The structure of positive linear symplectic relations

Nota di PAWEL URBANSKI *
presentata dal Socio straniero Wlodzimierz TULCZYJEW
nell'adunanza del 16 Gennaio 1985

Summary. *See the Introduction.*

0. Introduction

Symplectic relations have found extensive application in mathematical physics (see e.g. [1]). Results of a systematic study of linear symplectic relations are presented in a paper by Benenti and Tulczyjew [2]. In the present paper we define the concept of a positive linear symplectic relation and prove a theorem about the structure of positive relations. Results will be applied in symplectic control theory [3].

1. Symplectic vector spaces. Lagrangian subspaces.

A *symplectic vector space* is a pair (P, ω), where P is a real vector space of finite dimension and $\omega : P \times P \to R$ is a nondegenerate skew-symmetric bilinear form. The standard example of a symplectic vector space is provided by the direct sum $Q \oplus Q^*$ of a vector space Q and its dual space Q^* together with the canonical bilinear form ω defined by

$$\omega(q_1 \oplus f_1, q_2 \oplus f_2) = \langle q_2, f_1 \rangle - \langle q_1, f_2 \rangle .$$

Let (P, ω) be a symplectic vector space and let K be a subspace of P. The subspace of P defined by

$$K^{\S} = \{p \in P; \omega(p, p') = 0 \quad \text{for each} \quad p' \in K\}$$

* Department of Mathematica Methods in Physics - University of Warsaw - Hoza 74, 00-682 Warszawa, Poland.

is called the *symplectic polar* of K. The subspace K is said to be *isotropic* if $K \subset K^{\S}$, *coisotropic* if $K^{\S} \subset K$, *Lagrangian* if $K = K^{\S}$.

A Lagrangian subspace L of $Q \oplus Q^*$ is uniquely described by its *generating function* $F : C \to R$ defined on the image $C = pr_Q(L)$ of L by the canonical projection $pr_Q : Q \oplus Q^* \to Q$. The relation between L and the generating function F is expressed by

$$L = \{q \oplus f \in Q \oplus Q^* ; q \in C \text{ and } \langle q', f \rangle = \langle q', dF(q) \rangle \text{ for each } q' \in C\}$$

or by

$$F(q) = \frac{1}{2} \langle q, f \rangle ,$$

where f is any element of Q^* such that $q \oplus f \in L$. The differential dF of the quadratic function F is a linear mapping $dF : Q \to Q^*$ related to F by $F(q) = \frac{1}{2} \langle q, dF(q) \rangle$.

2. Symplectic relations. Reductions

Let (P, ω) and (P', ω') be symplectic vector space. A *symplectic relation* is a linear relation $\rho : P \to P'$ whose graph is a Lagrangian subspace of $(P \oplus P', (-\omega) \oplus \omega')$. It can be shown that the composition of two symplectic relations is a symplectic relation.

Let $\rho : P \to P'$ be a symplectic relation. For each subspace K of P we have

$$(\rho(K))^{\S} = \rho(K^{\S}) .$$

It follows that $\rho(0)$ is isotropic and $\rho(P)$ is coisotropic.

Let K be a coisotropic subspace of (P, ω). The vector space $P_{[K]} = K/K^{\S}$ and the projection $\omega_{[K]}$ of the symplectic form ω define a symplectic space $(P_{[K]}, \omega_{[K]})$. The canonical relation from P to $P_{[K]}$ is symplectic. It will be denoted by $\mathrm{red}_{(P, \omega; K)}$ and called the *symplectic reduction* of (P, ω) with respect to K. We have a structure theorem [2]:

THEOREM 2.1. - Let (P, ω) and (P', ω') be symplectic vector spaces and let $\rho : P \to P'$ be a symplectic relation. There exists a unique symplectic isomorphism ρ_0 such that

$$\rho = (\mathrm{red}_{(P', \omega'; \rho(P))})^{-1} \circ \rho_0 \circ \mathrm{red}_{(P, \omega; \rho^{-1}(P'))} .$$

Let $P = Q \oplus Q^*$ and $P' = Q' \oplus Q'^*$, and let ω and ω' denote the canonical symplectic forms. For each subspace S of $P' \oplus P$ we denote by $S^{\&}$ the subspace

$$S^{\&} = \{(q' \oplus f') \oplus (q \oplus f) \in P' \oplus P; \\ (q' \oplus f') \oplus (q \oplus (-f)) \in S\} .$$

A linear relation $\rho : P \to P'$ is symplectic if and only if $(\mathrm{graph}(\rho))^{\&}$ is a Lagrangian subspace of $(P \oplus P', \omega' \oplus \omega)$. The *generating function* of a symplectic relation $\rho : P \to P'$ is the generating function of the Lagrangian subspace $(\mathrm{graph}(\rho))^{\&}$. If L is a Lagrangian subspace of (P, ω) generated by a function $F : C \to R$ and $\rho : P \to P'$ is a symplectic relation generated by a function $G : D \to R$ then $L' = \rho(L)$ is a Lagrangian subspace of (P', ω'), $C' = pr_{Q'}(L')$ is the subspace

$$C' = \{q' \oplus Q';\ \text{there exists } q \oplus C \text{ such that } q \oplus q' \in D \\ \text{and } \langle \hat{q} \oplus 0, dG(q \oplus q')\rangle + \langle \hat{q}, dF(q)\rangle = 0 \\ \text{for each } \hat{q} \in C \text{ such that } \hat{q} \oplus 0 \in D\} \tag{2.1}$$

and L' is generated by the function $F' : C' \to R$ defined by

$$F'(q') = F(q) + G(q \oplus q') , \tag{2.2}$$

where q satisfies the condition stated in the definition of C'.

3. Positive Lagrangian subspaces

Spaces P and P' considered in this section and the subsequent are the direct sums $Q \oplus Q^*$ and $Q' \oplus Q'^*$, and ω and ω' denote the canonical symplectic forms.

DEFINITION 3.1. A Lagrangian subspace L of (P, ω) is said to be *positive* (*negative*) if its generating function is positive (negative). A symplectic relation $\rho : P \to P'$ is said to be *positive* (*negative*) if graph$(\rho)^{\&}$ is a positive (negative) Lagrangian subspace of $(P \oplus P', \omega \oplus \omega')$.

The following proposition is an immediate consequence of the composition properties of generating functions.

PROPOSITION 3.1. - The image $\rho(L)$ of a positive Lagrangian subspaces L of P by a positive symplectic relation $\rho : P \to P'$ is positive.

The set of positive Lagrangian subspaces of P is ordered by the relation $\geqslant$ defined by

$$L_1 \geqslant L_2 \text{ if } C_1 \subset C_2 \text{ and } F_1 \geqslant F_2 \,|\, C_1 ,$$

where $F_1 : C_1 \to R$ and $F_2 : C_2 \to R$ are generating functions of L_1 and L_2 respectively. The subspace $L_{\min} = Q \oplus 0$ is the minimal element in the set of positive Lagrangian subspaces of P and $L_{\max} = 0 \oplus Q^*$ is the maximal element.

THEOREM 3.1. - Let $\rho : P \to P'$ be a positive symplectic relation and let L_1 and L_2 be positive Lagrangian subspaces of P. If $L_1 \geqslant L_2$ then $\rho(L_1) \geqslant \rho(L_2)$.

Proof. - Let ρ, L_1 and L_2 be generated by $G : D \to R$, $F_1 : C_1 \to R$ and $F_2 : C_2 \to R$ respectively. Since these functions are positive and $C_1 \subset C_2$ it follows from (2.1) and (2.2) that $C_1' \subset C_2'$, where $C_1' = pr_{Q'}(\rho(L_1))$ and $C_2' = pr_{Q'}(\rho(L_2))$.

The point q in (2.1) is the minimum point of $F(q) + G(q \oplus q')$ for each q'. If q_1 and q_2 are related to F_1 and F_2 as q in (2.1) is related to F then

$$\begin{aligned} F_1'(q') &= F_1(q_1) + G(q_1 \oplus q') \geqslant \\ &\geqslant F_2(q_1) + G(q_1 \oplus q') \geqslant \\ &\geqslant F_2(q_2) + G(q_2 \oplus q') = F_2'(q'). \end{aligned}$$

Q.E.D.

4. Structure of positive symplectic relations.

In this section we give a proof of the following theorem.

THEOREM 4.1. - Let $\rho : P \to P'$ be a positive symplectic relation and let K denote $\rho^{-1}(P')$. The space Q can be represented as the direct sum $Q_1 \oplus Q_2$ of subspaces Q_1 and Q_2 such that if P_1 and P_2 denote the symplectic subspaces $Q_1 \oplus (Q_2)^\circ$ and $Q_2 \oplus (Q_1)^\circ$ of P and $K_1 = K \cap P_1$, $K_2 = K \cap P_2$ then $K = K_1 \oplus K_2$, $K_1 = pr_Q(K_1) \oplus \oplus pr_{Q^*}(K_1)$ and K_2 is a strictly negative Lagrangian subspace of P_2.

If Q_1 is a subspace of Q then $(Q_1)^\circ$ denotes the polar of Q_1 defined by

$$(Q_1)^\circ = \{f \in Q^* ; \langle q, f \rangle = 0 \quad \text{for each} \quad q \in Q_1\} .$$

The proof of the theorem is based on the following three lemmas.

LEMMA 4.1. - Let ρ and K be the objects introduced in Theorem 4.1. Then $K^\S$ is a negative isotropic subspace, i.e., $\langle q, f \rangle \leqslant 0$ for $q \oplus f \in K^\S$.

Proof. We have $K^\S = \rho^{-1}(0) \subset \rho^{-1}(L_{\max})$. Let $G : D \to R$ be the generating function of ρ. The generating function H of ρ^{-1} defined by $H(q' \oplus q) = -G(q \oplus q')$ is negative. It follows from (2.2) that if $q \oplus f \in K^\S$ then $\langle q, f \rangle = 2H(0 \oplus q) \leqslant 0$. Q.E.D.

LEMMA 4.2. - Let L_1, L_2 and L be positive Lagrangian subspaces of P. If $L_1 \geqslant L \geqslant L_2$ then $L \supset (L_1 \cap L_2)$.

Proof. Let F_1, F_2 and F be generating functions of L_1, L_2 and L defined on C_1, C_2 and C respectively. Then $C_1 \subset C \subset C_2$, $F_1 \geqslant F|C_1$ and $F \geqslant F_2|C$. If $q \in pr_Q(L_1 \cap L_2)$ then $F_1(q) = F_2(q) = F(q)$ and, since $F - F_2|C$ is positive, $dF(q) - d(F_2|C)(q) = 0$. It follows that $q \oplus f \in L_2$ implies $q \oplus f \in L$. Q.E.D.

LEMMA 4.3. - If $\rho : P \to P'$ is a positive symplectic relation then $\rho(L_{\min}) + \rho(L_{\max}) = \rho(P)$.

Proof. We denote by P_+ the set of positive elements of P defined by $P_+ = \{q \oplus f \in P; \langle q, f \rangle > 0\}$. This set in open in P. We assume that $K = \rho^{-1}(P')$ is not Lagrangian, i.e., $\dim(K) > \frac{1}{2}\dim(P) = n$. If K is Lagrangian then the lemma is trivial. Since $P_+ \cup 0$ contains an n-dimensional subspace, it follows that $P_+ \cap K$ is not empty and open in K. Hence $\rho(P_+)$ is not empty and open in $\rho(P)$. Let $p = q \oplus f \in P_+$ and let L_p be the positive Lagrangian subspace generated by the function $F_p : C_p \to R$ defined on $C_p = \{\hat{q} \in C;\ \hat{q} = aq$ for some $a \in R\}$ by

$F_p(aq) = \dfrac{1}{2} a^2 \langle q, f \rangle$. From Lemma 4.2 it follows that

$$(\rho(L_{\min}) \cap \rho(L_{\max})) \subset \rho(L_p) \subset (\rho(L_{\min}) \cap \rho(L_{\max}))^{\S}.$$

Since $\rho(P_+)$ is open in $\rho(P)$, we have

$$\rho(P_+) \subset (\rho(L_{\min}) \cap \rho(L_{\max}))^{\S} = \rho(P). \qquad \text{Q.E.D.}$$

Proof of Theorem 4.1. - From Lemma 4.1 we know that $K^{\S}$ is isotropic and negative. We introduce subspaces

$$Q_0 = \{q \in Q; q \oplus 0 \in K^{\S}\},$$
$$Q_0^{\#} = \{f \in Q^*; 0 \oplus f \in K^{\S}\},$$
$$\bar{K}_1^{\S} = Q_0 \dot{\oplus} Q_0^{\#} \subset K^{\S}.$$

Let $\bar{K}_2^{\S}$ be a complement of $\bar{K}_1^{\S}$ in $K^{\S}$. Since $K^{\S}$ is isotropic, we have inclusions

$$pr_Q(\bar{K}_2^{\S}) \subset (Q_0^{\#})^{\circ}, \quad pr_{Q^*}(\bar{K}_2^{\S}) \subset (Q_0)^{\circ}.$$

Let $q \in Q_0$ and $q \oplus f \in \bar{K}_2^{\S}$. Then $f \in Q_0^{\#}$ and $q \oplus f \in \bar{K}_1^{\S}$. It follows that $q = 0$ and $f = 0$. We conclude that $pr_Q(\bar{K}_2^{\S}) \cap Q_0 = 0$ and $pr_{Q^*}(\bar{K}_2^{\S}) \cap Q_0^{\#} = 0$. This implies that spaces Q and Q^* can be represented as direct sums $Q = Q_0 \oplus \bar{Q}_1 \oplus \bar{Q}_2$ and $Q^* = Q_0^{\#} \oplus \bar{Q}_1^{\#} \oplus \bar{Q}_2^{\#}$ of their subspaces such that

$$Q_0 \oplus \bar{Q}_2 = (Q_0^{\#})^{\circ} \text{ and } pr_Q(\bar{K}_2^{\S}) \subset Q_2,$$
$$Q_0^{\#} \oplus \bar{Q}_2^{\#} = (Q_0)^{\circ} \text{ and } pr_{Q^*}(\bar{K}_2^{\S}) \subset \bar{Q}_2^{\#},$$
$$Q_0^{\#} \oplus \bar{Q}_1^{\#} = (\bar{Q}_2)^{\circ} \text{ and } \bar{Q}_1 = (\bar{Q}_1^{\#} \oplus \bar{Q}_2^{\#})^{\circ}.$$

It follows that $\bar{Q}_2^{\#} = (Q_0 \oplus \bar{Q}_1)^{\circ}$. We see that the dual spaces of $\bar{Q}_2$ and $Q_0 \oplus \bar{Q}_1$ can be identified with $\bar{Q}_2^{\#}$ and $Q_0^{\#} \oplus \bar{Q}_1^{\#}$ respectively. Hence the vector space $\bar{P}_2 = \bar{Q}_2 \oplus \bar{Q}_2^{*}$ and $\bar{P}_1 = (Q_0 \oplus \bar{Q}_1) \oplus \oplus (Q_0^{\#} \oplus \bar{Q}_1^{*})$ are canonically symplectic. It is easily seen that the canonical identification of P with $\bar{P}_1 \oplus \bar{P}_2$ is a symplectomorphism. It follows that $\bar{K}_1^{\S} = K^{\S} \cap \bar{P}_1$, $\bar{K}_2^{\S} = K^{\S} \cap \bar{P}_2$ and $\bar{K}_1^{\S}$ and $\bar{K}_2^{\S}$ are isotropic subspaces of $\bar{P}_1$ and $\bar{P}_2$ respectively. Hence $K = \bar{K}_1 \oplus \bar{K}_2$, where $\bar{K}_1$ and $\bar{K}_2$ are symplectic polars of $\bar{K}_1^{\S}$ and $\bar{K}_2^{\S}$ in $\bar{P}_1$ and

P_2 respectively, and we have the canonical identification

$$(P_{[K]}, \omega_{[K]}) = (\bar{P}_{1[\bar{K}_1]} \oplus \bar{P}_{2[\bar{K}_2]}, \omega_{1[\bar{K}_1]} \oplus \omega_{2[\bar{K}_2]}) .$$

It is easily seen that $\bar{K}_2^{\S}$ is the graph of a bijection between $Q_2 = pr_Q(\bar{K}_2^{\S})$ and $Q_2^{\#} = pr_{Q^*}(\bar{K}_2^{\S})$. From Lemma 4.3 it follows that

$$\bar{K}_2 = \bar{K}_2^{\S} + \bar{K}_2 \cap (\bar{Q}_2 \oplus 0) + \bar{K}_2 \cap (0 \oplus \bar{Q}_2^{\#}) . \tag{4.1}$$

On the other hand we have $\bar{K}_2 \cap (\bar{Q}_2 \oplus 0) = (Q_2)^{\circ}$ and $\bar{K}_2 \cap (0 \oplus \bar{Q}_2^{\#}) = (Q_2)^{\circ}$, and a simple comparison of dimensions shows that the decomposition (4.1) is a direct sum. Hence, $Q_2 \cap (Q_2^{\#})^{\circ} = 0$. It follows that with $Q_1 = Q_0 + \bar{Q}_1 \oplus (Q_2^{\#})^{\circ}$ we obtain the required decompositions of Q and P with P_1 and P_2 identified with $\bar{P}_1 \oplus ((Q_2^{\#})^{\circ} \oplus (Q_2)^{\circ})$ and $Q_2 \oplus Q_2^{\#}$ respectively. Q.E.D.

It is obvious that a theorem analogous to Theorem 4.1 holds for negative symplectic relations and that an analogous decomposition of Q', P' and $K' = \rho(P)$ can be obtained.

DEFINITION 4.1. - The reduction of P with respect to a coisotropic subspace K is said to be

a) *special symplectic* if there exist subspaces Q_0 and $Q_0^{\#}$ of Q and Q^* respectively such that $K = Q_0 \oplus Q_0^{\#}$;

b) *essentially special symplectic* if there is a decomposition $Q = Q_1 \oplus Q_2$ and the corresponding decomposition $P = P_1 \oplus P_2$, with $P_1 = Q_1 \oplus (Q_2)^{\circ}$ and $P_2 = Q_2 \oplus (Q_1)^{\circ}$, such that $K = (K \cap P_1) + (K \cap P_2)$, $K \cap P_2$ is Lagrangian subspace of P_2 and the reduction of P_1 with respect to $K \cap P_1$ is special symplectic.

PROPOSITION 4.1 - Let the reduction of P with respect to a coisotropic subspace K be essentially special symplectic. The reduced space $P_{[K]}$ can be identified with the symplectic space $\underset{\sim}{Q} \oplus \underset{\sim}{Q}^*$ constructed from a linear space $\underset{\sim}{Q}$.

The following theorem is a consequence of Theorem 4.1 and Proposition 4.1.

THEOREM 4.2. - Let $\rho : P \to P'$ be a positive symplectic relation. Then all components in the decomposition

$$\rho = (\mathrm{red}_{(P', \omega'; \rho(P))})^{-1} \circ \rho_0 \circ \mathrm{red}_{(P, \omega; \rho^{-1}(P'))}$$

are positive and the reductions are essentially special symplectic.

THEOREM 4.3. - Let $\rho : P \to P'$ be a positive symplectic relation. Let L_1 and L_2 be positive Lagrangian subspaces of P such that $L_1 \geqslant L_2$. A positive lagrangian subspace L' of P' is the image of a positive Lagrangian subspace L satisfying $L_1 \geqslant L \geqslant L_2$ if and only if $\rho(L_1) \geqslant L \geqslant \rho(L_2)$.

The following lemma reduces the proof of the theorem to the case $L_1 = L_{\max}$ and $L_2 = L_{\min}$.

LEMMA 4.4 - Let L_1 and L_2 be positive Lagrangian subspaces of P such that $L_1 \geqslant L_2$. There exists a positive symplectic relation $\sigma : \widetilde{Q} \oplus \widetilde{Q}^* \to P$ such that $\sigma(L_{\max}) = L_1$ and $\sigma(L_{\min}) = L_2$.

Complete proofs of Lemma 4.4 and Theorem 4.3 will be given in a more extensive publication.

Aknowledgments

This paper is a contribution to the programme of study of symplectic geometry and its application in mathematica physics centered at Istituto di Fisica Matematica "J.-L. Lagrange" of Torino. This research programme has been made possible by the support from Consiglio Nazionale delle Ricerche.

I am indebted to Dr. Adam Smolski for his critical comments.

REFERENCES

[1] J. KIJOWSKI and W.M. TULCZYJEW, A sympletic framework for field theories., *Lecture Notes in Physics*, **107**, Springer, Berlin, 1979.

[2] S. BENENTI and W.M. TULCZYJEW, Relazioni lineari simplettiche, **Mem. Acc. Sci. Torino**, 5 (1981).

[3] W.M. TULCZYJEW and P. URBANSKI, Symplectic control theory, Mem. Acc. Sci. Torino, (to appear).