A pseudocategory of principal bundles

Nota del Socio straniero W.M. TULCZJEW*
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presentata nell'adunanza del 25 Novembre 1987

Summary. *Principal bundles and principal bundle homomorphisms form a category. Applications to mathematical physics require that this category be extended to include morphisms associated with differentiable relations. We define an extension which is not a category although it preserves some of the features of a category. Principal bundles with the additive group of real numbers as the structure group are needed as a basis for intrinsic formulations of Newtonian mechanics* [5] *and gauge independent formulations of the dynamics of charged particles* [2]. *The study of asymptotic classical limits of wave theories is also based on the geometry of principal bundles.*

This note is a contribution to a programme of study of geometric foundations of physical theories conducted jointly with Professor Benenti at Istituto di Fisica Matematica "J.-L. Lagrange" of Torino.

Sommario. *Fibrati principali e omomorfismi di fibrati principali formano una categoria. Le applicazioni alla fisica matematica richiedono un'estensione di questa categoria che includa morfismi associati a relazioni differenziabili. Definiamo una estensione che non è una categoria sebbene essa conservi alcune caratteristiche di una categoria.*

1. The pseudocategory of differentiable relations

Let M and M' be differential manifolds and let R be a submanifold of the product manifold $M' \times M$. The triple $\rho = (M', M, R)$ is called a *differentiable relation* from M to M'. The manifolds M and M' are the *domain* and the *codomain* respectively of the relation ρ.

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Differentiable relations are composed as set theoretical relations. The composition of differentiable relations is not always a differentiable relation.

We denote by $\mathrm{M}^\infty(M', M)$ the set of all differentiable relations from M to M'. Differentiable relations are morphisms of the *pseudocategory of differentiable relations* denoted by M^∞. Differential manifolds are objects of this pseudocategory.

The *transposition functor* is a contravariant functor which assigns to each manifold M the same manifold and to each relation $\rho = (M', M, R)$ the *transpose relation* $\rho^t = (M, M', R^t)$, where R^t denotes the set $\{(x, x') \in M \times M'; (x', x) \in R\}$. The transposition functor is an involution.

The pseudocategory M^∞ has several subcategories which are true categories. We denote by C^∞ the category of differentiable mappings.

The composition $\rho = \pi \circ \iota^t$ of the transpose ι^t of an embedding $\iota : N \to M$ with a differential fibration $\pi : N \to M'$ is a differentiable relation from M to M' called a *differentiable reduction.* The transpose $\iota \circ \pi^t$ of a differentiable reduction $\pi \circ \iota^t$ is a differentiable relation from M' to M, called a *differentiable coreduction.*

Let $\rho = \pi \circ \iota^t$ and $\rho' = \pi' \circ \iota'^t$ be reductions constructed from embeddings $\iota : N \to M$, $\iota' : N' \to M'$ and differential fibrations $\pi : N \to M'$, $\pi : N' \to M''$. The mapping ι restricted to $\bar{N} = \pi^{-1}(\iota'(N'))$ is an embedding $\bar{\iota} : \bar{N} \to M$. The mapping π restricted to $\bar{N}$ induces a differential fibration $\tau : \bar{N} \to \iota'(N')$. The embedding $\iota' : N' \to M'$ is a diffeomorphism onto its image $\iota'(N')$. Let $\lambda : \iota'(N') \to N'$ denote the inverse of this diffeomorphism. The composition $\pi' \circ \lambda \circ \tau$ is a differential fibration $\bar{\pi} : \bar{N} \to M''$. The differentiable reduction $\bar{\rho} = \bar{\pi} \circ \bar{\iota}^t$ is easily seen to be the composition of the relations ρ and ρ' . It follows that differentiable reductions form a category. This category will be denoted by R^∞, and the category of differentiable coreductions will be denoted by CR^∞.

The *product functor* $\times$ is a covariant functor of two arguments which assigns to manifolds M and N the product manifold $M \times N$, and to relations $\rho = (M', M, R)$ and $\sigma = (N', N, S)$ the *product relation* $\rho \times \sigma = (M \times N', M \times N, R \times S)$. Products of more than two factors are constructed in a similar way.

2. Principal bundles

A *principal bundle* is denoted by (Z, M, ζ, G, σ), where Z is the *bundle space,* M is the *base,* $\zeta : Z \to M$ is the *bundle projection,* G

is the *structure group* and $\sigma : G \times Z \to Z$ is the *group action*. A principal bundle with the additive group $\mathbf{R}$ of real numbers for the structure group will be denoted by $\mathbf{Z} = (Z, M, \zeta, \sigma)$. Only bundles with the structure group $\mathbf{R}$ will be considered. The group $\mathbf{R}$ considered as a bundle with a single fibre and the canonical group action will be denoted by $\mathbf{I}$. The set $\{1\}$ is the base of this bundle.

The *conjugate* of a bundle $\mathbf{Z} = (Z, M, \zeta, \sigma)$ is the bundle $\bar{\mathbf{Z}} = (Z, M, \zeta, \bar{\sigma})$, where $\bar{\sigma}$ is defined by $\bar{\sigma}(r, z) = \sigma(-r, z)$.

Given a principal bundle (Z, M, ζ, G, σ) and a group epimorphism $\chi : G \to G'$, we construct the *reduced bundle* $(Z', M, \zeta', G', \sigma')$, where Z' is the space of orbits in Z of the kernel of χ, and ζ' and σ' are the result of an obvious canonical reduction applied to ζ and σ. The structure group of the product bundle $\mathbf{Z} \times \mathbf{Z}' = (Z \times Z', M \times M', \zeta \times \zeta', \sigma \times \sigma')$ of two bundles $\mathbf{Z} = (Z, M, \zeta, \sigma)$, and $\mathbf{Z}' = (Z, M', \zeta', \sigma')$ is the product group $\mathbf{R} \times \mathbf{R}$, and the codiagonal mapping $\delta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by $\delta(r, s) = r + s$ is a group epimorphism. The reduced bundle is denoted by $\mathbf{Z} \otimes \mathbf{Z}' = (Z \otimes Z', M \times M', \zeta \otimes \zeta', \sigma \otimes \sigma')$ and called the *tensor product* of bundles $\mathbf{Z} = (Z, M, \zeta', \sigma)$ and $\mathbf{Z}' = (Z', M', \zeta', \sigma')$. We denote by $z \otimes z'$ the orbit of (z, z') in $Z \otimes Z'$.

The construction of the tensor product is easily extended to an arbitrary number of factors. The tensor product is always constructed by reducing a product bundle with respect to a codiagonal epimorphism.

3. Principal bundle morphism and functors

Let $\mathbf{Z} = (Z, M, \zeta, \sigma)$ and $\mathbf{Z}' = (Z', M', \zeta', \sigma')$ be principal bundles. The action $\sigma' \times \sigma$ of the group $\mathbf{R} \times \mathbf{R}$ in $Z' \times Z$ restricted to the diagonal $\{(r', r) \in \mathbf{R} \times \mathbf{R};\ r' = r\}$ will be called the diagonal action of $\mathbf{R}$.

A *principal bundle morphism* from a bundle $\mathbf{Z} = (Z, M, \zeta, \sigma)$ to a bundle $\mathbf{Z}' = (Z', M', \zeta', \sigma')$ is a quadruple $(\mathbf{Z}', \mathbf{Z}, \phi, \psi)$, where ϕ and ψ are differentiable relations $\phi = (M', M, F)$ and $\psi = (Z', Z, G)$, and the following conditions are satisfied.

(1) $(\zeta' \times \zeta)(G) = F$.

(2) G is invariant under the diagonal action of $\mathbf{R}$.

(3) The quotient set of G by the diagonal action is locally the image of a section over F of the quotient bundle.

We will refer to the class of all principal bundles with the class of principal bundle morphisms as the *pseudocategory of principal bundles.* Principal bundles are *objects* of this pseudocategory. The pseudocategory of principal bundles will be denoted by PB.

We say that morphisms $(\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ and $(\mathbf{Z}'', Z', \phi', \psi')$ *compose as differentiable relations* if $(\mathbf{Z}'', \mathbf{Z}', \phi', \psi') \circ (\mathbf{Z}', \mathbf{Z}, \phi, \psi) = (\mathbf{Z}'', \mathbf{Z}, \phi' \circ \phi, \psi' \circ \psi)$. Principal bundle morphisms do not in general compose as differentiable relations. Classes of special morphisms which compose as differentiable relations and form true categories will be defined in the present section. Composition of general morphisms will be introduced in Section 4.

A principal bundle morphism $(\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ is said to be a *principal bundle homomorphism* from a bundle $\mathbf{Z} = (Z, M, \zeta, \sigma)$ to a bundle $\mathbf{Z}' = (Z', M', \zeta', \sigma')$ if $\phi : M \to M'$ and $\psi : Z \to Z'$ are differentiable mappings. Principal bundle homomorphisms compose as differentiable mappings and form category denoted by PBH.

A principal bundle homomorphism $(\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ is said to be an *embedding* if ϕ is an embedding. A homomorphism $(\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ is said to be an *fibration* if ϕ is a differential fibration. A homomorphism $(\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ is an *isomorphism* if ϕ is a diffeomorphism.

Let $\mathbf{Z} = (Z, M, \zeta, \sigma)$ be a principal bundle and let L be a submanifold of M. A principal bundle $\mathbf{Y} = (Y, L, \eta, \rho)$ is obtained by restricting ζ and σ to the submanifold $Y = \zeta^{-1}(L)$. The canonical injectins $\lambda : L \to M$ and $\mu : Y \to Z$ define an embedding $(\mathbf{Z}, \mathbf{Y}, \lambda, \mu)$ called the *restriction* of $\mathbf{Z} = (Z, M, \zeta, \sigma)$ to L.

The *transposition functor* in the pseudocategory of principal bundles assigns to each bundle $\mathbf{Z}$ the same bundle $\mathbf{Z}$ and to each morphism $\rho = (\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ the morphism $\rho^t = (\mathbf{Z}, \mathbf{Z}', \phi^t, \psi^t)$. The transposition functor is an involution.

A principal bundle morphism $(\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ is said to be a *reduction* if it is the composition $\pi \circ \gamma^t$ of the transpose γ^t of an embedding $\gamma = (\mathbf{Z}, \mathbf{Y}, \alpha, \beta)$ with a fibration $\pi = (\mathbf{Z}', \mathbf{Y}, \kappa, \lambda)$ defined by $\pi \circ \gamma^t = (\mathbf{Z}', \mathbf{Z}, \kappa, \alpha^t, \lambda \circ \beta^t)$. Principal bundle reductions compose as differential relations. The cateogry of reductions will be denoted by PBR. A principal bundle morphism $(\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ is said to be a *coreduction* if it is the transpose of a reduction. Each coreduction is a composition of the transpose of a fibration with an embedding. The category of coreductions will be denoted by PCR. The transposition functor is a true contravariant functor from the category PBR to the category PCR and from the category PCR to the category PBR.

The *conjugation functor* in the pseudocategory of principal bundles assigns to each bundle $\mathbf{Z}$ the conjugate bundle $\mathbf{Z}^c = \overline{\mathbf{Z}}$, if $\mathbf{Z} \neq \mathbf{I}$ and $\mathbf{Z}^c = \mathbf{I}$ if $\mathbf{Z} = \mathbf{I}$ To each morphism $\rho = (\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ this functor assigns the morphism $\rho^c = (\mathbf{Z}'^c, \mathbf{Z}^c, \phi, \epsilon^t_{Z'} \circ \psi \circ \epsilon_Z)$, where $\epsilon_Z = 1_Z$ if $Z \neq I$ and $\epsilon_I = (I, I, R)$ with $R = \{(r, r') \in I \times I; r' = -r\}$. The conjugation functor is an involution.

The *duality functor* assigns to each bundle $\mathbf{Z}$ the *dual bundle* $\mathbf{Z}^* = \mathbf{Z}$ if $\mathbf{Z} \neq \mathbf{I}$ $\mathbf{Z}^* = \mathbf{I}$ if $\mathbf{Z} = \mathbf{I}$. To each morphism $\rho = (\mathbf{Z}', \mathbf{Z}, \phi, \psi)$, the duality functor assigns the morphism $\rho^* = (\mathbf{Z}^*, \mathbf{Z}'^*, \phi^t, \epsilon^t_Z, \psi^t, \epsilon_{Z'})$.

The *tensor product functor* is a functor of several arguments. In the case of two arguments it assigns to bundles $\mathbf{Z} = (Z, M, \zeta, \sigma)$, $\mathbf{Z}' = (Z', M', \zeta', \sigma')$ the bundle $\mathbf{Z} \otimes \mathbf{Z}' = (Z \otimes Z', M \times M', \zeta \otimes \zeta', \sigma \otimes \sigma')$, and to morphisms $\rho = (\mathbf{Z}, \mathbf{Y}, \phi, \psi)$, $\rho' = (\mathbf{Z}', \mathbf{Y}', \phi', \psi')$ the morphism $\rho \otimes \rho' = (\mathbf{Z} \otimes \mathbf{Z}', \mathbf{Y} \otimes \mathbf{Y}', \phi \times \phi', \psi \otimes \psi')$, where $\psi \otimes \psi'$ is the differentiable relation defined below. If $\psi = (Z, Y, G)$ and $\psi' = (Z', Y', G')$ then $\psi \times \psi' = (Z \times Z', Y \times Y', G \times G')$. We denote by $G \otimes G'$ the submanifold of $(Z \otimes Z') \times (Y \otimes Y')$ obtained by applying to $G \times G'$ the reductions with respect to the codiagonal homomorphisms described in Section 2. The differentiable relation $\psi \otimes \psi'$ is the relation $(Z \otimes Z', Y \otimes Y', G \otimes G')$.

The conjugation functor and the tensor product functor are true covariant functors in the categories PBH, PBR and PCR. The duality functor is a contravariant functor from PBR to PCR and from PCR to PBR. A number of natural equivalences between functors can be defined:

(1) There is a natural equivalence between $\mathbf{Z} \otimes \mathbf{Z}'$ and $\mathbf{Z}' \otimes \mathbf{Z}$.

(2) There are natural equivalences between the products $(\mathbf{Z} \otimes \mathbf{Z}') \otimes \mathbf{Z}''$, $\mathbf{Z} \otimes (\mathbf{Z}' \otimes \mathbf{Z}'')$ and $\mathbf{Z} \otimes \mathbf{Z}' \otimes \mathbf{Z}''$.

(3) There is a natural equivalence between $(\mathbf{Z} \otimes \mathbf{Z}')^*$. and $\mathbf{Z}^* \otimes \mathbf{Z}'^*$.

(4) There is a natural equivalence between $(\mathbf{Z} \otimes \mathbf{Z}')^c$ and $\mathbf{Z}^c \otimes \mathbf{Z}'^c$.

(5) There is a natural equivalence between $\mathbf{Z}^{c*}$ and $\mathbf{Z}^{*c}$.

(6) There is a natural equivalence between $\mathbf{Z} \otimes \mathbf{I}$ and $\mathbf{Z}$.

4. Composition of morphisms

A morphism $\kappa : \mathbf{I} \to \mathbf{Z}$ is called a *state*. Let $\nu : \mathbf{Z} \to \mathbf{Z}'$ be an embedding. The state κ composes with ν^t as a differentiable relation and $\nu^t \circ \kappa$ is a state.

Let $\kappa = (\mathbf{Z}, \mathbf{I}, \lambda, \mu)$ be a state and let $\pi = (\mathbf{Z}', \mathbf{Z}, \phi, \psi)$ be a fibration Let $\lambda = (M, \{1\}, \{1\} \times N)$ and $\mu = (Z, \mathbf{R}, K)$. We introduce sets $\bar{N} = \zeta^{-1}(N)$, $K(r) = \{z \in Z; (z, r) \in K\}$ and $K' \subset Z' \times \mathbf{R}$ such that $(z', r) \in K'$ if $\psi^{-1}(z') \cap \bar{N}$ is a submanifold of Z tangent to $K(r)$. Let N' be the projection of K' onto M'. If $\kappa' = (\mathbf{Z}', \mathbf{I}, \lambda', \mu')$ with $\lambda' = (M', \{1\}, \{1\} \times N')$ and $\mu' = (Z', \mathbf{R}, K')$ is a state then κ' is by definition the composition $\pi \circ \kappa$ of the state κ with the fibration π.

For each object $\mathbf{Z}$ we introduce the canonical state $\delta_{\mathbf{Z}} = (\mathbf{Z} \times \mathbf{Z}^*, I, \delta, \lambda)$, where $\delta = (M \times M, \{1\}, D)$, $D = \{((x, x'), 1) \in ((M \times M) \times \{1\}); \; x' = x\}$, $\lambda = (Z \times Z, I, L)$ and $L = \{(z \times z', r) \in (Z \times Z) \times \mathbf{R}; \; z = \sigma(r, \epsilon_Z(z'))\}$. The δ_{Z^*} is a reduction.

Let ρ be a morphism from $\mathbf{Z}$ to $\mathbf{Z}'$. The state graph$(\rho) = (\rho \times 1_{Z^*}) \circ \delta_Z$ is called the *graph* of ρ. The state δ_Z is the graph of 1_Z.

We observe that each state $\kappa : \mathbf{I} \to \mathbf{Z}' \times \mathbf{Z}^*$ is the graph of a unique morphism. Let $\rho : \mathbf{Z} \to \mathbf{Z}'$ and $\rho' : \mathbf{Z}' \to \mathbf{Z}''$ be morphisms. If $(1_{Z''} \otimes \delta_{Z'} \otimes 1_Z) \circ (\text{graph}(\rho') \otimes \text{graph}(\rho))$ is a state then it is by definition the composition $\rho \circ \rho'$ of morphisms ρ and ρ'.

The definition of composition introduced here is consistent with the special cases defined earlier.

Graphs of morhisms have the following properties:

i) $\text{graph}(\text{graph}(\rho)) = \text{graph}(\rho)$,

ii) $(\text{graph}(\rho))^c = \text{graph}(\rho^c)$,

iii) $\text{graph}(\rho) \otimes \text{graph}(\rho') = \text{graph}(\rho \otimes \rho')$.

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