ON THE CANONICAL STRUCTURE OF THE $\lambda\varphi^4$ FIELD THEORY

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(*Received October 24, 1977*)

The proof of existence of a differentiable structure in the space of states in $\lambda\varphi^4$ theory is given. This structure renders possible applications of the Kijowski–Szczyrba formalism.

Introduction

The canonical structure in classical theory is one of the most important objects on the way leading to correspondence quantum field theory. Usually the principal elements of this structure (e.g. Poisson brackets) are rather postulated than deduced. Kijowski and Szczyrba [1] have proposed a method allowing to deduce the infinite-dimensional canonical structure from the finite-dimensional formalism. Their method is based on the concept of the finite-dimensional multisymplectic space—a natural (for the field theory) generalization of the phase space in mechanics.

In order to apply the Kijowski–Szczyrba construction one must have an appropriate structure of a differentiable manifold in the space of states. Namely, the model space should be a space of initial data with a suitable differential structure. This structure should have the following properties:

(1) Different initial surfaces should lead to the same differentiable structure in the space of solutions of field equations.

(2) Lie brackets of vector fields exist.

For smooth solutions Kijowski and Szczyrba propose an idea of the "inductive differentiable manifold". Unfortunately, in the case of the $\lambda\varphi^4$ theory there are difficulties with such structure. It should be replaced by another one by admitting a larger class of solutions, so this is what we do.

Results

We shall consider the $\lambda\varphi^4$ field theory, i.e. the equation $(\Box + m^2)\varphi = \lambda\varphi^3$, $\lambda < 0$, in the four-dimensional space-time. In the space $\mathscr{H}$ of solutions of this equation we shall introduce a differentiable structure with the model space $\mathscr{E}(\sigma)$ of initial data given on a surface $\sigma = \{t = \chi(x)\}$. Elements of $\mathscr{E}(\sigma)$ are pairs of functions (f, g) where f is the value of the field, g its time derivative on σ. We equip $\mathscr{E}(\sigma)$ with the energy norm

$$\|(f,g)\|^2 = \int\left\{f^2m^2+\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i}\right)^2+(1-|\operatorname{grad}\chi|^2)g^2\right\}d^3x.$$

Of course we take $\mathscr{E}(\sigma) \simeq H^1(R^3)\times H^0(R^3)$ where H^i are Sobolev spaces. The fact that the differentiable structure does not depend on the choice of σ follows from the following

THEOREM. *Let σ_1, σ_2 be asymptotically constant space-like surfaces. Then the natural mapping*

$$\mathscr{E}(\sigma_1) \ni (f_1, g_1) \to (f_2, g_2) \in \mathscr{E}(\sigma_2),$$

where $f_2 = u|_{\sigma_2}$, $g_2 = u_t|_{\sigma_2}$ *and* $(\Box+m^2)u = \lambda u^3$ *with initial values* (f_1, g_1), *is a* C^∞-*diffeomorphism.*

Now, we can construct (following [1]) a symplectic form Γ on $\mathscr{H}$. Let us fix $\sigma = (t = \varphi(x))$, then the element of the tangent space $T_u(\mathscr{H})$, $u\in\mathscr{H}$ can be represented by $X_u = (Q, P) \in \mathscr{E}(\sigma)$. We have ([1])

$$\Gamma_u(X_u^1, X_u^2) = \int\left\{(P_1-(\operatorname{grad}Q_1|\operatorname{grad}\varphi))Q_2-(P_2-(\operatorname{grad}Q_2|\operatorname{grad}\varphi))Q_1\right\}d^3x.$$

Γ does not depend on the choice of σ and is continuous in the energy norm. Hence Γ gives us a symplectic structure, weakly nondegenerate, i.e. the canonical mapping b: $T(\mathscr{H}) \to T^*(\mathscr{H})$ is an injection (but not surjection: duality between $T_u\mathscr{H}$ and $T_u^*\mathscr{H}$ is equivalent to that in L^2, but as for the last one, $T_u^*\mathscr{H} \simeq (H^1\oplus H^0)' = H^{-1}\oplus H^0$. On the other hand, the image of the tangent space under b is equal to $H^0\oplus H^1$).

Two physical quantities, i.e. smooth functions $G, F\in C^1(\mathscr{H})$ such that $dG, dF\in H^0\oplus \oplus H^1$, generate canonical fields X_F and X_G. Let us take any spacelike surface and the coordinate system connected with it. From the definition $\Gamma(X_F, X) = X(F) = dF(X)$. Now, we have

$$X_u = (Q, P) \quad \text{and} \quad (X_F)_u = (Q_F, P_F),$$

so

$$\Gamma_u(X_F, X) = \int\{[P_F-(\operatorname{grad}Q_F|\operatorname{grad}\varphi)Q]-[P-(\operatorname{grad}Q|\operatorname{grad}\varphi)]Q_F\}d^3x$$

and on the other hand

$$dF(X) = \int\left(Q\frac{\delta F}{\delta u(x)}+P\frac{\delta F}{\delta u_t(x)}\right)d^3x,$$

where $\delta F/\delta u(x)$ and $\delta F/\delta u_t(x)$ are variational derivatives with respect to the first and to the second component in $\mathscr{E}(\sigma)$ respectively.

Let us introduce new coordinates in $\mathscr{E}(\sigma)$:

$$\mathscr{E}(\sigma) \ni (f, g) \to \left(f, g-(\operatorname{grad}f|\operatorname{grad}\varphi)\right) =: (\tilde{u}, \tilde{u}_n).$$

We have

$$\frac{\delta F}{\delta u(x)} = \frac{\delta F}{\delta\tilde{u}(x)}-(\operatorname{grad}(\cdot)|\operatorname{grad}\varphi)\frac{\delta F}{\delta\tilde{u}_n(x)}, \quad \frac{\delta F}{\delta u_t(x)} = \frac{\delta F}{\delta u_n(x)}.$$

Hence

$$dF(X) = \int \left\{ Q \frac{\delta F}{\delta \tilde{u}(x)} + [P - (\text{grad}\, Q|\text{grad}\, \varphi)] \frac{\delta F}{\delta \tilde{u}_n(x)} \right\} d^3x.$$

Comparing with $\Gamma_u(X_F, X)$ we get

$$\frac{\delta F}{\delta \tilde{u}(x)} = P_F - (\text{grad}\, Q_F|\text{grad}\, \varphi) =: P_F^n,$$

$$\frac{\delta F}{\delta \tilde{u}_n(x)} = -Q_F.$$

The Poisson bracket can be written in the following form:

$$\{F, G\}_u = -\Gamma_u(X_F, X_G) = \int \left\{ \left[\frac{\delta F}{\delta \tilde{u}(x)} \right]_u \left[\frac{\delta G}{\delta \tilde{u}_n(x)} \right]_u - \left[\frac{\delta G}{\delta \tilde{u}(x)} \right]_u \left[\frac{\delta F}{\delta \tilde{u}_n(x)} \right]_u \right\} d^3x$$

which coincides with standard formulas.

Proof of the theorem

In the sequel we assume that $\sigma_1 = (t = 0)$ and $\sigma_2 = (t = \varphi(x))$. Since σ_2 is asymptotically constant there exist t_1 and t_2 such that $t_1 \leqslant \varphi(x) \leqslant t_2$. For simplicity let us take $t_1 > 0$.

A. *Continuity.* Let us start with

LEMMA 1. *Let u be a solution of wave equation with smooth initial data (with compact support) (f, g). Then* $\|u\|_{\sigma_2} \leqslant \text{const}\|u\|_{\sigma_1}$.

Proof: We have the explicit formula for the solution of wave equation:

$$4\pi u(x, t) = \int_{|\xi|=1} f(x+\xi t)\, d\xi + t \int_{|\xi|=1} (\text{grad} f(x+\xi t)|\xi)\, d\xi + t \int_{|\xi|=1} g(x+\xi t)\, d\xi.$$

Hence

$$(4\pi)^2 u^2(x, t) \leqslant 3\left\{ \int_{|\xi|=1} f(x+\xi t)\, d\xi \right\}^2 + 3t^2 \left\{ \int_{|\xi|=1} (\text{grad} f(x+\xi t)|\xi)\, d\xi \right\}^2 + 3t^2 \left\{ \int_{|\xi|=1} g(x+\xi t)\, d\xi \right\}^2$$

$$\leqslant 12\pi \int f^2(x+\xi t)\, d\xi + 12t^2\pi \int_{|\xi|=1} (\text{grad} f(x+\xi t)^2 d\xi + 12\pi t^2 \int_{|\xi|=1} g^2(x+\xi t)\, d\xi.$$

We obtain

$$\int u^2(x, \varphi(x))\, d^3x \leqslant \text{const} \int \left\{ \int f(x+\xi\varphi(x))^2 d\xi + \ldots \right\} d^3x.$$

Let us introduce new variables $y = x + \varphi(x)\xi$. Hence $dy = |1 + (\text{grad}\varphi|\xi)|\, dx$, but $|1 + (\text{grad}\varphi|\xi)| \geqslant 1 - \|\text{grad}\varphi\|$ so since σ_2 is spacelike: $1 - |\text{grad}\varphi|^2 > 0$, $1 - |\text{grad}\varphi| > 0$, and as it is asymptotically constant, we have

$$\frac{1}{|1+(\mathrm{grad}\,\varphi|\xi)|} \leqslant \frac{1}{1-|\mathrm{grad}\,\varphi|} \leqslant A^2 < \infty .$$

Now our inequality takes the form

$$\int u^2(x_1\varphi(x))dx \leqslant 3A^2\left\{\int f^2(x)d^3x+t_2^2\int|\mathrm{grad} f(x)|^2d^3x+t_2^2\int g(x)^2d^3x\right\}.$$

Now, we use the energy equality and obtain

$$\int\left\{m^2u^2(x)+\sum_{i=1}^{3}\left(\frac{\partial u(x,\varphi(x))}{\partial x_i}\right)^2+(1-|\mathrm{grad}\,\varphi|^2)u_t^2\right\}d^3x$$

$$\leqslant 3A^2\int m^2\{f^2(x)+t_2^2|\mathrm{grad} f(x)|^2+t_2^2g(x)\}d^3x+\int\{|\mathrm{grad} f(x)|^2+g(x)^2\}d^3x$$

$$\leqslant 3A^2(m^2+t_2^2)\,\|u\|_{\sigma_1}^2, \quad \text{q.e.d.}$$

In order to solve the problem of continuity, we have to analyse the difference $w := u_1-u_2$. From now we assume that initial data, and in consequence u_i, are smooth functions.

Recall that u_i are solutions of the integral equation

$$u(x,t) = v(x,t)-\int_0^t(t-\tau)M\{m^2u-\lambda u^3|x,\tau;t-\tau\}d\tau,$$

where $M\{f|x,t;r\} = \frac{1}{4\pi}\int\limits_{|\xi|=1}f(x+r\xi,t)d\xi$ and v is a solution of the free wave equation.

Thus we have

$$|w(x,t)| \leqslant |v_1(x,t)-v_2(x,t)|+b\int_0^t(t-\tau)M\{(1+u_1^2+u_2^2)|w||x,\tau;t-\tau\}d\tau$$

and

$$\left\{\int w(x,\varphi(x))^2d^3x\right\}^{1/2} \leqslant \left\{\int_{\sigma_2}|v_1-v_2|^2d^3x\right\}^{1/2}+$$

$$+b\left\{\int\left\{\int_0^{\varphi(x)}(\varphi(x)-\tau)\frac{1}{4\pi}\left(\int\limits_{|\xi|=1}f\big(x+(\varphi(x)-\tau)\xi,\tau\big)d\xi\right)d\tau\right\}^2d^3x\right\}^{1/2}$$

$$(f = (1+u_1^2+u_2^2)|w|)$$

$$\leqslant \|v_1-v_2\|_{\sigma_2}+b\int_0^{t_2}(t_2-\tau)\frac{1}{4\pi}\left\{\int\left\{\int\limits_{|\xi|=1}f\big(x+(\varphi(x)-\tau)\xi,\tau\big)d\xi\right\}^2d^3x\right\}^{1/2}d\tau$$

$$\leqslant \|v_1-v_2\|_{\sigma_2}+b\int_0^{t_2}(t_2-\tau)\left\{\frac{1}{4\pi}\int\int\limits_{|\xi|=1}f^2\big(x+(\varphi(x)-\tau)\xi,\tau\big)d\xi\,d^3x\right\}d\tau$$

$$\leqslant 2A(m+t_2)\|w\|_{\sigma_1}+bA\int\limits_0^{t_2}(t_2-\tau)\left\{\int(1+u_1^2+u_2^2)^2w^2d^3x\right\}^{1/2}d\tau$$

$$\leqslant 2A(m+t_2)\|w\|_{\sigma_1}+2Ab\int\limits_0^{t_2}(t_2-\tau)\left\{\int w^2d^3x\right\}^{1/2}d\tau+2Ab\int\limits_0^{t_2}(t_2-\tau)\left\{\int(u_1^2+u_2^2)^2w^2d^3x\right\}^{1/2}d\tau$$

$$\leqslant 2A(m+t_2)\|w\|_{\sigma_1}+2Abt_2\int\limits_0^{t_2}\left[\int w^2d^3x\right]^{1/2}d\tau+2Abt_2\int\limits_0^{t_2}\left[\int(u_1^2+u_2^2)^3d^3x\right]^{1/3}\left[\int w^6d^3x\right]^{1/6}d\tau$$

$$\leqslant 2A(m+t_2)\|w\|_{\sigma_1}+2Abt_2\int\limits_0^{t_2}\left[\int w^2d^3x\right]^{1/2}d\tau+2Abt_2\int\limits_0^{t_2}\left\{\left[\int u_1^6d^3x\right]^{1/2}+\right.$$

$$\left.+\left[\int u_2^6d^3x\right]^{1/3}\right\}\left[\int|\operatorname{grad}w|^2d^3x\right]^{1/2}d\tau\leqslant 2A(m+t_2)\|w\|_{\sigma_1}+ct_2\int\limits_0^{t_2}\|w\|_\tau d\tau.$$

Since w satisfies the equation $(\Box+m^2)w-w(u_1^2+u_1u_2+u_2^2)=0$, therefore

$$0=\frac{\partial}{\partial t}\{w_t^2+|\operatorname{grad}w|^2\}-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}(w_tw)+w_tw(m^2-u_1^2+u_1u_2+u_2^2).$$

Integrating this identity over the domain $0\leqslant t\leqslant\varphi(x)$, $x\in R^3$ we obtain

$$\int\left\{\sum_{i=1}^{3}\left(\frac{\partial w}{\partial x_i}(x,\varphi(x))\right)^2+(1-|\operatorname{grad}\varphi|^2)w_t^2(x,\varphi(x))\right\}d^3x-$$

$$-\int\{|\operatorname{grad}w(x,0)|^2+w_t^2(x,0)\}d^3x\leqslant 2\int\left\{\int\limits_0^{t_2}w_tw(-m^2+u_1^2+u_1u_2+u_2^2)dt\right\}d^3x$$

$$\leqslant 2b\int\left\{\int\limits_0^{t_2}|w_t||w|(1+u_1^2+u_2^2)dt\right\}d^3x\leqslant 2b\int\left[\int w_t^2d^3x\right]^{1/2}\left[\int w^2(1+u_1^2+u_2^2)^2d^3x\right]^{1/2}dt.$$

We have already estimated the last term, so

$$\int\left\{\sum_{i=1}^{3}\left(\frac{\partial w(x,\varphi(x))}{\partial x_i}\right)^2+(1-|\operatorname{grad}\varphi|^2)w_t(x,\varphi(x))\right\}d^3x\leqslant\|w\|_{\sigma_1}^2+c\int\limits_0^{t_2}\|w\|_t^2dt$$

($\|w\|_t$ stands for the energy norm on the surface $t=\text{const}$). In the estimations above we have used Schwarz, Hölder and Sobolev inequalities.

Combining all the inequalities obtained we get

$$\|w\|_{\sigma_2}^2\leqslant c(1+t_2^2)\|w\|_{\sigma_1}^2+c(1+t_2^3)\int\limits_0^{t_2}\|w\|_t^2dt.$$

If we put $\varrho(t) := \sup_{0\leqslant\varphi\leqslant t_2} ||w||_\sigma^2$ the following inequality holds

$$\varrho(t) \leqslant c(1+t_2^2)||w||_{\sigma_1}^2 + c(1+t_2^3)\int_0^{t_2}\varrho(t)\,dt$$

and we obtain

$$||w||_{\sigma_2}^2 \leqslant \varrho(t) \leqslant C(1+t_2^2)\exp(t_2+t_2^4)||w||_{\sigma_1}^2.$$

Since the estimation above is uniform with respect to the energy norm, continuity follows immediately.

B. *Differentiability.* We shall prove existence and continuity of the Gâteaux derivative. Let us introduce the following notations

u_0—solution corresponding to the data (f, g),

u_s—the solution corresponding to the data $(f+s\chi, g+s\psi)$,

$v_s := \frac{1}{s}(u_s - u_0)$.

v_s satisfies the equation $(\Box + m^2)v_s = (u_s^2 + u_s u_0 + u_0^2)v_s$ with the initial data (χ, ψ) on σ_1. We are going to show that $v_s \to v_0$ uniformly with respect to the energy norm. The following lemma will be needed.

LEMMA 2. *Let be* $(\Box + m^2)u = V(x, t)u + F(x, t)$. *The following inequality holds*

$$(||u||_{\sigma_2})^2 \leqslant \left\{(||u||_{\sigma_1})^2 + \int_0^{t_2} ||F||_{L^2}^2 dt\right\}\exp\left(\int_0^{t_2} a(t)\,dt\right)$$

where $a(t) = c||V||_{L^3} + 1$.

Proof: As in point A we have

$$(||u||_{\sigma_2})^2 = (||u_{\sigma_1}||)^2 + 2\int\left\{\int_0^{\varphi(x)} V(x,t)u(x,t)u_t(x,t)\,dt\right\}d^3x + 2\int\int_0^{\varphi(x)} F(x,t)u_t(x,t)\,dt\,d^3x$$

$$\leqslant (||u||_{\sigma_1})^2 + 2\int_0^{t_2}\int |Vuu_t|\,d^3x\,dt + 2\int_0^{t_2}\int |Fu_t|\,d^3x\,dt$$

$$\leqslant (||u||_{\sigma_1})^2 + 2\int_0^{t_2} ||V^2u^2||_{L^2}||u||_t\,dt + 2\int_0^{t_2} ||F||_{L^2}||u||_t\,dt$$

$$\leqslant (||u||_{\sigma_1})^2 + 2\int_0^{t_2} ||u||_t||V||_{L^3}||u||_{L^6}\,dt + 2\int_0^{t_2} ||F||_{L^2}||u||_t\,dt$$

$$\leqslant (||u||_{\sigma_1})^2 + c\int_0^{t_2} ||V||_{L^3}||u||_t^2\,dt + 2\int_0^{t_2} ||F||_{L^2}||u||_t\,dt$$

$$\leqslant (\|u\|_{\sigma_1})^2 + c\int_0^{t_2} \|V\|_{L^3}\|u\|_t^2\,dt + \int_0^{t_2} \{\|F\|_{L^2}^2 + \|u\|_t^2\}\,dt$$

$$\leqslant (\|u\|_{\sigma_1})^2 + \int_0^{t_2} \|F\|_{L^2}^2\,dt + \int_0^{t_2} a(t)\|u\|_t^2\,dt.$$

As before we obtain

$$\|u\|_{\sigma_2}^2 \leqslant \Big\{\|u\|_{\sigma_1}^2 + \int_0^{t_2} \|F\|_{L^2}^2\,dt\Big\}\exp\Big(\int_0^{t_2} a(t)\,dt\Big), \qquad \text{q.e.d.}$$

In order to apply Lemma 2 we need one estimation

$$\|u_s^2 + u_s u_0 + u_0^2\|_{L^3} \leqslant 4\Big\{\int u_s^6 d^3x + \int u_0^6 d^3x\Big\}^{\frac{1}{3}} \leqslant c\{\|u_s\|_t^2 + \|u_0\|_t^2\}.$$

Now we have by Lemma 2 and A

$$\|v_s\|_{\sigma_2} \leqslant \|v_s\|_{\sigma_1}\exp(t_2\,\mathrm{const}) = \|(\chi, \psi)\|_{\sigma_1}\exp(\mathrm{const}\,t_2).$$

On the other hand, $v_s - v_0$ satisfies the equation

$$(\Box + m^2)(v_s - v_0) = 3\lambda u_0^2(v_s - v_0) + 3\lambda(u_s + 2u_0)(u_s - u_0)v_s$$

with zero initial values.

From the Lemma 2 we infer that $v_s \to v_0$. In fact

$$\int\Big\{(u_s + 2u_0)(u_s - u_0)v_s\Big\}^2 d^3x \leqslant \Big\{\int (u_s + 2u_0)^3 v_s^3\,d^3x\Big\}^{\frac{1}{3}}\Big\{\int (u_s - u_0)^6 d^3x\Big\}^{\frac{1}{6}}$$

$$\leqslant c\Big\{\int (u_s + 2u_0)^6 d^3x\Big\}^{\frac{1}{6}}\Big\{\int v_s^6\,d^3x\Big\}^{\frac{1}{6}}\|u_s - u_0\|_t \leqslant c(\|u_s\|_t + 2\|u_0\|_t)\|u_s - u_0\|_t \xrightarrow[s\to 0]{} 0.$$

From the obtained estimations it follows that there exists a Gâteaux derivative which is continuous.

COROLLARY 1. *The mapping in the theorem is of class* C^1 *for* $\sigma_1 = \{t = 0\}$ *and* $\sigma_2 = \{t = \varphi(x)\}$.

COROLLARY 2. *The derivative at each point is an isomorphism and hence the mapping is locally reversible.*

In fact, in the proof of Lemma 2 we can change the roles of σ_1 and σ_2.

COROLLARY 3. *The mapping in the theorem is globally reversible.*

This follows from reversibility for the smooth data ([4]).

This completes the proof for C^1. Analogously, we obtain smoothness of higher order (see e.g. [2]).

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