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NONHOMOGENEOUS MIXED NEUMANN PROBLEM FOR NONLINEAR WAVE EQUATION

Summary: *Symplectic approach to the mixed problem is presented. As an application the proof of existence of generalized and classical solutions is given.*

Introduction.

In an unpublished paper [1] W.M. Tulczyjew outlined a symplectic framework for linear field theories. This framework based on concepts derived from physics contains a new approach to variational problems in the theory of partial differential equations. Applications of the framework to linear elliptic equation are given in reference [6]. Many elements of the general framework apply to linear as well as nonlinear equations. Other elements can be easily extended. In the present paper the framework is used to obtain a variational formulation of nonhomogeneous mixed Neumann problem for nonlinear hyperbolic equations. Within the framework we make use of the Ritz-Faedo-Galerkin method to prove existence and uniqness of classical and generalized solutions. The Klein – Gordon equation with a nonlinear term λx^3 is used as an explicite example. Solutions of the homogeneous mixed Dirichlet problem can be found in references [3] – [5].

The success of the Ritz-Faedo-Galerkin method applied to initial-boundary value problem depends crucially on the correct choice of the topology in the space of initial values. It can be easily seen that arguments used in references [3] – [5], based on interpolation technics do not extend to nonhomogeneous problems. The value of the symplectic approach consists in suggesting a natural · choice of Sobolev spaces of initial data in all

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cases. This choice is consistent with the physical meaning of initial data. The physical aspect is ignored in usual approaches to differential equations.

The basic technic of Ritz-Faedo-Galerkin method is that of apriori estimates. The version of this technic used in references [3] – [5] has to be significantly modified and improved in order to be applicable to the nonhomogeneous Neumann case. Modifications given in the present paper regard mainly the proof of existence of classical solutions (see III, 2, 3). In the case of generalized solutions the required modifications are less significant.

The paper begins with a short review of the unpublished paper [1]. This is followed by an example and a general discussion of the topology of the spaces of data (Chapter I). Chapter II deals with generalized solutions. Classical solutions are considered in Chapter III.

0. Notation.

Let Ω be bounded domain in R^3, with the smooth boundary $\partial\Omega$.
$\Omega_t := \Omega x \{t\} \subset D := \Omega x [0, T] \subset R^4$
$(,)$ – scalar product in $L^2(\Omega)$,
$(,)_t$ – scalar product in $L^2(\Omega_t)$,
$H^s(\Omega)$ – Sobolev space with index $s \geqslant 0$
$H^{-s}(\Omega) = (H^s(\Omega))'$, $\| \ \|_s$ – norm in H^s.
x – functions on D (or Ω_t),
$f = f\, dt \wedge dx^1 \wedge dx^2 \wedge dx^3$ – 4-form on D,
$p = p^0\, dx^1 \wedge dx^2 \wedge dx^3 + \ldots$ 3-form on D,
$(,)_{\Omega_t}$: $((p, f), x \to \int_{\partial\Omega_t} xp - \int_{\partial\Omega_t} xf$,
$y = p, f$.

I. Variational formulation of the problem.

1. Symplectic spaces.

In this section we shall describe the symplectic space associated with the curret $\mathbb{T}$ given by Ω_t and vector field $\frac{\partial}{\partial t}$ on it:

$<\mathbb{T},f> = \int_{\Omega_t} f \lrcorner \frac{\partial}{\partial t}$. We use the procedure described in [1]. According to it we have the mapping

$$<,>_{\mathbb{T}}: (y,x) \rightarrow <\mathbb{T}, d(xp) - xf> =$$

$$= \int_{\Omega_t} (d(xp) - xf) \lrcorner \frac{\partial}{\partial t} = \int_{\Omega_t} \left\{ \frac{\partial}{\partial t}(xp) - d\left(xp \lrcorner \frac{\partial}{\partial t}\right) - xf \lrcorner \frac{\partial}{\partial t} \right\} =$$

$$= - \int_{\partial\Omega_t} xp \lrcorner \frac{\partial}{\partial t} + \int_{\Omega_i} \left\{ \frac{\partial}{\partial t}(xp) - xf \lrcorner \frac{\partial}{\partial t} \right\} =$$

$$= \int_{\partial\Omega_i} x\, pd\sigma + \int_{\Omega_t} x(\partial_t p^\circ - f)dv + \int_{\Omega_t} \partial_t x\, p^\circ\, dv$$

Symplectic space is obtained as a direct sum of quotient spaces by left and right kernels of the bilinear form $<,>_{\mathbb{T}}$. We shall denote them by $X_{\mathbb{T}}$ and $Y_{\mathbb{T}}$ respectively. Duality between $X_{\mathbb{T}}$ and $Y_{\mathbb{T}}$ will be denote by the same symbol $<,>_{\mathbb{T}}$. Symplectic structure is the canonical structure in a dual pair. We see that an element $x_{\mathbb{T}} = [x]$ of $X_{\mathbb{T}}$ can be described by the pair $(x|_{\Omega_t}, \partial_t x_{\Omega_t})$ and an element $y_{\mathbb{T}} = [p,f]$ by the triple (p on $\partial\Omega_t$, $(\partial_t p^\circ - f)|_{\Omega_t}, p^\circ|_{\Omega_t}$).

2. Dynamics.

Elements of $X_{\mathbb{T}} \times Y_{\mathbb{T}}$ compatible with dynamics form a Lagrangian subspace $\Delta_{\mathbb{T}}$. Its generating function $W_{\mathbb{T}}$ is the following (see [1]):

$W_{\mathbb{T}}(x_{\mathbb{T}}) = <\mathbb{T}, L\circ j^1 x>$ where x is a representative of $x_T, j^1 x$ its first jet, L is Lagrangian density.

Thus we have:

$$(x_{\mathbb{T}}, y_{\mathbb{T}}) \in \Delta_{\mathbb{T}} \Leftrightarrow <x'_{\mathbb{T}}, y'_{\mathbb{T}}>_{\mathbb{T}} = \delta W(x_{\mathbb{T}}, x'_{\mathbb{T}}) \qquad x'_{\mathbb{T}} \in X_{\mathbb{T}}.$$

This is variational formulation we are interested in.

3. Nonlinear Klein - Gordon equation (λx^4 case).

In the following we shall deal with the following Lagrangian:

$$L\circ j^1 x = \frac{1}{2}\left\{ - m^2 x^2 + (\partial_t x)^2 - \sum_{i=1}^{3} (\partial_i x)^2 + \frac{\lambda}{2} x^4 \right\} \quad , \lambda > 0$$

Lagrange form of field equations is:

$$p^{\circ} = \partial_t x \ , \ p^i = -\partial_i x \ \ i = 1, 2, 3 \ ,$$
$$\partial_t p^{\circ} + \partial_i p^i - f = -m^2 x + \lambda x^3 \ .$$

In this case Δ_{II} is given by the equation

$$\delta W(x_{\mathrm{II}}, x'_{\mathrm{II}}) = \int_{\Omega_t} \left\{ \partial_t x \ \partial_t x' - \sum_{i=1}^{3} \partial_i x \ \partial_i x' - m^2 x x' - \lambda\, x^3 x' \right\} =$$

$$= \int_{\partial\Omega_t} x\, p + \int_{\Omega_t} (\partial_t p^{\circ} - f) x' + \int_{\Omega_t} p^{\circ} \partial_t x' \quad \forall x'_{\mathrm{II}} \in X_{\mathrm{II}} \tag{3.1}$$

(x', x represent x'_{II}, x_{II}).
Taking x'_{II} such that $x'|_{\Omega_t} = 0$ we get $p^{\circ} = \partial_t x$.
Taking x_{II} such that $\partial_t x'|_{\Omega_t} = 0$ we get

$$\int_{\Omega_t} \left(-\sum_{i=1}^{3} \partial_i x \ \partial_i x' - m^2 x x' - \lambda\, x^3 x'\right) = \int_{\partial\Omega_t} x' p + \int_{\Omega_i} (\partial_t p^{\circ} - f) x'.$$

Now, we can rewrite our equation in an equivalent form:

$$\int_{\Omega_t} \left\{ \partial_t^2 x x' + \sum_{i=1}^{3} \partial_i x \ \partial_i x' + m^2 x x' + \lambda x^3 x' \right\} = \int_{\Omega_t} f x' - \int_{\partial\Omega_t} p x' \tag{3.2}$$

or

$$(\partial_t^2 x, x') + b(x, x') + (\lambda x^3, x') = -(y, x')_{\Omega_t}$$

where $b(x, x') = \int_{\Omega_t} \left\{ \sum_{i=1}^{3} \partial_i x \ \partial_i x' + m^2 x x' \right\}$.

4. Topology.

We saw that canonical conjugate pairs are:
x and $\partial_t p^{\circ} - f$, $x|_{\partial\Omega}$ and p, $\partial_t x$ and p°.
It means, that if $x \in H^s(\Omega_t)$, $x|_{\partial\Omega_t} \in H^{s-\frac{1}{2}}(\partial\Omega_t)$, $\partial_t x \in H^{s'}(\Omega_t)$ i.e. $X_{\mathrm{II}} = H^s(\Omega_t) x H^{s'}(\Omega_t)$ $(s > \frac{1}{2})$, then conjugate elements should be treated as elements of:

$$f - \partial_t p^0 \in H^{-s}(\Omega_t) \ , \ p \in H^{-s+\frac{1}{2}}(\partial\Omega_t) \ , \ p^{\circ} \in H^{-s}(\Omega_t)$$

in order to obtain $Y_{\mathrm{II}} = (X_{\mathrm{II}})'$.

It seems to be natural to put $-s_1 - \frac{1}{2} = -s + \frac{1}{2}$ i.e. $s = s_1 + 1$. In the following we shall consider the case $s = 1$ only, i.e.

$$x \in H^1(\Omega_t),\ f \in H^{-1}(\Omega_t),\ \partial_t x \in H^0(\Omega_t) = L^2(\Omega_t)\ ,\ p \in H^{-\frac{1}{2}}(\partial\Omega_t).$$

5. Formulation of the mixed Neumann problem.

Given: p, f, x_o, x_o. We are looking for an element $x \in H^1(D)$ such that:

(i) (3.2) is satisfied for each $t \in [0, T]$ (5.1)
(ii) $x(0) = x_o$, $\partial_t x(0) = x_o$

x satisfying (5.2) is called "a generalized solution of the mixed Neumann problem".

assuming that the expressions below have the sence, the formulation above is equivalent to the following one

(i) $(\partial_t^2 - \Delta)x + m^2 x + \lambda x^3 = f$ in D
(ii) $\partial_n x = -p$ on $\partial\Omega x[0, T]$ (5.2)
(ii) $x(0) = x_o$, $\partial_t x(0) = x_o$

II. Generalized solution of the mixed Neumann problem.

In order to prove existence we use Galerkin-Faedo method.

1. Approximate solution.

First we choose a basis $w_1, w_2, ..., w_m, ...$ in $H^1(\Omega)$. We define approximate solution $x_m(t)$ by:

$x_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i$, where functions $g_{im}(t)$ satisfy system of equations

$$(\partial_t^2 x_m, w_i) + b(x_m, w_i) + \lambda(x_m^3, w_i) = -(y, w_i)_{\Omega_t} \quad i = 1, ..., m \qquad (1.1)$$

with initial data $g_{im}(0)=\xi_{im}$, $\frac{d}{dt}g_{im}(0)=\eta_{im}$ where

$$\sum_{i=1}^{m}\xi_{im}w_i\to x_0\ ,\ \sum_{i=1}^{m}\eta_{im}w_i\to x_0 \text{ in } H^1(\Omega)\ ,\ H^0(\Omega)$$

This means that g_{im} are given as solutions of systems of ordinary differential equations. By well known theorems such systems have unique local solutions provided $y=(p,f)$ where

$$p\in H^1(0,T;H^{-\frac{1}{2}}(\partial\Omega))\quad ,\quad f\in H^0(0,T;H^0(\Omega)).$$

Global existence follows from the a priori estimates.

2. A priori estimates.

From (1.1) it follows that

$$(\partial_t^2x_m,\partial_tx_m)+b(x_m,\partial_tx_m)+\lambda(x_m^3,\partial_tx_m)=-(y,\partial_tx_m)_{\Omega_t}$$

and

$$\frac{d}{dt}\left\{\frac{1}{2}(\partial_tx_m,\partial_tx_m)+\frac{1}{2}b(x_m,x_m)+\frac{1}{4}\lambda(x_m^2,x_m^2)+\int_{\partial\Omega_t}px_m\right\}=$$

$$=\int_{\partial\Omega_t}\partial_tpx_m+(f,\partial_tx_m).$$

Hence for $\lambda>0$

$$W^2(t):=(\partial_tx_m(t),\partial_tx_m(t))+$$
$$+b(x_m(t),x_m(t))\leqslant(\partial_tx_m(0),\partial_tx_m(0))+b(x_m(0),x_m(0))+$$
$$+\frac{1}{2}\lambda(x_m^2(0),x_m^2(0))+\int_{\partial\Omega_0}px_m-\int_{\partial\Omega_1}px_m+\int_0^t\int_{\partial\Omega_\tau}\partial_\tau px_m\,d\tau+$$
$$+\int_0^t(f,\partial_\tau x_m(\tau))d\tau.$$

But $(x_m^2(0),x_m^2(0))\leqslant c_1\|x_m(0)\|_1^4$ (Sobolev inequality),

$$\left|\int_{\partial\Omega_\tau}px_m\right|\leqslant\sup_{\tau\in[0,T]}\|\ \|_{-\frac{1}{2}}\|x_m(\tau)\|_1=c_2\|x_m(\tau)\|_1$$

and $|\int_0^t \int_{\partial\Omega_\tau} \partial_\tau p x_m|^2 \leqslant \int_0^T \|p(\tau)\|^2_{-\frac{1}{2}} d\tau \int_0^t \|x_m(\tau)\|_1^2 d\tau =$

$= c_3 \int_0^t \|x_m(\tau)\|_1 d\tau.$

Thus we have:

$$(W(t) - \frac{1}{2} c_2)^2 \leqslant W^2(0) + c_1 W^4(0) + c_2 W(0) + c_4 + c_5 \int_0^t (W(\tau) - \frac{1}{2} c_2)^2 d\tau.$$

where c_i depends on T and norms of p and f only. By Gronwall Lemma $W(t)$ is uniformely bounded on $[0, T]$ and by standard arguments we have global existence of x_m.

In the same way we can get estimates for the solution of the problem I. (3.2).

3. Generalized solution.

By standart compactness arguments (for details see [3]) we have that w_m converges (weakly) in $H^1(D)$ and the limit satisfies (3.2) (or rather (3.1)). Also uniqueness is easy to prove. Thus we have proved the following:

Theorem 1. For $p \in H^1(0, T; H^{-\frac{1}{2}}(\partial\Omega))$, $f \in H^0(0, T; H^0(\Omega))$ there exist exactly one solution $x \in H^1(D)$ of I. (3.1) such that $x(0) = x_0 \in H^1(\Omega)$, $p^0(0) = x_0 \in H^0(\Omega)$.

Remark. One can treat f and p jointly taking $y \in H^1(0, T; H^{-1}(\Omega))$.

III. Classical solution.

By well known embedding theorem it is sufficient to show that the generalized solution is in $H^5(D)$, i.e. in $H^0(0, T; H^5(\Omega)) \cap H^5(0, T; H^0(\Omega))$.

1. A priori estimates of the time derivatives of approximate solutions.

Suppose $w_i \in H^5(\Omega)$ $i = 1, 2, \ldots$, $p \in H^5(0, T; H^{-\frac{1}{2}}(\partial\Omega))$, $f \in H^3(D)$. We know, that g_{im} are differentiable and the following equality holds:

$$(\partial_t^{k+2} x_m, w_i) + b(\partial_t^k x_m, w_i) + \lambda(\partial_t^k x_m^3, w_i) = -(\partial_t^k y, w_i) \qquad k = 1,\dots,4, \tag{1.1}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\Big\{(\partial_t^{k+1} x_m, \partial_t^{k+1} x_m) + b(\partial_t^k x_m, \partial_t^k x_m)\Big\} = = -\lambda(\partial_t^k x_m^3, \partial_t^{k+1} x_m) - (\partial_t^k y, \partial_t^{k+1} x_m)_\Omega \tag{1.2}$$

We have to prove uniform boundedness of $(\partial_t^k x_m^3, \partial_t^{k+1} x_m)$.

$2|(\partial_t^k x_m^3, \partial_t^{k+1} x_m)| \leqslant \|\partial_t^k x_m^3\|^2 + \|\partial_t^{k+1} x_m\|^2$, but $\|\partial_t^k x_m^3\|^2 \leqslant 3\|\partial_t^k x_m x_m^2\| +$ terms with derivatives of an order $\leqslant k-1$, and $\|\partial_t^k x_m x_m^2\|^2 \leqslant c\|\partial_t^k x_m\|_1^2 \|x_m\|_1^4$.

Since we have already proved boundedness of $\|x_m\|_1$ and $\|\partial_t x_m\|$ then by induction, using (1.2) and Gronwall Lemma (as in section II) $\|\partial_t^k x_m(t)\|_1$ and $\|\partial_t^{k+1} x_m(t)\|$ are bounded if bounded are $\|\partial_t^k x_m(0)\|_1$ and $\|\partial_t^{k+1} x_m(0)\|$.

2. Boundedness of $\|\partial_t^k x_m(0)\|_1$.

From now we assume that data are compatible, i.e.

$$\partial_n(\partial_t^k x(0)) = \partial_t^k p(0) \quad \text{on } \partial\Omega_o. \tag{2.1}$$

($\partial_t^k x(0)$ can be expressed by x_o, x_o using I. (5.2)).

Moreover we assume that $\sum_{i=1}^{m} \xi_{im} w_i \to x_o$ in $H^5(\Omega)$ and

$$\sum_{i=1}^{m} \eta_{im} w_i \to x_o \quad \text{in } H^4(\Omega). \tag{2.2}$$

Let A be a self-adjoint, positively definite operator on $H^0(\Omega)$ such that $\|x\|_5 = \|Ax\|$. Of course norm $\|x\|_1$ is equivalent to $\|A^{1/5}x\|$. Let w_i be eigenfunction of A, $i = 1, 2\dots$.

Now, we pass to the proof:

$k = 0, 1$ — evident,

$k = 2$.

$(\partial_t^2 x_m(0), w_i) = -b(x_m(0), w_i) - \lambda(x_m^3(0), w_i) - (y(0), w_i)_{\Omega_0} = ((-m^2 +$

$+\Delta)x_m(0), w_i + (f, w_i) - \lambda(x_m^3(0), w_i) - \int_{\partial\Omega_0} w_i(\partial_n x_m(0) + p(0)) \qquad i = 1, ..., m.$

But $\partial_n x_m(0) \to -p(0)$ $(m \cdot \to \infty)$ and we can write $\partial_t^2 x_m(0) = P_m(-m^2 + +\Delta)x_m(0) + P_m f - P_m x_m^3(0) + \epsilon_m$ where P_m is ortogonal projection onto subspace of $H^0(\Omega)$ spanned by $w_1, ..., w_m$; ϵ_m are small in $H^5(\Omega)$. Hence

$$\|\partial_t^2 x_m(0)\|_3 \leqslant \|P_m(-m^2 + \Delta)x_m(0)\|_3 + \|P_m f\|_3 + \lambda\|P_m x_m^3(0)\|_3 + \epsilon$$

But $P_m A = A\, P_m$ and

$\|\partial_t^2 x_m(0)\|_3 \leqslant \|(-m + \Delta)\, x_m(0)_3 + \|f\|_3 + \lambda\|x_m^3(0)\|_3 + \epsilon \leqslant c\|x_m(0)\|_5 + + \|f(0)\|_3 + c_1\|x_m(0)\|_4^3 + \epsilon$. Now, boundedness follows from (2.2).

$k = 3$

Using (2.1) we have, as in the case $k = 2$, that

$$\|\partial_t^3 x_m(0)\|_2 \leqslant \|(-m^2 + \Delta)\,\partial_t x_m(0)\|_2 + \|\partial_t f(0)\|_2 + \|\partial_t x_m^3(0)\|_2 + \epsilon \leqslant$$
$$\leqslant c\|\partial_t x_m(0)\|_4 + \|\partial_t f(0)_2 + c_1(\|\partial_t x_m(0)\|_3 + \|x_m(0)\|_3)^3 + \epsilon$$

$k = 4$

$$\|\partial_t^4 x_m(0)\|_1 \leqslant c\|\partial_t^2 x_m(0)\|_3 + \|\partial_t^2 f(0)\|_1 +$$
$$+ c_2(\|\partial_t^2 x_m(0)\|_2 + \|\partial_t x_m(0)\|_2 + \|x_m(0)\|_2)^3\, \epsilon$$

$k = 5$

$$\|\partial_t^5 x_m(0)\| \leqslant c\|\partial_t^3 x_m(0)\|_2 + \|\partial_t^3 f(0)\| + c_3(\|\partial_t^3 x_m(0)\|_1 + ...)^3 + \epsilon$$

3. Classical solution.

Results of previous sections show existence of $\partial_t^k x$ $k = 1, ..., 5$. Moreover, $\partial_t^k x \in H^0(0, T; H^0(\Omega))$.
It remains to prove that $x \in H^0(0, T; H^5(\Omega))$.
We have

$$(\partial_t^2 x, w) + b(x, w) + \lambda(x^3, w) = -(y, w)_{\Omega_t} \qquad w \in H^1(\Omega).$$

Hence

$$b(x, w) = -(\partial_t^2 x, w) + \lambda(x^3, w) - (y, w)_{\Omega_t} \tag{3.1}$$

Let $p \in H^0(0, T; H^{\frac{1}{2}}(\partial\Omega))$ and $f \in H^0(0, T; H^0(\Omega))$ then right-hand side in (3.1) is continuous with respect to

$(w, w|_{\partial\Omega}) \in H^0(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ $(\partial_t^2 x \in H^1(\Omega), x^3 \in H^0(\Omega))$.
On the other hand $b(\dot{x}, w) = (\Lambda x, w)_\Omega$ where (see [6])

$$\Lambda x = (\partial_n x|_{\partial\Omega}, (-\Delta + m^2)x.$$

It follows that $\Lambda x \in H^{\frac{1}{2}}(\partial\Omega) \times H^0(\Omega)$. But Λ is an isomorphism between $H^s(\Omega)$ and $H^{s-\frac{3}{2}}(\partial\Omega) \times H^{s-2}(\Omega)$ $(s \geq 2)$ (see [2], [6]). It means that $x(t) \in H^2(\Omega)$ and $x \in H^0(0, T; H^2(\Omega))$.
Next step:

$\partial_t^2 x = -(-\Delta + m^2)x - \lambda x^3 + f$ and (3.1) implies $b((-\Delta + m^2)x, w) = b(\lambda x^3 - f, w) - (\partial_t^4 x, w) - \lambda(\partial_t^2 x^3, w) - (\partial_t^2 y, w)$. As before we get $(-\Delta + m^2)x \in H^1(\Omega)$ and because

$$\partial_n x|_{\partial\Omega} = -p \in H^{\frac{3}{2}}(\partial\Omega)$$

we have, by isomorphism theorem, that $x \in H^3(\Omega)$. Repeating twice this procedure we obtain $x \in H^0(0, T; H^5(\Omega))$.
Now, we can formulate results as a theorem

Theorem 2. Let $f \in H^3(D)$, $p \in H^5(0, T; H^{-\frac{1}{2}}(\partial\Omega)) \cap H^0(0, T; H^{\frac{7}{2}}(\partial\Omega))$, $x_0 \in H^5(\Omega)$, $\dot{x}_0 \in H^4(\Omega)$.
Suppose that this data are compatible, then there exists exactly one solution x of the problem II. (5.2) and $x \in C^2(D)$.

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