace X. These reductions correspond to homogeneous Dirichlet-type boundary value problems for potentials.

3. Dirichlet-type reductions

Let $\overset{\circ}{X}{}^{k,1}$ be the closure in $X^{k,1}$ of $C_0^\infty(\Omega; \Lambda^k T^*M)$.

Definition 3.1. *The reduction with respect to the closed subspace $X \subset X^{k,1}$ is said to be of Dirichlet-type if*

$$\overline{\overset{\circ}{X}{}^{k,1} + \ker d_k} \supset X \supset \overset{\circ}{X}{}^{k,1}.$$

Proposition 3.1. *Let Ω have a smooth boundary. Then the image of Λ_X is closed for X equal to $\overline{\overset{\circ}{X}{}^{k,1} + \ker d_k}$.*

The proof of this proposition is an immediate consequence of Proposition 2.2.

Proposition 3.2. *Let Ω have a smooth boundary. If $X = \overset{\circ}{X}{}^{k,1}$ then the image of Λ_X is closed.*

Proof. According to Proposition 2.1 it is enough to know that $d_k(\overset{\circ}{X}{}^{k,1})$ is closed. But this is indeed the case.[5] Q.E.D.

Since $d_k(\overset{\circ}{X}{}^{k,1})$ is closed it follows that $\overset{\circ}{X}{}^{k,1} + \ker d_k = d_k^{-1}(d_k(\overset{\circ}{X}{}^{k,1}))$ is closed.

Theorem 3.1. *Let Ω have a smooth boundary. For each Dirichlet-type reduction the image of the reduced mapping Λ_X is closed.*

Proof. We have $d_k(X) = d_k(\overset{\circ}{X}{}^{k,1})$ because $d_k(\overset{\circ}{X}{}^{k,1} + \ker d_k) = d_k(\overset{\circ}{X}{}^{k,1})$. Since $d_k(\overset{\circ}{X}{}^{k,1})$ is closed it follows that the image of Λ_X is closed. Q.E.D.

It is well known[4] that in the case of a smooth boundary $\overset{\circ}{X}{}^{k,1}$ is the kernel of the mapping

$$b\colon X^{k,1} \to H^{1/2}(\partial\Omega; \Lambda^k T^*M),$$

where $b(\phi)$ is the restriction of ϕ to the boundary. The following theorem gives a similar characterization of $\overset{\circ}{X}{}^{k,1} + \ker d_k$.

Theorem 3.2. *Let $\alpha\colon \partial\Omega \to \Omega$ denote the canonical injection and let $H^k(\Omega)$ be the kth de Rham cohomology group. Then*

$$\overset{\circ}{X}{}^{k,1} + \ker d_k = \{X^{k,1} \ni A : A \in \ker(d_k \alpha^*)$$

and the cohomology class of $\alpha^* A$ is in $\alpha^*(H^k(\Omega))\}$.

Proof. Let X denote the space $\{A \in \overset{\circ}{X}{}^{k,1} : A \in \ker(d_k\alpha^*)$ and the cohomology class of $\alpha^* A$ is in $\alpha^*(H^k(\Omega))\}$. Since X and $\overset{\circ}{X}{}^{k,1} + \ker d_k$ are closed it is sufficient to prove inclusions of dense subspaces of these spaces. Let A_1 and A_2 be smooth sections in $\overset{\circ}{X}$ and $\ker d_k$, respectively, and let $A = A_1 + A_2$. Since $\mathrm{d}\alpha^* A = \mathrm{d}\alpha^* A_2 = \alpha^* \mathrm{d}A_2 = 0$ and $[\alpha^* A] = \alpha^*[A_2]$ it follows that $A \in X$. Consequently $\overset{\circ}{X}{}^{k,1} + \ker d_k \subset X$ because smooth sections are dense in $\ker d_k$ and $\overset{\circ}{X}{}^{k,1}$.

We show that smooth sections are dense in X. We choose a neighbourhood Ω_1 of $\partial\Omega$ in Ω diffeomorphic to $\partial\Omega \times [0, 1[$. We denote by Ω_ε the inverse image of $\partial\Omega \times [0, \varepsilon[$ by the diffeomorphism from Ω_1 to $\partial\Omega \times [0, 1[$. Each $A \in X^{k,1}$ is decomposed into a sum $A = A_1 + A_2$, where $\operatorname{supp} A_2 \subset \Omega_1$ and $A_1 = A$ in $\Omega_{1/2}$. A_2 can be regularized in a manner preserving the vanishing of A_2 in $\Omega_{1/4}$. A_1 is considered a k-form on $\partial\Omega \times [0, 1[$ and represented in the form $A_1 = \omega_1 \wedge \mathrm{d}t + \omega_2$ where t is the co-ordinate in $[0, 1[$ and ω_2 does not contain $\mathrm{d}t$. The form ω_1 is regularized in an arbitrary way and

ω_2 is regularized by a regularization procedure commuting with exterior differentiation and satisfying the additional requirement that the regularization pulled back to the boundary again commutes with the exterior differential (see 2 and 4 for details of this procedure). The resulting regularization of A preserves X. It follows that smooth sections are dense in X.

Let $A \in X$ be a smooth form. There is a smooth, closed k-form A_3 such that $\alpha^*(A - A_3) = \mathrm{d}\phi$ for a smooth $(k-1)$-form ϕ on $\partial\Omega$. We can choose a smooth $(k-1)$-form $\tilde{\phi}$ on Ω such that $\tilde{\phi}_{|\partial\Omega} = \phi$ and $\mathrm{d}\tilde{\phi}_{|\partial\Omega} = (A - A_3)_{|\partial\Omega}$. We do it in the following way: using decomposition and representation of k-forms as described above we have that, in the neighbourhood of $\partial\Omega$, $A - A_3 = \mathrm{d}t \wedge \omega_1 + \omega_2$ where ω_2 does not contain $\mathrm{d}t$. At the boundary we put $\tilde{\phi} = t\omega_1 + \phi$, i.e. coefficients of the second component do not depend on t. It follows that

$$\mathrm{d}\tilde{\phi}_{|\partial\Omega} = \mathrm{d}t \wedge \omega_{1\,|\partial\Omega} + \mathrm{d}\phi = (A - A_3)_{|\partial\Omega},$$

since $\mathrm{d}\phi = \alpha^*(A - A_3) = \omega_{2\,|\partial\Omega}$. We see that

$$\alpha^*\mathrm{d}\tilde{\phi} = \mathrm{d}\alpha^*\tilde{\phi} = \mathrm{d}\phi = \alpha^*(A - A_3)$$

and $A - (A_3 + d\tilde{\phi}) \in \mathring{X}^{k,1}$. Since $\mathrm{d}A_3 = 0$ we obtain the required decomposition

$$A = A - (A_3 + \mathrm{d}\tilde{\phi}) + (A_3 + \mathrm{d}\tilde{\phi}),$$

with $A - (A_3 + d\tilde{\phi}) \in \mathring{X}^{k,1}$ and $(A_3 + \mathrm{d}\tilde{\phi}) \in \ker d_k$. Q.E.D.

Remark. If M is a Euclidean space of dimension 3 the $\mathrm{d}\alpha^* A$ corresponds to the normal component of the field $\mathbf{B}$ (case $k = 1$) and to the tangent component of $\mathbf{E}$ (case $k = 0$). Hence Theorem 3.1 gives the Fredholm alternative for the homogeneous boundary value problems of electro- and magneto-statics:

$$\operatorname{div} \mathbf{D} = \rho \in H^{-1}(\Omega, \mathbb{R}),$$

$$\operatorname{curl} \mathbf{E} = 0, \quad \text{in } \Omega$$

$$E_s = 0, \quad \text{on } \partial\Omega$$

and

$$\operatorname{curl} \mathbf{H} = \mathbf{j} \in H^{-1}(\Omega, \mathbb{R}^3).$$

$$\operatorname{div} \mathbf{B} = 0, \quad \text{in } \Omega$$

$$B_n = 0, \quad \text{on } \partial\Omega.$$

4. Inhomogeneous Dirichlet-type boundary value problems

Inhomogeneous Dirichlet-type boundary value problems are usually treated by a method of reduction described in Sections 2 and 3. Since these reductions disregard sources on the boundary $\partial\Omega$ the complete statics Σ cannot be analysed by this method. We propose a different approach in which the complete subspace Σ is described in terms of the graphs of self-adjoint operators whose images are closed. These operators are chosen to solve Dirichlet-type boundary value problems.

Let X be a subspace of $X^{k,1}$ satisfying $\mathring{X}^{k,1} + \ker d_k \supset X \supset \mathring{X}^{k,1}$ as in Section 3. We define a space $\Sigma_1 = \{(A \oplus y) \oplus (J \oplus z) \in (X^{k,1} \oplus Y) \oplus (X_{k,1} \oplus Y^*)\colon y = \Phi(A)$ and $y + \Phi^* z = \Lambda A\}$, where $Y = X^{k,1}_{/X}$ and $\Phi\colon X^{k,1} \to Y$ is the canonical projection. The space Σ_1 is the graph of the mapping

$$\Lambda_1 : X^{k,1} \oplus Y^* \to X_{k,1} \oplus Y : A \oplus z \mapsto (\Lambda(A) - \Phi^* z) \oplus \Phi(A).$$

Theorem 4.1. *Let the images of* Λ_1 *and* Λ_X *be closed. Then the image of* Λ_1 *is closed and* $\ker \Lambda_1$ *is isomorphic to* $\ker \Lambda_X$.

Proof. It follows from the construction of Λ_1 that Λ_1 is self-adjoint with respect to the pairing

$$(X^{k,1} \oplus Y^*) \oplus (X_{k,1} \oplus Y) \ni (A \oplus z, J \oplus y) \mapsto \langle A, J \rangle - \langle y, z \rangle. \tag{1}$$

Let $A \oplus z \in \ker \Lambda_1$ then $\Phi(A) = 0$, $\Lambda A = \Phi^* z$ and, consequently, $\Lambda_X A = 0$. Since Φ^* is an injection z is uniquely determined by A. Now, let $A \in \ker \Lambda_X$ then $\Lambda A = \Phi^* z$ for some $z \in Y^*$ and $A \oplus 0 \oplus 0 \oplus z \in \Sigma_1$. It follows that $A \oplus z \in \ker \Lambda_1$. We have shown that $\ker \Lambda_1$ and Λ_X are isomorphic.

In order to prove that $\operatorname{im} \Lambda_1$ is closed it is sufficient to show that $\operatorname{im} \Lambda_1 = (\ker \Lambda_1)^\circ$ where the polar is taken with respect to the pairing (1). The inclusion $\operatorname{im} \Lambda_1 \subset (\ker \Lambda_1)^\circ$ follows from Λ_1 being self-adjoint. Let $J \oplus y \in (\ker \Lambda_1)^\circ$. This means that

$$\langle A, J \rangle - \langle y, z \rangle = 0 \quad \text{for each } A \oplus z \in \ker \Lambda_1. \tag{2}$$

Since Φ is surjective there is an element A_1 such that $y = \Phi(A_1)$. Hence

$$A \oplus y \oplus \Lambda A_1 \oplus 0 \in \Sigma_1. \tag{3}$$

It follows that

$$\langle A, \Lambda A_1 \rangle - \langle y, z \rangle = 0 \quad \text{for each } A \oplus z \in \ker \Lambda_1.$$

From (2) and (4) it follows that $\langle A, J - \Lambda A_1 \rangle = 0$ for each $A \in \ker \Lambda_X$. Since $\operatorname{im} \Lambda_X = (\ker \Lambda_X)^\circ$ there exist $A_2 \in X^{k,1}$ and $z_1 \in Y^*$ such that $\Lambda A_2 = J - \Lambda A_1 + \Phi^* z_1$. Hence

$$A_2 \oplus 0 \oplus (J - \Lambda A_1) \oplus (\Phi^* z_1) \in \Sigma_1. \tag{5}$$

(3) and (5) imply $(A_1 + A_2) \oplus y \oplus J \oplus (\Phi^* z_1) \in \Sigma_1$. Q.E.D.

In the particular cases of $X = \mathring{X}^{k,1}$ and $X = \mathring{X}^{k,1} + \ker d_k$ the quotient space $Y = X^{k,1}_{/X}$ can be given an explicit description. In the first case the space Y is the space of boundary values of potentials, and in the second case Y can be identified with the subspace of exact forms in $H^{-1/2}(\partial\Omega; \Lambda^{k+1} T^*(\partial\Omega))$. These forms represent in the case of magneto(electro)-statics the normal (tangent) component of magnetic induction (electrostatic field).

5. Hodge theorems

Frequently a boundary value problem is posed in terms of fields rather than potentials, i.e. the operator $d_k^* \hat{\Lambda}$ is considered instead of $d_k^* \hat{\Lambda} d_k$. For example instead of $\mathbf{E} = \mathrm{d}\phi$ in electrostatics we consider $\operatorname{curl} \mathbf{E} = \mathbf{0}$ and $\operatorname{div} \mathbf{B} = 0$ instead of $\mathbf{B} = \operatorname{curl} \mathbf{A}$ in magnetostatics. This means that the operator $d_k^* \hat{\Lambda}$ is restricted to a closed subspace of $X^{k+1,0}$. In the following we discuss two interesting cases:

(i) $d_k^* \hat{\Lambda}$ is restricted to the kernel of the mapping $d_{k+1}: X^{k+1,0} \to X^{k+2,-1}$ (div $\mathbf{B}=0$ in magnetostatics).

(ii) $d_k^* \hat{\Lambda}$ is restricted to the kernel of the mapping $(\delta_{k+1})^*: X^{k+1,0} \to \tilde{X}_{k+2,1}$ where $\tilde{X}_{k+2,1}$ and $\delta_{k+1}: \tilde{X}^{k+2,1} \to \tilde{X}^{k+1,0} = X_{k+1,0}$ are spaces and operators corresponding to the dual complex of densities (div $\mathbf{B}=0$, $B_n=0$ in magnetostatics).

For a fixed k we denote by Z^{k+1} the subspace defined in (i) and by Z_o^{k+1} the subspace defined in (ii).

Proposition 5.1.

(i) *The kernel of $d_k^* \hat{\Lambda}$ restricted to Z^{k+1} is isomorphic to the quotient space Z^{k+1}/B^{k+1}, where $B^{k+1} = \overline{d_k X^{k,1}}$.*

(ii) *The kernel of $d_k^* \hat{\Lambda}$ restricted to Z_o^{k+1} is isomorphic to the quotient space Z_o^{k+1}/B_o^{k+1}, where*

$$B_o^{k+1} = B^{k+1} \cap Z_o^{k+1}.$$

Proof. It is enough to recall (Theorem 1.1) that the kernel of $d_k^* \hat{\Lambda}$ restricted to $\operatorname{im} d_k$ and, consequently, restricted to B^{k+1} is trivial. Q.E.D.

The following proposition refers to the reduction with respect to the subspace $\mathring{X}^{k,1}$.

Proposition 5.2. *Let $i: \mathring{X}^{k,1} \to X^{k,1}$ be the canonical injection. The kernel of $i^* d_k^* \hat{\Lambda}$ restricted to Z_o^{k+1} is isomorphic to the quotient space $Z_o^{k+1}/\mathring{B}^{k+1}$, where $\mathring{B}^{k+1} = \overline{d_k \mathring{X}^{k,1}}$.*

Proof. We observe that $C_o^\infty(\Omega; \Lambda^{k+1} T^* M) \cap \ker d_{k+1}$ is a subspace of Z_o^{k+1}. It follows that $\mathring{B}^{k+1}$ is a subspace of Z_o^{k+1}. Then we argue as in the proof of Proposition 5.1. Q.E.D.

Let M be equipped with a Riemannian metric tensor and let $\hat{\Lambda}$ be the canonical identification of $X_{k+1,0}$ and $X^{k+1,0}$ defined by the metric. We recognize elements of the subspace $\ker d_k^* \hat{\Lambda} \cap Z^{k+1}$ as harmonic Neumann $(k+1)$-forms and elements of $\ker i^* d_k^* \hat{\Lambda} \cap Z_o^{k+1}$ as harmonic Dirichlet $(k+1)$-forms. This justifies the following definition.

Definition 5.1. *Ω is a Hodge domain if the following conditions are satisfied for each k:*

(i) *The dimension of Z^k/B^k is equal to the dimension of $H^k(\Omega)$*

(ii) *The dimension of $Z_o^k/\mathring{B}^k$ is equal to the dimension of $H_c^k(\Omega)$—the de Rham compact cohomology class.*

Proposition 5.3. *Let Ω be a Hodge domain. Then the dimension of the kernel of $d_k^* \hat{\Lambda}$ restricted to Z_o^{k+1} is equal to $\dim a(H_c^{k+1}(\Omega))$ where $a: H_c^{k+1} \to H^{k+1}$ is the canonical mapping.*

The following theorem gives a non-trivial example of a Hodge domain.

Theorem 5.1. *A domain Ω with the local segment property[1] is a Hodge domain.*

Proof. We will prove only statement (ii) of Definition 5.1. The same method can be used to prove a similar statement for the dual complex of densities. Then (i) follows by duality.

Let U be an open neighbourhood of Ω. An element C of $\tilde{X}_{l,1}$ defines a current $\tilde{C}$ in U by the formula

$$\langle D, \tilde{C} \rangle = \langle D_{|\Omega}, C \rangle.$$

The following formula holds for $B \in X^{k,0}$:

$$(\delta_k^* B)\tilde{} = d_k \tilde{B} \tag{6}$$

Suppose that there exists a sequence R_ε of regularizing operators on U (in the sense of de Rham[9]) satisfying

$$\text{for } B \in X^{l,0} \quad R_\varepsilon \tilde{B} \xrightarrow[\varepsilon \to \infty]{} \tilde{B} \quad \text{in } H^0(U; \Lambda^l T^* M), \tag{7}$$

$$\text{for } B \in X^{l,0} \quad \operatorname{supp} R_\varepsilon \tilde{B} \subset \operatorname{int} \Omega. \tag{8}$$

We assume also the existence of a sequence S_ε of homotopy operators associated with R_ε, i.e. operators satisfying the following formula:

$$R_\varepsilon B - B = S_\varepsilon(\mathrm{d}B) + \mathrm{d} S_\varepsilon B. \tag{9}$$

The operators S_ε are assumed to have property

$$\text{for } B \in X^{l,0} \quad \operatorname{supp} S_\varepsilon \tilde{B} \subset \Omega. \tag{10}$$

Let B be an element of $X^{k,0}$ such that $\delta_k^* = 0$. From (6) and (9) it follows that $d_k R_\varepsilon \tilde{B} = 0$. The property (8) implies that

$$R_\varepsilon \tilde{B} = H_\varepsilon + d_{k-1} A_\varepsilon, \tag{11}$$

where H_ε represents a compact cohomology class of $\operatorname{int}\Omega$ and A_ε is a smooth k-form on U with support in $\operatorname{int}\Omega$.

Because of (9) we have

$$H_\varepsilon - H_{\varepsilon'} = d_{k-1}(S_\varepsilon \tilde{B} - S_{\varepsilon'} \tilde{B} + A_\varepsilon + A_{\varepsilon'}),$$

where the right-hand side is the differential of a form with support in Ω (this follows from the property (10)). Now, we observe that for each open domain Ω_1 such that $\bar{\Omega}_1 \subset \operatorname{int}\Omega$ there exists a didiffeomorphism $\Psi: U \to U$ such that $\Psi(\Omega) \subset \operatorname{int}\Omega$ and $\Psi(p) = p$ for each $p \in \Omega_1$ (for the construction of Ψ see below). It follows that for $\bar{\Omega}_1 \supset \operatorname{supp}(H_\varepsilon - H_{\varepsilon'})$ we have

$$H_\varepsilon - H_{\varepsilon'} = \Psi^*(H_\varepsilon - H_{\varepsilon'}) = d_{k-1} \Psi^*(S_\varepsilon \tilde{B} - S_{\varepsilon'} \tilde{B} + A_\varepsilon - A_{\varepsilon'})$$

and, consequently, H_ε and $H_{\varepsilon'}$ represent the same compact cohomology class. Hence, we can take $H_{\varepsilon'} = H$ and formula (11) assumes the form

$$R_\varepsilon \tilde{B} = H + d_{k-1} A_\varepsilon.$$

The property (7) implies that $B = H + A$, where $A \in \mathring{B}^k$. It remains to show that if $H \in \mathring{B}^k$, then H represents the zero compact cohomology class, but this follows from the well-known theorem of de Rham.[9]

Constructions of R_ε, S_ε and Ψ.

In order to construct R_ε and S_ε we use a modification of a construction due to de Rham.[9,2] First, we construct a localized regularizer in $\mathbb{R}^m$. Let

$\phi:]-1, 1[\to \mathbb{R}$ be a smooth function which is symmetric, non-decreasing on $]-1, 0]$ and satisfies

$$\phi(t)=1 \quad \text{for } |t|<1/2, \ \phi(t)=\exp((1-t^2)^{-1}-1) \quad \text{for } |t|>3/4. \tag{12}$$

We define a mapping $\Phi: D \to \mathbb{R}^m$, where

$$D=\{\mathbf{x}\in \mathbb{R}^m: |x^m|<1, r(\mathbf{x})=((x^1)^2+(x^2)^2+(x^{m-1})^2<1\},$$

by

$$\Phi(\mathbf{x})=\phi(|x^m|)\phi(r(\mathbf{x})).$$

It is evident that Φ is a diffeomorphism. Now, we define a mapping

$$\Theta: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m: (\mathbf{x}, \mathbf{y}) \mapsto \begin{cases} \Phi^{-1}(\Phi(\mathbf{x})-\mathbf{y}), & \text{for } \mathbf{x}\in D, \\ \mathbf{x}, & \text{for } \mathbf{x}\notin D, \end{cases}$$

and we show that it is smooth. It is enough to show (24.12.5 in Reference 2) that the mapping

$$F: \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^m): \mathbf{x} \mapsto \begin{cases} -(D\Phi(\mathbf{x}))^{-1}, & \text{for } \mathbf{x}\in D, \\ 0, & \text{for } \mathbf{x}\notin D \end{cases}$$

is smooth. For this purpose we observe that the matrix elements of $-(D\Phi(\mathbf{x}))^{-1}$ have the form

$$[-(D\Phi(\mathbf{x}))^{-1}]_{ij}=-(\phi(|x^m|)\phi(r(\mathbf{x})))^{-1}\left[\delta_{ij}-\frac{\phi(a_i(\mathbf{x}))x^i x^j}{\phi(a_i(x))a_i(\mathbf{x})}\right.$$
$$\left.\times\left(\frac{\phi(x^m)}{\phi(x^m)}x^m+\frac{\phi(r)}{\phi(r)}r+1\right)^{-1}\right],$$

where

$$a_i(x)=\begin{cases} x^m, & i=m, \\ r(\mathbf{x}), & i\neq m \end{cases}.$$

Since the function $t \to \phi(t)/\phi(t)$ vanishes near zero and is rational near 1 we conclude that F is smooth. Let $g_{\varepsilon,\delta}$ be a family of non-negative functions on $\mathbb{R}^m$ approximating the Dirac distribution such that $\int g_{\varepsilon,\delta}=1$ and the support of $g_{\varepsilon,\delta}$ is contained in the ball of radius δ centred at $(0, \ldots, 0, -\varepsilon)$. We define a localized regularizer in $\mathbb{R}^m$ by

$$R_{\varepsilon,\delta}(B_I \mathrm{d}x^I)=\left(\int g_{\varepsilon,\delta}(\mathbf{y})B_I(\Theta(\cdot, \mathbf{y}))\,\mathrm{d}\mathbf{y}\right)\mathrm{d}x^I, \tag{13}$$

where I is a multi-index. Now, we choose a homotopy $\tau: [0, 1]\times\mathbb{R}^m \to \mathbb{R}^m$ connecting identity and zero mappings on $\mathbb{R}^m$:

$$\tau_s(\mathbf{y})=\begin{cases} (2(s-1/2)\mathbf{y}', y^m), & \text{for } 1/2\leqslant s\leqslant 1, \\ (0, 2sy^m), & \text{for } 0\leqslant s\leqslant 1/2, \end{cases}$$

where $\mathbf{y}'$ denotes $(y^1, y^2, \ldots, y^{m-1})$. The corresponding homotopy operator $S_{\varepsilon,\delta}$ is of the form (see 24.12.7.10 in Reference 2)

$$S_{\varepsilon,\delta}(B_I \mathrm{d}x^I)=\left(\int g_{\varepsilon,\delta}(\mathbf{y})\left(\int_0^1 B_I(\Theta(., \tau(s, \mathbf{y}))\omega^I(., \tau(s, \mathbf{y}))\,\mathrm{d}s\right)\mathrm{d}y\right. \tag{14}$$

where ω_k^I is a differential form with coefficients depending only on θ and derivatives of θ such that $\omega^I(\mathbf{x}, .)=0$ for $\mathbf{x}\notin D$.

Now, we define local regularizers in U (24.12.8 in Reference 2). The segment property of Ω implies the existence of a locally finite covering $\{O_i\}$ of U with the following properties:

$$\left.\begin{array}{l}O_i \text{ is the domain of a map } \phi_i \text{ such that } \phi_i(O_i) \\ =\{\mathbf{x}\in\mathbb{R}^m: r'<2, |x^m|<2\}=: D' \text{ and } \{\phi_i^{-1}(D)\} \text{ is the covering of } U, \\ \text{for } O_i \text{ such that } O_i\cap\partial\Omega\neq\varnothing \text{ the set } \phi_i(O_i\cap\partial\Omega) \text{ is the graph of} \\ \text{a uniformly continuous function } f_i: \{\mathbf{x}': |\mathbf{x}'|<2\}\to]-1/4, \ 1/4[, \\ \text{and } \phi_i(O_i\cap\partial\Omega)=\{\mathbf{x}\in D': x^m\leqslant f_i(\mathbf{x}')\}.\end{array}\right\} \quad (15)$$

Let $h_i: U\to\mathbb{R}$ be a smooth function such that $\operatorname{supp} h_i\subset O_i$ and $h_i(p)=1$ for $p\in\phi_i^{-1}(D)$. We define a local regularizer $R_{\varepsilon,\delta}$:

$$R^i_{\varepsilon,\delta}B=(\phi_i^*)^{-1}(R_{\varepsilon,\delta}\phi_i^*(h_iB))+(1-h_i)B, \tag{16}$$

and a homotopy operator $S_{\varepsilon,\delta}$:

$$S^i_{\varepsilon,\delta}B=(\phi_i^*)^{-1}(S_{\varepsilon,\delta}\phi_i^*(h_iB)) \tag{17}$$

As in References 9 and 2 we define a regularizer

$$\mathsf{R}^i_{\varepsilon,\delta}=R^1_{\varepsilon,\delta}R^2_{\varepsilon,\delta}\ldots R^i_{\varepsilon,\delta} \quad \text{and} \quad \mathsf{R}_{\varepsilon,\delta}=\lim_{i\to\infty}\mathsf{R}^i_{\varepsilon,\delta}. \tag{18}$$

$$\mathsf{S}^i_{\varepsilon,\delta}=R^1_{\varepsilon,\delta}\ldots R^{i-1}_{\varepsilon,\delta}S^i_{\varepsilon,\delta} \quad \text{and} \quad \mathsf{S}_{\varepsilon,\delta^n}=\Sigma_i\mathsf{S}^i_{\varepsilon,\delta}. \tag{19}$$

The properties (7) and (9) follow from the general properties of the construction (24.12.9 in Reference 2). A particular choice of ε and δ is needed in order to satisfy (8) and (10). Because of (12) and (15) the set $\Phi\circ\phi_i(O_i\cap\partial\Omega)$ is the graph of an uniformly continuous function $\tilde{f}_i: \mathbb{R}^{m-1}\to\mathbb{R}$ and, consequently, for each $\varepsilon>0$ there exist $\delta_i(\varepsilon)$ such that the inequality

$$|(\mathbf{x}',f_i(\mathbf{x}')+\varepsilon)-\mathbf{y}|<\delta_i(\varepsilon) \text{ implies } y^m>\tilde{f}(\mathbf{y}'). \tag{20}$$

Since Ω is compact we can assume that there is only a finite number of elements of the covering $\{O_i\}$ intersecting $\partial\Omega$. Thus we can set $\delta(\varepsilon)=\inf\delta_i(\varepsilon)$, $\mathsf{R}_\varepsilon=\mathsf{R}_{\varepsilon,\delta(\varepsilon)}$. From the property (20) and the definitions (13) and (16) it follows that if $\operatorname{supp} B\subset\Omega$ then $\operatorname{supp}\mathsf{R}_{\varepsilon,\delta(\varepsilon)}B\subset\Omega$ and $(\operatorname{supp}\mathsf{R}_{\varepsilon,\delta(\varepsilon)}B)\cap O_i$ is contained in the interior of Ω. The property (38) of R_ε is evident in view of definition (18). The above choice of δ guaranties that the integral

$$\int_0^1 B_I(\Theta(\phi_i(p),\tau(s,\mathbf{y}))\omega^I_k(\phi_i(p),\tau(s,\mathbf{y}))\,\mathrm{d}s$$

which appears in the formula (14) is equal to zero for $p\notin\Omega$, provided that $\operatorname{supp} B\subset\Omega$ and, consequently, $\operatorname{supp} S^i_\varepsilon B\subset\Omega$. Hence $\operatorname{supp}\mathsf{S}^i_\varepsilon B\subset\Omega$ and $\operatorname{supp}\mathsf{S}_\varepsilon B\subset\Omega$.

In order to construct Ψ we notice that if $\bar{\Omega}_1\subset\operatorname{int}\Omega$ then there exists a real number a, $1/4>a>0$, such that for $\mathbf{x}\in D$ and $0\leqslant t\leqslant 1$ $(\mathbf{x}',f_i(\mathbf{x}')-at)\notin\phi_i(\Omega_1\cap O_i)$. Let g_i be a smooth approximant of f_i, i.e.

$$\operatorname*{supp}_{|\mathbf{x}'|<2}|f_i(\mathbf{x}')-g_i(\mathbf{x}')|<\frac{a}{2}.$$

It follows, that the mapping Φ sends the image of g_i into the image of a smooth function $\tilde{g}_i$ and $\sup|\tilde{f}_i(\mathbf{x}')-\tilde{g}_i(\mathbf{x}')<|a/2$. Hence a mapping $\tilde{\Psi}_i$: $\mathbb{R}^m\to\mathbb{R}^m$ defined by the formula

$$\tilde{\Psi}_i(\mathbf{x})=\begin{cases}\left(\mathbf{x}', x^m+c\exp\left[-\left((x^m-g_i(\mathbf{x}'))^2\right)-\left(\left(\frac{a}{4}\right)^2\right)^{-1}\right]\right), & \text{for } |x^m-g_i(\mathbf{x}')|<\frac{a}{4},\\ \mathbf{x}, \quad \text{for } |x^m-g_i(\mathbf{x}')|>\frac{a}{4} & \end{cases}$$

is a diffeomorphism for sufficiently small c, and $\tilde{\Psi}_i(\Phi\circ\phi_i(\Omega\cap O_i))\subset \operatorname{int}\Phi\circ\phi_i(\Omega\cap O_i)$, $\tilde{\Psi}_i(\mathbf{x})=\mathbf{x}$ for $\mathbf{x}\in\Phi\circ\phi_i(\Omega_1\cap O_i)$. These properties of $\tilde{\Psi}_i$ and differentiability of Θ imply that the mapping

$$\Psi_i\colon U\to U\colon p\mapsto\begin{cases}(\Phi\circ\phi_i)^{-1}\circ\tilde{\Psi}_i\circ(\Phi\circ\phi_i)(p) & \text{for } p\in O_i\\ p \quad \text{for } p\notin O_i\end{cases}$$

is a diffeomorphism such that $\Psi_i(\Omega\cap O_i)\subset\operatorname{int}\Omega$ and $\Psi_i(p)=p$ for $p\in\Omega_1$. It follows, that the mapping $\Psi=\lim_{i\to\infty}\Psi_i\ldots\Psi_2\Psi_1$ has the needed properties. This completes the proof of the theorem. Q.E.D.

References

1. Adams, R. A., *Sobolev Spaces*, Academic Press, 1975.
2. Dieudonne, J., *Elements d'Analyse, vol. 9*, Gauthier-Villars, 1979.
3. Friedrichs, K. O., 'Differential forms on Riemannian manifolds', *Comm. Pure Appl. Math.*, **8**, 551–590 (1955).
4. Lions, J. L. and Magenes, E., *Nonhomogeneous Boundary Value Problems*, Springer, 1972.
5. Morrey, C. B., *Multiple Integrals in the Calculus of Variations*, Springer, 1966.
6. Picard, R., 'Randwertaufgaben der verallgemeinerten Potentialtheorie', *Math. Meth. in the Appl. Sci.*, **3**, 218–228 (1981).
7. Picard, R., 'On the boundary value problems of electro- and magneto-statics', *Preprint No. 442*, Bonn (1981).
8. Picard, R., 'An elementary proof for a compact imbedding results in generalized electromagnetic theory', *Math. Z.*, **187**, 151–164 (1984).
9. de Rham, G., *Variétes Differentiables*, Hermann, 1973.
10. Saranen, J., 'On generalized harmonic fields in domains with anisotropic nonhomogeneous media', *J. Math. Anal. Appl.*, **88**, 104–115 (1982).
11. Saranen, J., 'On electric and magnetic problems for vector fields', *J. Math. Anal. Appl.*, **91**, 255–275 (1983).
12. Tulczyjew, W. M., 'A symplectic framework of linear field theories', *Ann. Math. Pura Appl.*, **130**, 177–195 (1982).
13. Urbanski, P., 'A symplectic approach to field theories of elliptic type', *Publ. Fis. Mat.*, **4**, (1983).
14. Weck, N., 'Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries', *J. Math. Anal. Appl.*, **46**, 410–437 (1974).
15. Zajaczkowski, W., 'Existence and regularity properties of some elliptic systems in domain with edges', to appear in *Dissertationes Mathematicae*.