STRINGS, GERBES, AND ALL THAT
1. GENERALITIES

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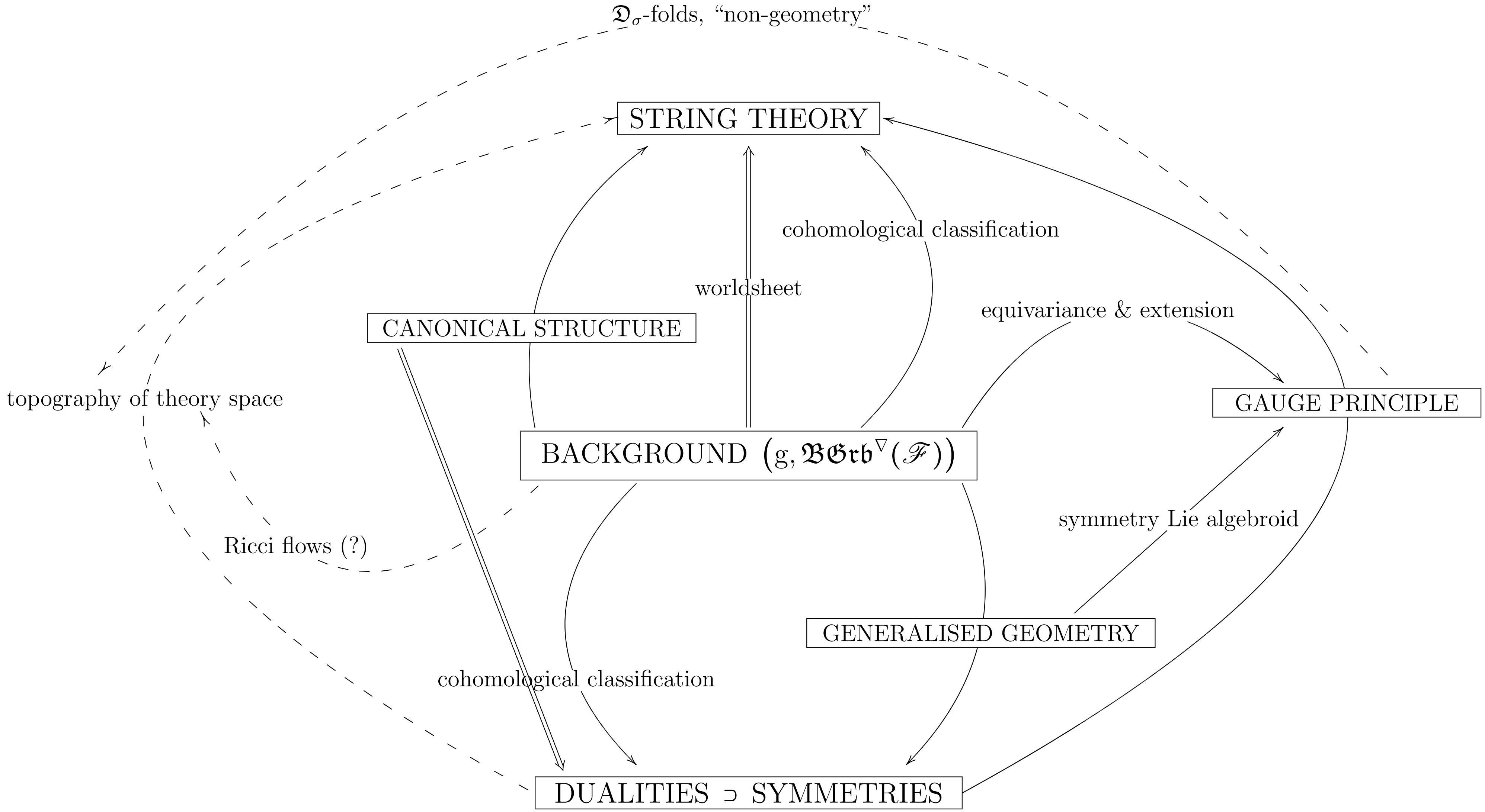
The topology of the subject:



I. A motivating analogy

I.1. The lagrangean description of a charged point-like particle

A theory of C^1-maps $x : \Gamma \to M$ determined by the PLA for

$$S_p[x] = -\tfrac{1}{2} \int_\Gamma \mathrm{g}_x(\dot{x}, \dot{x}) + S_{\rm top}[x] \,, \qquad S_{\rm top}[x] = \text{"} \int_\Gamma x^* \mathsf{d}^{-1}\mathrm{F} \text{"} \,,$$

- Γ – **WORLD-LINE**, assumed closed, $\partial\Gamma = \varnothing$;
- M – **TARGET SPACE**;
- (g, F) – **BACKGROUND** fields:
  - $\mathrm{g} \in \mathrm{Sym}_2(M)$ – metric on M;
  - $\mathrm{F} \in Z^2(M)$ – electromagnetic field strength.

Problem: $H^2(M) \ni [\mathrm{F}] \neq 0 \qquad \Longrightarrow \qquad \neg\exists\, \mathsf{d}^{-1}\mathrm{F}$

Solution: Abstractly, we need

$$\text{LINE BUNDLE} \quad \mathbb{C} \hookrightarrow \mathcal{L} \xrightarrow{\pi_{\mathcal{L}}} M \,, \qquad \mathrm{curv}(\nabla_L) = \pi_{\mathcal{L}}^* \mathrm{F} \,,$$

$$S_{\rm top}[x] = -\mathsf{i} \log \mathrm{Hol}_{\mathcal{L}}(x) \equiv [x^* \mathcal{L}] \in H^1\big(\Gamma, \mathrm{U}(1)\big) \cong \mathrm{U}(1) \,.$$

Less so, given a **GOOD OPEN COVER** $\mathcal{O}_M = \{\mathcal{O}_i\}_{i \in I}$ of M,

cohomological constraints: $$\begin{cases} \mathrm{F}|_{\mathcal{O}_i} =: \mathsf{d}A_i \,, & A_i \in \Omega^1(\mathcal{O}_i) \\ (A_j - A_i)|_{\mathcal{O}_{ij}} =: \mathsf{i}\,\mathsf{d} \log g_{ij} \,, & g_{ij} \in \mathrm{U}(1)_{\mathcal{O}_{ij}} \end{cases} \,,$$

'quantum' constraints: $$(g_{ij}\, g_{jk})|_{\mathcal{O}_{ijk}} = g_{ik}|_{\mathcal{O}_{ijk}}$$

yield a local formula for $S_{\rm top}[x]$.

I.2. The canonical structure of the theory

The **PHASE SPACE**: $\mathsf{P}_p \cong \mathsf{T}^*M \xrightarrow{\pi} M$, equipped with

SYMPLECTIC STRUCTURE $\quad \Omega_p = \mathsf{d}\theta_{\mathsf{T}^*M} + \pi^*\mathrm{F}\,, \qquad \theta_{\mathsf{T}^*M}$ – canonical 1-form on T^*M.

The **HILBERT SPACE** $\mathcal{H}_p = \Gamma_{\rm pol.}(\mathcal{L}_p)$ of the theory determined by

PRE-QUANTUM BUNDLE $\quad \mathbb{C} \hookrightarrow \pi^*\mathcal{L} \otimes (\mathsf{P}_p \times \mathbb{C}) =: \mathcal{L}_p \to \mathsf{P}_p\,,$

where the second tensor factor has the (global) connection 1-form θ_{T^*M}.

I.3. (Internal) Symmetries of the theory

Infinitesimally, generated by $\mathscr{K} \in \Gamma(\mathsf{T}M)$,

$$x^\mu \mapsto x^\mu + \epsilon\,\mathscr{K} \lrcorner\, \mathsf{d}x^\mu \qquad \Longrightarrow \qquad \delta_{\epsilon\mathscr{K}} S_p[x] = \epsilon\left(-\tfrac{1}{2}\int_\Gamma (\mathscr{L}_{\mathscr{K}}\mathrm{g})_x(\dot{x},\dot{x}) + \int_\Gamma x^*(\mathscr{K} \lrcorner\, \mathrm{F})\right).$$

Conclusion: Symmetries described by $\mathfrak{K} := \mathscr{K} \oplus k \in \Gamma_p\big(\mathsf{E}^{(1,0)}M\big) \subset \Gamma\big(\mathsf{E}^{(1,0)}M\big)$,

GENERALISED TANGENT BUNDLE $\quad \mathsf{E}^{(1,0)}M := \wedge^1\mathsf{T}M \oplus \wedge^0\mathsf{T}^*M \to M\,,$

$$\Gamma_p\big(\mathsf{E}^{(1,0)}M\big) := \left\{\, \mathscr{V} \oplus v \in \Gamma\big(\mathsf{E}^{(1,0)}M\big) \quad \middle|\quad \mathsf{d}v + \mathscr{K} \lrcorner\, \mathrm{F} = 0 \quad \wedge \quad \mathscr{L}_{\mathscr{K}}\mathrm{g} = 0 \,\right\}.$$

Observation: $\Gamma\big(\mathsf{E}^{(1,0)}M\big)$ admits

F-TW. VINOGRADOV BRACKET $\quad [\,\mathscr{V}_1 \oplus v_1\,,\, \mathscr{V}_2 \oplus v_2\,]^{\mathrm{F}}_{\mathrm{V}} = [\,\mathscr{V}_1\,,\,\mathscr{V}_2\,] \oplus (\mathscr{V}_1 \lrcorner\, v_2 - \mathscr{V}_2 \lrcorner\, v_1 + \mathscr{V}_1 \lrcorner\, \mathscr{V}_2 \lrcorner\, \mathrm{F})$

with the properties:

- $[\,\cdot\,,\,\cdot\,]^{\mathrm{F}}_{\mathrm{V}} \;:\; \Gamma_p\big(\mathsf{E}^{(1,0)}M\big)^2 \to \Gamma_p\big(\mathsf{E}^{(1,0)}M\big)$;
- $\Big(\Gamma_p\big(\mathsf{E}^{(1,0)}M\big), [\,\cdot\,,\,\cdot\,]^{\mathrm{F}}_{\mathrm{V}}\Big) \overset{\rm hom.}{\hookrightarrow} \Big(\Gamma\big(\mathsf{E}^{(1,0)}\mathsf{P}_p\big), [\,\cdot\,,\,\cdot\,]^{\Omega_p}_{\mathrm{V}}\Big)$.

II. <u>The two-dimensional (non-linear) σ-model</u>

A theory of C^1-maps $X : \Sigma \to M$ determined by the PLA for

$$S_\sigma[X;\gamma] = -\tfrac{1}{2}\int_\Gamma \mathrm{g}_X(\mathsf{d}X \overset{\wedge}{,} \star_\gamma \mathsf{d}X) + S_{\rm top}[X]\,, \qquad\qquad S_{\rm top}[X] = \text{``}\int_\Sigma X^*\mathsf{d}^{-1}\mathrm{H}\text{''}\,,$$

- Γ – **WORLD-SHEET**, assumed closed, $\partial\Sigma = \varnothing$;
- M – **TARGET SPACE**;
- (g,H) – **BACKGROUND** fields:
  - $\mathrm{g} \in \mathrm{Sym}_2(M)$ – metric on M;
  - $\mathrm{H} \in Z^3(M)$ – torsion field strength.

<u>Problem:</u> $H^3(M) \ni [\mathrm{H}] \neq 0 \qquad \Longrightarrow \qquad \neg\exists\, \mathsf{d}^{-1}\mathrm{H}$

<u>N.B.</u> The choice of (g,H) is severely constrained by the requirement of a non-anomalous symmetry $\mathscr{D}iff^0(\Sigma)\ltimes\mathrm{Weyl}(\gamma)$ of the quantum theory.

<u>Solution:</u> [Alvarez '85;Gawędzki '86] Locally, given a good open cover $\mathcal{O}_M = \{\mathcal{O}_i\}_{i\in I}$ of M,

CURVINGS $\quad B_i \in \Omega^2(\mathcal{O}_i)\,,\qquad$ CONNECTIONS $\quad A_{ij} \in \Omega^1(\mathcal{O}_{ij})\,,$

TRANSITION FUNCTIONS $\quad g_{ijk} \in \mathrm{U}(1)_{\mathcal{O}_{ijk}}\,,$

$$\begin{aligned} &\mathrm{H}|_{\mathcal{O}_i} =: \mathsf{d}B_i \\ &(B_j - B_i)|_{\mathcal{O}_{ij}} =: \mathsf{d}A_{ij} \\ &(A_{jk} - A_{ik} + A_{ij})|_{\mathcal{O}_{ijk}} =: \mathsf{i}\,\mathsf{d}\log g_{ijk} \\ &(g_{jkl}\cdot g_{ikl}^{-1}\cdot g_{ijl}\cdot g_{ijk}^{-1})|_{\mathcal{O}_{ijkl}} = 1 \end{aligned} \qquad \text{mod} \qquad \begin{aligned} &B_i \mapsto B_i + \mathsf{d}\Pi_i \\ &A_{ij} \mapsto A_{ij} + (\Pi_j - \Pi_i)|_{\mathcal{O}_{ij}} - \mathsf{i}\,\mathsf{d}\log\chi_{ij} \\ &g_{ijk} \mapsto g_{ijk}\cdot(\chi_{jk}^{-1}\cdot\chi_{ik}\cdot\chi_{ij}^{-1})|_{\mathcal{O}_{ijk}} \end{aligned}$$

define – for a triangulation $\triangle_\Sigma$ of Σ subordinate to $\mathcal{O}_M$ wrt. X –

$$S_{\rm top}[X] = \sum_{p\in\triangle_\Sigma}\left[\int_p X_p^* B_{i_p} + \sum_{e\subset p}\left(\int_e X_e^* A_{i_p i_e} - \mathsf{i}\sum_{v\in e}\log X^* g_{i_p i_e i_v}(v)^{\varepsilon_{pev}}\right)\right].$$

III. Abelian (bundle) gerbes with connection

III.1. The local (cohomological) description

Recall: given a sheaf $\mathcal{S}$ over M, and a cover $\mathcal{O}_M = \{\mathcal{O}_i\}_{i\in I}$, we have

$$\textbf{ČECH OPERATOR}\qquad \check{\delta}^{(p)} \;:\; \check{C}^p(\mathcal{O}_M,\mathcal{S}) \to \check{C}^{p+1}(\mathcal{O}_M,\mathcal{S})$$

$$: \;(c_{[i_0 i_1 \ldots i_p]}) \mapsto \left(\sum_{k=0}^{p+1} (-1)^k\, c_{i_0 i_1 \ldots \widehat{i_k} \ldots i_{p+1}}\big|_{\mathcal{O}_{i_0 i_1 \ldots i_{p+1}}}\right)$$

defined on sections $c_{[i_0 i_1 \ldots i_p]} \in \mathcal{S}(\mathcal{O}_{i_0 i_1 \ldots i_p})$.

Consider the **DELIGNE COMPLEX** of differential sheaves

$$\mathcal{D}(2)^\bullet \;:\; \underline{\mathrm{U}(1)}_M \xrightarrow{\frac{1}{\mathsf{i}}\,\mathsf{d}\log\equiv\mathsf{d}^{(0)}} \underline{\Omega}^1(M) \xrightarrow{\mathsf{d}\equiv\mathsf{d}^{(1)}} \underline{\Omega}^2(M)\,.$$

Its Čech extension defines the **ČECH–DELIGNE BICOMPLEX**

$$\begin{array}{ccccccc}
\check{C}^0(\mathcal{O}_M,\underline{\Omega}^2) & \xrightarrow{\check{\delta}^{(0)}} & \check{C}^1(\mathcal{O}_M,\underline{\Omega}^2) & \xrightarrow{\check{\delta}^{(1)}} & \check{C}^2(\mathcal{O}_M,\underline{\Omega}^2) & \xrightarrow{\check{\delta}^{(2)}} & \cdots \\
\uparrow{\scriptstyle \mathsf{d}^{(1)}} & & \uparrow{\scriptstyle \mathsf{d}^{(1)}} & & \uparrow{\scriptstyle \mathsf{d}^{(1)}} & & \\
\check{C}^0(\mathcal{O}_M,\underline{\Omega}^1) & \xrightarrow{\check{\delta}^{(0)}} & \check{C}^1(\mathcal{O}_M,\underline{\Omega}^1) & \xrightarrow{\check{\delta}^{(1)}} & \check{C}^2(\mathcal{O}_M,\underline{\Omega}^1) & \xrightarrow{\check{\delta}^{(2)}} & \cdots\,. \\
\uparrow{\scriptstyle \mathsf{d}^{(0)}} & & \uparrow{\scriptstyle \mathsf{d}^{(0)}} & & \uparrow{\scriptstyle \mathsf{d}^{(0)}} & & \\
\check{C}^0(\mathcal{O}_M,\underline{\mathrm{U}(1)}) & \xrightarrow{\check{\delta}^{(0)}} & \check{C}^1(\mathcal{O}_M,\underline{\mathrm{U}(1)}) & \xrightarrow{\check{\delta}^{(1)}} & \check{C}^2(\mathcal{O}_M,\underline{\mathrm{U}(1)}) & \xrightarrow{\check{\delta}^{(2)}} & \cdots
\end{array}$$

Def$^{\underline{\mathrm{n}}}$: *The* ***DELIGNE HYPERCOHOMOLOGY*** *is the cohomology of the Čech–Deligne bicomplex, i.e. the cohomology of the diagonal subcomplex (additive notation!)*

$$A_M^\bullet \;:\; A_M^0 \xrightarrow{D_{(0)}} A_M^1 \xrightarrow{D_{(1)}} \ldots\,, \qquad\qquad A_M^r = \bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=r}} \check{C}^p\big(\mathcal{O}_M,\mathcal{D}(2)^q\big)\,,$$

$$\textit{\textbf{DELIGNE DIFFERENTIAL}}\qquad D_{(r)}\big|_{\check{C}^p(\mathcal{O}_M,\mathcal{D}(2)^q)} = \mathsf{d}^{(q)} + (-1)^{q+1}\,\check{\delta}^{(p)}\,.$$

Def$^{\underline{n}}$: *Given a good open cover* $\mathcal{O}_M = \{\mathcal{O}_i\}_{i\in I}$ *of a differentiable manifold* M*, a Deligne 2-cocycle*

$$(B_i, A_{ij}, g_{ijk}) =: b \in A^2_M\,, \qquad\qquad D_{(2)}b = 0$$

defines an ***ABELIAN GERBE WITH CONNECTION*** $\mathcal{G}$*.*
Its ***EQUIVALENCE CLASS*** $[\mathcal{G}] \in \mathbb{H}^2\big(M, \mathcal{D}(2)^\bullet\big)$ *is*

$$b \sim b + D_{(1)}p\,, \qquad\qquad p = (\Pi_i, \chi_{ij}) \in A^1_M\,.$$

The Classification Theorem: The set $\mathcal{W}(M;\mathrm{H})$ of equivalence classes of gerbes of curvature $\mathrm{H} \in Z^3_{2\pi\mathbb{Z}}(M)$ over M is a $\check{H}^2\big(M, \mathrm{U}(1)\big)$-torsor wrt. the action

$$\big([(c_{ijk})], [(B_i, A_{ij}, g_{ijk})]\big) \mapsto [(B_i, A_{ij}, c_{ijk}\cdot g_{ijk})]\,.$$

Idea of proof: Clearly, $\mathcal{W}(M;\mathrm{H})$ is a $\mathcal{W}(M;0)$-torsor wrt. the action

$$\big([(\beta_i, \alpha_{ij}, \gamma_{ijk})], [(B_i, A_{ij}, g_{ijk})]\big) \mapsto [(B_i + \beta_i, A_{ij} + \alpha_{ij}, g_{ijk}\cdot\gamma_{ijk})]\,.$$

Furthermore, $\mathcal{W}(M;0) \cong \check{H}^2\big(M, \mathrm{U}(1)\big)$.

Implication: Cohomological classification of (inequivalent) σ-models for given (g, H), e.g., for $M = \mathrm{G}$ a simple compact 1-connected Lie group,

$$\forall_{\mathsf{k}\in\mathbb{N}}\ \exists!\ \mathcal{G}_{\mathsf{k}} = \mathcal{G}_1^{\otimes\mathsf{k}}\,, \quad \mathrm{curv}(\mathcal{G}_{\mathsf{k}}) = \tfrac{\mathsf{k}}{12\pi}\,\mathrm{tr}_{\mathfrak{g}}(\theta_L \wedge \theta_L \wedge \theta_L)\,.$$

Here, $\mathcal{G}_1$ is the so-called **BASIC GERBE**.

Physical result: Given $[(c_{ijk})] \in H^2\big(\Sigma, \mathrm{U}(1)\big)$ s.t. $\mathcal{W}(\Sigma;0) \ni [X^*\mathcal{G}] = [(0, 0, c_{ijk})]$, the triviality of the Bokshteyn homomorphism $H^2\big(\Sigma, \mathrm{U}(1)\big) \cong H^3\big(\Sigma, 2\pi\mathbb{Z}\big)$ implies the existence of $[(\rho)] \in H^2(\Sigma, \mathbb{R})/H^2(\Sigma, 2\pi\mathbb{Z})$ s.t.

$$[X^*\mathcal{G}] = [(\rho, 0, 1)] \equiv [I_\rho]\,, \qquad \text{and} \qquad S_{\mathrm{top}}[X] = \int_\Sigma \rho \equiv -\mathsf{i}\,\log\mathrm{Hol}_{\mathcal{G}}(X)\,.$$

III.1$\frac{1}{2}$. Constructions – examples & un bout d'histoire:

- basic gerbes over SC1C Lie groups: SU(2) [Gawędzki '86], SU(N) [Chatterjee '98], G arbitrary [Meinrenken '02]

- gerbes over SCCnsC Lie groups $\mathrm{G} \cong \widetilde{\mathrm{G}}/Z$, $Z \subset Z(\widetilde{\mathrm{G}})$ ($\widetilde{\mathrm{G}}$ is a simply connected cover) $\overset{1:1}{\longleftrightarrow}$ gerbes over $\widetilde{\mathrm{G}}$ with the Z-equivariant structure [Gawędzki & Reis '02,'03]

- gerbes over orientifolds $\widetilde{\mathrm{G}}/(\mathbb{Z}_2 \ltimes Z)$ of SC1C Lie groups $\overset{1:1}{\longleftrightarrow}$ gerbes over $\widetilde{\mathrm{G}}$ with the twisted $\mathbb{Z}_2 \ltimes Z$-equivariant structure [Schreiber, Schweigert & Waldorf '05;Gawędzki, Suszek & Waldorf '07,'08]

- multiplicative gerbes over SCC Lie groups [Carey, Johnson, Murray, Stevenson & Wang '04;Waldorf '08;Gawędzki & Waldorf]

- (continuously) equivariant gerbes for the gauged σ-model, in particular, over semi-simple compact connected Lie groups [Gawędzki, Suszek & Waldorf '10]

- gerbe modules for D-branes [Carey, Johnson & Murray '02]

- maximally symmetric modules of gerbes over SCC Lie groups [Gawędzki & Reis '02,'03;Gawędzki '04]

- gerbes bi-modules for bi-branes, and maximally symmetric bi-modules of gerbes over SC1C Lie groups [Fuchs, Schweigert & Waldorf '07]

- the full gerbe 2-category for the multi-phase σ-model [Runkel & Suszek '08]

- inter-bi-branes for the maximally symmetric bi-branes of gerbes over SC1C Lie groups vs Verlinde fusion rings [Runkel & Suszek '09,'10]

- transgression [Gawędzki '86] and geometric quantisation of the WZW model [Felder, Gawędzki & Kupiainen '88]

- canonical interpretation of the gerbe 2-category, and 2-categorially twisted generalised geometry [Suszek '10]

III.2. The geometric construction – bundle gerbes

Problem: The lack of a *natural* choice of a good open cover.

Solution-Def$^{\underline{n}}$: [Murray '94] *An* ***ABELIAN BUNDLE GERBE WITH CONNECTION*** $\mathcal{G}$ *of curvature* H *over a manifold* M *is a quadruple* (YM, B, L, μ) *in which*

- $\pi_{YM} : YM \to M$ *is a* ***surjective submersion****;*
- *the* ***curving 2-form*** $B \in \Omega^2(YM)$ *satisfies*

$$\pi_{YM}^* \mathrm{H} = \mathsf{d}B \,;$$

- $\mathbb{C} \hookrightarrow L \to Y^{[2]}M$ *is a* ***line bundle*** *over the fibred product*

$$Y^{[2]}M \equiv YM \times_M YM := \{ \ (y_1, y_2) \in YM \times YM \quad | \quad \pi_{YM}(y_1) = \pi_{YM}(y_2) \ \} \,,$$

$$\mathrm{pr}_i(y_1, y_2) = y_i \,, \ i \in \{1, 2\}$$

with connection ∇_L *of curvature*

$$\mathrm{curv}(\nabla_L) = \mathrm{pr}_2^* B - \mathrm{pr}_1^* B \,;$$

- μ *engenders a* ***groupoid structure*** *on* $L \rightrightarrows YM$ *via*

$$\mu \ : \ L_{1,2} \otimes L_{2,3} \xrightarrow{\cong} L_{1,3} \,, \qquad L_{i,j} := (\mathrm{pr}_i \times \mathrm{pr}_j)^* L$$

over $Y^{[3]}M$ *that is associative in the sense of*

$$\begin{array}{ccc} L_{1,2} \otimes L_{2,3} \otimes L_{3,4} & \xrightarrow{\mathrm{id}\otimes\mu_{2,3,4}} & L_{1,2} \otimes L_{2,4} \\ {\scriptstyle \mu_{1,2,3}\otimes\mathrm{id}} \downarrow & \circlearrowright & \downarrow {\scriptstyle \mu_{1,2,4}} \\ L_{1,3} \otimes L_{3,4} & \xrightarrow[\mu_{1,3,4}]{} & L_{1,4} \end{array} \,.$$

IV. The canonical structure of the σ-model

The **CONFIGURATION SPACE** of the theory is given by $\mathsf{L}M = C^\infty(\mathbb{S}^1, M)$, and we have the natural identification

$$\text{PHASE SPACE} \qquad \mathsf{P}_\sigma = \mathsf{T}^*\mathsf{L}M \xrightarrow{\pi_{\mathsf{T}^*\mathsf{L}M}} \mathsf{L}M\,.$$

Denote by $\mathrm{ev} : \mathsf{L}M \times \mathbb{S}^1 \to M$ the standard evaluation map, and by $\theta_{\mathsf{T}^*\mathsf{L}M}$ the canonical 1-form on $\mathsf{T}^*\mathsf{L}M$. Using the first-order formalism of Gawędzki–Kijowski–Szczyrba–Tulczyjew, we derive

$$\text{(PRE-)SYMPLECTIC FORM} \qquad \Omega_\sigma = \delta\theta_{\mathsf{T}^*\mathsf{L}M} + \pi^*_{\mathsf{T}^*\mathsf{L}M} \int_{\mathbb{S}^1} \mathrm{ev}^*\mathrm{H} \in Z^2(\mathsf{P}_\sigma)\,.$$

It gives us access to the Poisson-bracket algebra of hamiltonian functions $C^\infty(\mathsf{P}_\sigma, \mathbb{R})$, or – equivalently – to

Def^n: The **CANONICAL VINOGRADOV STRUCTURE** on a symplectic manifold (P, Ω) is the triple $\big(\mathsf{E}^{(1,0)}\mathsf{P}, [\cdot,\cdot]^\Omega_{\mathrm{V}}, \alpha_{\mathsf{TP}}\big)$ composed of

- the **GENERALISED TANGENT BUNDLE OF TYPE** $(1,0)$

$$\mathsf{E}^{(1,0)}\mathsf{P} := \wedge^1\mathsf{TP} \oplus \wedge^0\mathsf{T}^*\mathsf{P} \to \mathsf{P}\,;$$

- the **Ω-TWISTED VINOGRADOV BRACKET**

$$[\mathscr{V} \oplus f, \mathscr{W} \oplus g]^\Omega_{\mathrm{V}} := [\,\mathscr{V}\,,\,\mathscr{W}\,] \oplus \big(\mathscr{V} \lrcorner\, \mathsf{d}g - \mathscr{W} \lrcorner\, \mathsf{d}g + \mathscr{V} \lrcorner\, \mathscr{W} \lrcorner\, \Omega\big)\,;$$

- the **ANCHOR MAP** $\alpha_{\mathsf{TP}} \equiv \mathrm{pr}_{\mathsf{TP}} : \mathsf{E}^{(1,0)}\mathsf{P} \to \mathsf{TP}$.

Observation: The bracket closes on

$$\text{HAMILTONIAN SECTIONS} \qquad \mathscr{H} \oplus h =: \mathfrak{X}_h \in \ker\big(\mathscr{V} \oplus f \mapsto \delta f + \mathscr{V} \lrcorner\, \Omega\big)\,,$$

$$[\mathfrak{X}_{h_1}, \mathfrak{X}_{h_2}]^\Omega_{\mathrm{V}} = \mathfrak{X}_{\{\,h_1\,,\,h_2\,\}_\Omega}\,.$$

IV.1. Reminder on the KGST formalism & geometric (pre)quantisation

To a theory given in terms of an action functional ($D = \dim \mathscr{M}$)

$$S[\phi^A] = \int_{\mathscr{M}} \mathsf{d}^D x\, \mathcal{L}(x^\mu, \phi^A, \xi^B_\nu)|_{\xi^B_\nu = \partial_\nu \phi^B}\,, \qquad \mathsf{d}^D x = \mathsf{d}x^1 \wedge \mathsf{d}x^2 \wedge \cdots \wedge \mathsf{d}x^D$$

on sections $(\phi^A)^{A \in \overline{1,N}}$ of the **CONFIGURATION BUNDLE** $\pi_{\mathscr{F}} : \mathscr{F} \to \mathscr{M}$, we associate the **CARTAN FORM** on the first-jet bundle $J^1\mathscr{F} \to \mathscr{M}$

$$\Theta(x^\mu, \phi^A, \xi^B_\nu) = \big(\mathcal{L} - \xi^C_\lambda \tfrac{\delta \mathcal{L}}{\delta \xi^C_\lambda}\big)(x^\mu, \phi^A, \xi^B_\nu)\, \mathsf{d}^D x + \tfrac{\delta \mathcal{L}}{\delta \xi^C_\lambda}(x^\mu, \phi^A, \xi^B_\nu)\, \delta\phi^C \wedge \big(\partial_\lambda \lrcorner\, \mathsf{d}^D x\big)\,.$$

The latter has the all-important properties:

(i) the PLA for the functional

$$S_\Theta[\Psi] := \int_{\mathscr{M}} \Psi^* \Theta\,, \qquad \Psi \in \Gamma(J^1\mathscr{F})$$

yields the Euler–Lagrange equations of S;

(ii) upon defining a functional

$$S_{12}[\Psi_{\rm cl.}] := \int_{\mathscr{M}_{12}} \big(\Psi_{\rm cl.}|_{\mathscr{M}_{12}}\big)^* \Theta\,,$$

for a region $\mathscr{M}_{12} \subset \mathscr{M}$ cobounded by two homotopic Cauchy surfaces $\mathscr{C}_1$ and $\mathscr{C}_2$, we readily establish

$$\delta S_{12}[\Psi_{\rm cl.}] = \Xi_{\mathscr{C}_2}[\Psi_{\rm cl.}] - \Xi_{\mathscr{C}_1}[\Psi_{\rm cl.}]\,,$$

and so Θ canonically defines a closed 2-form

$$\Omega[\Psi_{\rm cl.}] := \delta \Xi_{\mathscr{C}}[\Psi_{\rm cl.}]\,, \qquad \mathscr{C} \in [\mathscr{C}_1]_{\rm hom.}$$

on the space $\mathsf{P}_{([\mathscr{C}_1]_{\rm hom.})}$ of extremal sections of $J^1\mathscr{F}$, i.e. also a symplectic structure on the phase space $\overline{\mathsf{P}}_{([\mathscr{C}_1]_{\rm hom.})}$ of the field theory.

Let, next, $\mathbb{C} \hookrightarrow \mathcal{L} \xrightarrow{\pi_{\mathcal{L}}} \overline{\mathsf{P}}$ be a line bundle with connection $\nabla_{\mathcal{L}}$ of curvature $\pi_{\mathcal{L}}^*\Omega$ and, for a choice $\{\mathcal{O}_i\}_{i\in\mathscr{I}}$ of an open cover of $\overline{\mathsf{P}}$, fix local data $(\theta_i, \gamma_{ij}) \in A^1_{\overline{\mathsf{P}}}$ of $\mathcal{L}$, so that

$$D_{(1)}(\theta_i, \gamma_{ij}) = (\Omega|_{\mathcal{O}_i}, 0, 1)$$

PREQUANTISATION assigns to every $h \in C^\infty(\overline{\mathsf{P}})$ a collection $\widehat{O}_h = (\widehat{h}_i)_{i\in\mathscr{I}}$ of local (linear) operators on $\Gamma(\mathcal{L})$,

$$\widehat{h}_i := -\mathsf{i}\mathscr{L}_{\mathscr{X}_h} - \mathscr{X}_h \lrcorner\, \theta_i + h|_{\mathcal{O}_i} \quad \text{satisfying} \quad \left[\widehat{O}_{h_1}, \widehat{O}_{h_2}\right] = -\mathsf{i}\,\widehat{O}_{\{h_1,h_2\}_\Omega}\,.$$

IV.2. Prequantisation via transgression

Given a choice $\mathcal{O}_M = \{\mathcal{O}_i^M\}_{i\in\mathscr{I}_{\mathscr{M}}}$ of a good open cover of M, consider the non-empty open sets

$$\mathcal{O}_{\mathfrak{i}} = \{\ X \in \mathsf{L}M \quad | \quad \forall_{e,v\in\triangle(\mathbb{S}^1)}\ :\ X(e) \subset \mathcal{O}_{i_e}^M \quad \wedge \quad X(v) \in \mathcal{O}_{i_v}^M\ \}\,,$$

with the index $\mathfrak{i}$ given by a pair $\big(\triangle_{\mathbb{S}^1},\phi\big)$ consisting of a choice $\triangle_{\mathbb{S}^1}$ of the triangulation of $\mathbb{S}^1$, with its edges e and vertices v, and a choice $\phi\ :\ \triangle_{\mathbb{S}^1} \to \mathscr{I}_M\ :\ f \mapsto i_f$ of the assignment of indices of $\mathcal{O}_M$ to elements of $\triangle_{\mathbb{S}^1}$. By varying these two choices arbitrarily, all of $\mathsf{L}M$ is covered, thus yielding an **OPEN COVER** $\mathcal{O}_{\mathsf{L}M} = \{\mathcal{O}_{\mathfrak{i}}\}_{\mathfrak{i}\in\mathscr{I}_{\mathsf{L}M}}$ of $\mathsf{L}M$.

The above choice of an open cover of the configuration space of the σ-model, together with the corresponding choice of local data for $\mathcal{G}$, is the basis of a constructive proof of the following remarkable

Th$^{\underline{\mathrm{m}}}$: [Gawędzki '86] An abelian (bundle) gerbe $\mathcal{G}$ over a differentiable manifold M with connection of curvature $\mathrm{H} \in Z^3(M, 2\pi\mathbb{Z})$ canonically induces a line bundle $\mathbb{C} \hookrightarrow \mathcal{L}_{\mathcal{G}} \to \mathsf{L}M$, termed the **TRANSGRESSION BUNDLE**, with connection $\nabla_{\mathcal{L}_{\mathcal{G}}}$ of curvature

$$\mathrm{curv}(\nabla_{\mathcal{L}_{\mathcal{G}}}) = \int_{\mathbb{S}^1} \mathrm{ev}^*\mathrm{H}\,,$$

and the assignment $\mathcal{G} \to \mathcal{L}_{\mathcal{G}}$ defines a cohomology map

$$\textsc{transgression map} \qquad \mathbb{H}^2\big(M, \mathcal{D}(2)^\bullet\big) \to \mathbb{H}^1\big(\mathsf{L}M, \mathcal{D}(1)^\bullet\big)\,.$$

Corollary: The torsion gerbe $\mathcal{G}$ over the target space M of the σ-model canonically determines

$$\textsc{pre-quantum bundle} \qquad \mathcal{L}_\sigma := \pi_{\mathsf{T}^*\mathsf{L}M}^*\mathcal{L}_{\mathcal{G}} \otimes \big(\mathsf{P}_\sigma \times \mathbb{C}\big) \to \mathsf{P}_\sigma$$

over the phase space $\mathsf{P}_\sigma = \mathsf{T}^*\mathsf{L}M \xrightarrow{\pi_{\mathsf{T}^*\mathsf{L}M}} \mathsf{L}M$ of the σ-model, in which the trivial tensor factor comes with the (global) connection 1-form $\theta_{\mathsf{T}^*\mathsf{L}M}$.

Explicitly, taking the induced open cover $\{\mathcal{O}^*_{\mathsf{i}}\}_{\mathsf{i}\in\mathscr{I}_{\mathsf{L}M}}$, $\mathcal{O}^*_{\mathsf{i}} := \pi^{-1}_{\mathsf{T}^*\mathsf{L}M}(\mathcal{O}_{\mathsf{i}})$ of P_σ, we find **LOCAL DATA** of $\mathcal{L}_\sigma$ in the form

$$\begin{aligned}\theta_{\sigma,\mathsf{i}} &:= \theta_{\mathsf{T}^*\mathsf{L}M}|_{\mathcal{O}^*_{\mathsf{i}}} + \pi^*_{\mathsf{T}^*\mathsf{L}M}E_i \in \Omega^1(\mathcal{O}^*_{\mathsf{i}})\,,\\ \gamma_{\sigma,\mathsf{ij}} &:= \pi^*_{\mathsf{T}^*\mathsf{L}M}G_{\mathsf{ij}} \in \mathrm{U}(1)_{\mathcal{O}^*_{\mathsf{ij}}}\,,\end{aligned}$$

where $(E_{\mathsf{i}}, G_{\mathsf{ij}}) \in A^1_{\mathsf{L}M}$ are local data of $\mathcal{L}_{\mathcal{G}}$ associated with $\mathcal{O}_{\mathsf{L}M}$,

$$\begin{aligned}E_{\mathsf{i}}[X] &:= -\sum_{e\in\triangle_{\mathbb{S}^1}}\int_e X^*_e B_{i_e} - \sum_{v\in\triangle_{\mathbb{S}^1}} X^* A_{i_{e_+(v)}i_{e_-(v)}}(v)\,,\\ G_{\mathsf{ij}}[X] &:= \prod_{\overline{e}\in\overline{\triangle}_{\mathbb{S}^1}} \mathrm{e}^{-\mathsf{i}\int_{\overline{e}} X^*_{\overline{e}} A_{i_{\overline{e}}j_{\overline{e}}}} \cdot \prod_{\overline{v}\in\overline{\triangle}_{\mathbb{S}^1}} X^*\big(g_{i_{\overline{e}_+(\overline{v})}i_{\overline{e}_-(\overline{v})}j_{\overline{e}_+(\overline{v})}} \cdot g^{-1}_{j_{\overline{e}_+(\overline{v})}j_{\overline{e}_-(\overline{v})}i_{\overline{e}_-(\overline{v})}}\big)(\overline{v})\,.\end{aligned}$$

These satisfy the standard cohomological identities

$$E_{\mathsf{j}} - E_{\mathsf{i}} = \mathsf{i}\delta\log G_{\mathsf{ij}}\,, \qquad\qquad G_{\mathsf{jk}}\cdot G^{-1}_{\mathsf{ik}}\cdot G_{\mathsf{ij}} = 1\,,$$

and transform as

$$(E_{\mathsf{i}}, G_{\mathsf{ij}}) \mapsto (E_{\mathsf{i}}, G_{\mathsf{ij}}) + D_{(0)}(H_{\mathsf{i}})\,,$$

with

$$H_{\mathsf{i}}[X] = \prod_{e\in\triangle_{\mathbb{S}^1}} \mathrm{e}^{\mathsf{i}\int_e X^*_e \Pi_{i_e}} \cdot \prod_{v\in\triangle_{\mathbb{S}^1}} X^*\chi^{-1}_{i_{e_+(v)}i_{e_-(v)}}(v)$$

under "gauge transformations"

$$(B_i, A_{ij}, g_{ijk}) \mapsto (B_i, A_{ij}, g_{ijk}) + D_{(1)}(\Pi_i, \chi_{ij})\,.$$

Consider the symplectic space ($\mathsf{P} = \mathsf{T}^*\mathsf{L}M$, for now)

$$\left(\mathsf{P}_\sigma \times \mathsf{P}_\sigma, \Omega_\sigma^- := \mathrm{pr}_1^*\Omega_\sigma - \mathrm{pr}_2^*\Omega_\sigma\right), \qquad \mathrm{pr}_i \;:\; \mathsf{P}_\sigma \times \mathsf{P}_\sigma \to \mathsf{P}_\sigma \text{ canonical},$$

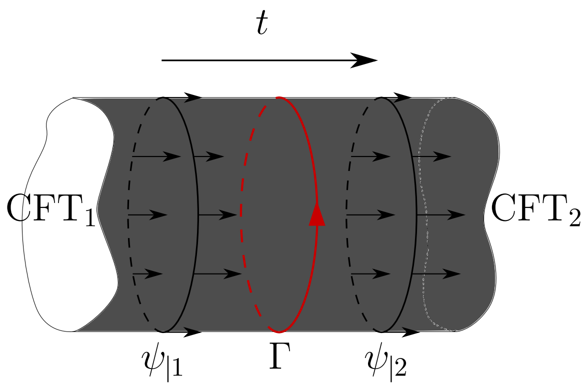
together with the pair of line bundles (with connection) $\mathcal{L}_{\sigma,i} := \mathrm{pr}_i^*\mathcal{L}_\sigma$.

Def$^{\underline{\mathrm{n}}}$: *A* ***PREQUANTUM DUALITY*** *of the σ-model is a pair* $(\mathfrak{J}_\sigma, \mathfrak{D}_\sigma)$ *composed of*

(i) *a graph* $\mathfrak{J}_\sigma \subset \mathsf{P}_\sigma \times \mathsf{P}_\sigma$, *isotropic wrt.* Ω_σ^- *and such that the difference* $\mathscr{H}_\sigma^- := \mathrm{pr}_1^*\mathscr{H}_\sigma - \mathrm{pr}_2^*\mathscr{H}_\sigma$ *of pullback hamiltonian densities* $\mathscr{H}_\sigma$ *of the σ-model satisfies* $\mathscr{H}_\sigma^-|_{\mathfrak{J}_\sigma} = 0$*;*

(ii) *an isomorphism* $\mathfrak{D}_\sigma \;:\; \mathcal{L}_{\sigma,1} \xrightarrow{\cong} \mathcal{L}_{\sigma,2}$.

World-sheet intuition: Consider an oriented closed time-like discontinuity contour Γ at $t = t_0$, or a **DEFECT LINE**. The limiting field configurations $\psi_{|1}(\varphi) := \lim_{\eta\to 0^-}(X, \mathsf{p})(t_0 + \eta, \varphi)$ and $\psi_{|2}(\varphi) := \lim_{\eta\to 0^+}(X, \mathsf{p})(t_0 + \eta, \varphi)$ define a (local) correspondence between states in P_σ.



Formalisation: Pass from C^1-maps X from Σ to M (with $(\mathrm{g}, \mathcal{G})$) to *patchwise* C^1-maps $X : \wp \to M$, $\wp \in \mathfrak{P}_\Sigma$ with extensions $X : \Gamma \to M \times M$

$$\lim_{\eta\to 0^-} X(t_0 + \eta, \varphi) = \mathrm{pr}_1 \circ X(\varphi), \qquad \lim_{\eta\to 0^+} X(t_0 + \eta, \varphi) = \mathrm{pr}_2 \circ X(\varphi)$$

and additional structure

$$\Phi \;:\; \mathrm{pr}_1^*\mathcal{G} \xrightarrow{\cong} \mathrm{pr}_2^*\mathcal{G} .$$

V.1. Bi-branes vs. dualities

Generalisation: A σ-model for patchwise C^1-maps $X : \wp \to M$ with discontinuities at a **DEFECT QUIVER** $\Gamma := \bigsqcup_{e\in\mathfrak{E}_\Gamma} \ell_e,\ \ell_e \cong \mathbb{S}^1$ embedded in $\Sigma = \bigsqcup_{\wp\in\mathfrak{P}_\Sigma} \wp$, and, for some $Q \xrightarrow{\iota_\alpha} M,\ \alpha \in \{1,2\}$ with ι_α smooth,

$$X\ :\ \wp \to M\,, \qquad\qquad X\ :\ \ell_i \to Q\,, \qquad \begin{cases} X_{|1} = \iota_1 \circ X \\ X_{|2} = \iota_2 \circ X \end{cases} .$$

Derivation:

(i) Choose $\mathcal{O}_M$ and $\triangle_\Sigma$ subordinate to it and such that $\triangle_\Sigma|_\Gamma = \triangle_\Gamma$.

$$S^{(0)}_{\rm top}[(X\,|\,\Gamma)] = -\mathsf{i} \sum_{p\in\mathfrak{P}_\Gamma} {\rm Hol}_{\mathcal{G}}(X_p)\,.$$

(ii) Independence of $S_{\rm top}$ of the choice of $(B_i, A_{ij}, g_{ijk}) \in A^2_M$ implies the need for $(P_i, K_{ij}) \in A^1_Q$ associated with a choice $\mathcal{O}_Q = \{\mathcal{O}^Q_i\}_{i\in\mathscr{I}_Q}$ and such that, for $\mathcal{O}_Q$ chosen so that the ι_α are covered by index maps $\phi_\alpha : \mathscr{I}_M \to \mathscr{I}_Q$ such that $\iota_\alpha(\mathcal{O}^M_i) \subset \mathcal{O}^Q_{\phi_\alpha(i)}$,

$$(P_i, K_{ij}) \mapsto (P_i, K_{ij}) + \iota_2^*(\Pi_{\phi_2(i)}, \chi_{\phi_2(i)\phi_2(j)}) - \iota_1^*(\Pi_{\phi_1(i)}, \chi_{\phi_1(i)\phi_1(j)})$$

whenever $(B_i, A_{ij}, g_{ijk}) \mapsto (B_i, A_{ij}, g_{ijk}) + D_{(1)}(\Pi_i, \chi_{ij})$. Then,

$$S^{(1)}_{\rm top}[(X\,|\,\Gamma)] = S^{(0)}_{\rm top}[(X\,|\,\Gamma)] + \sum_{e\in\triangle_\Gamma} \left(\int_e X_e^* P_{i_e} - \mathsf{i} \sum_{v\subset e} \varepsilon_{ev} \log X^* K_{i_e i_v}(v) \right).$$

N.B. For $\partial\Gamma = \emptyset$, we may further allow proper "gauge freedom" (or cohomological equivalences)

$$(P_i, K_{ij}) \mapsto (P_i, K_{ij}) - D_{(0)}(W_i)\,, \qquad (W_i) \in A^0_Q\,.$$

(iii) Invariance of $S_{\rm top}$ under changes of $i_\cdot : \triangle_\Gamma \to \mathscr{I}_Q$ calls for a global $\omega \in \Omega^2(Q)$ such that (for $\check\iota_\alpha = (\iota_\alpha, \phi_\alpha)$)

$$\check\iota_1^*(B_i, A_{ij}, g_{ijk}) - \iota_2^*(B_i, A_{ij}, g_{ijk}) + D_{(1)}(P_i, K_{ij}) = (\omega|_{\mathcal{O}^Q_i}, 0, 1)\,.$$

Recapitulation: To defect lines, we associate a $\mathcal{G}$**-BI-BRANE**

$$\mathcal{B} := \big(Q, \iota_\alpha, \omega, \Phi \mid \alpha \in \{1,2\}\big)\,, \qquad \iota_\alpha \;:\; Q \to M\,, \qquad \Phi \;:\; \iota_1^*\mathcal{G} \xrightarrow{\cong} \iota_2^*\mathcal{G} \otimes I_\omega\,,$$

with ι_α smooth and Φ a $(\iota_1^*\mathcal{G}, \iota_2^*\mathcal{G})$**-BI-MODULE**, i.e. a distinguished

Def$^{\underline{\rm n}}$: [Carey, Mickelsson & Murray '97] A ***STABLE ISOMORPHISM*** *between abelian bundle gerbes with connection* $(Y^a M, B^a, L^a, \mu^a)$, $a \in \{1,2\}$ *(over the same manifold* M*) is a pair* $\Phi = (E, \alpha)$ *composed of*

- *a* ***line bundle*** $\mathbb{C} \hookrightarrow E \to Y^1M \times_M Y^2M =: Y^{1,2}M$ *with connection* ∇_E *of curvature*

$$\mathrm{curv}(\nabla_E) := \mathrm{pr}_2^* B^2 - \mathrm{pr}_1^* B^1\,;$$

- *an isomorphism*

$$\alpha \;:\; (L^1)_{1,3} \otimes E_{3,4} \xrightarrow{\cong} E_{1,2} \otimes (L^2)_{2,4}$$

over $(Y^{1,2})^{[2]}M$*, compatible with the* μ_i *in the sense of*

$$\begin{array}{ccccc}
 & & (L^1)_{1,3} \otimes E_{3,4} \otimes (L^2)_{4,6} & & \\
 & \nearrow^{\mathrm{id}\otimes\alpha_{3,4,5,6}} & & \searrow^{\alpha_{1,2,3,4}\otimes\mathrm{id}} & \\
(L^1)_{1,3} \otimes (L^1)_{3,5} \otimes E_{5,6} & & \circlearrowright & & E_{1,2} \otimes (L^2)_{2,4} \otimes (L^2)_{4,6}\,. \\
 & \searrow_{(\mu_1)_{1,3,5}\otimes\mathrm{id}} & & \swarrow_{\mathrm{id}\otimes(\mu_2)_{2,4,6}} & \\
 & (L^1)_{1,5} \otimes E_{5,6} & \xrightarrow[\alpha_{1,2,5,6}]{} & E_{1,2} \otimes (L^2)_{2,6} &
\end{array}$$

In the newly defined two-dimensional field theory, we still need to impose

DEFECT GLUING CONDITION $\quad \Omega^1(Q) \ni \mathsf{p}_{|1} \circ \iota_{1\,*} - \mathsf{p}_{|2} \circ \iota_{2\,*} - X_*\widehat{t} \lrcorner\, \omega(X) \stackrel{!}{=} 0\,,$

where $\widehat{t} \in \mathsf{T}_p\Gamma \subset \mathsf{T}_p\Sigma$ and $\widehat{n} = \gamma^{-1}\big(\widehat{t} \lrcorner \mathrm{Vol}(\Sigma, \gamma)\big)$, and where $\mathsf{p}_{|\alpha} = \mathrm{g}(X_{|\alpha})(X_{|\alpha\,*}\widehat{n}, \cdot)$.

Returning to σ-model dualities, we find

Th$^{\underline{\mathrm{m}}}$: [rrS '10] A $\mathcal{G}$-bi-brane $(Q, \iota_\alpha, \omega, \Phi \mid \alpha \in \{1,2\})$, in conjunction with the DGC canonically define a prequantum duality of the σ-model iff

(i) the induced loop maps $\widetilde{\iota}_\alpha \ : \ \mathsf{L}Q \to \mathsf{L}M \ : \ X \mapsto \iota_\alpha \circ X, \ \alpha \in \{1,2\}$ are surjective submersions (onto connected components of M);

(ii) the network-field configuration $(X \,|\, \Gamma)$ is **TOPOLOGICAL**, i.e. the energy-momentum tensor

$$T^{ab} := \tfrac{2}{\sqrt{|\det\gamma|}} \tfrac{\delta S_\sigma}{\delta \gamma_{ab}}$$

is continuous across Γ ($\Longleftarrow$ DGC);

(iii) extra conditions (technical) are satisfied.

Idea of proof: Consider the subspace

$$\mathfrak{I}_\sigma(\mathcal{B}) := \left\{ \ (\psi_1, \psi_2) \in \mathsf{P}_\sigma \times \mathsf{P}_\sigma \quad \middle| \quad \begin{cases} (X_1, X_2) \in (\iota_1 \times \iota_2)(\mathsf{L}Q) \\ \exists_{X \in (\iota_1 \times \iota_2)^{-1}\{(X_1,X_2)\}} \ : \ \mathrm{DGC}_\omega(\psi_1, \psi_2, X) = 0 \end{cases} \right\}.$$

We readily establish the following:

(i) $\mathsf{T}\mathfrak{I}_\sigma(\mathcal{B}) \subset \mathsf{T}(\mathsf{P}_\sigma \times \mathsf{P}_\sigma)$ is isotropic wrt. Ω_σ^-;

(ii) explicit expressions for local data of the isomorphism $\mathfrak{D}_\sigma(\mathcal{B})$ can be found – these are determined by local data of Φ;

(iii) the remaining conditions ensure that $\mathfrak{I}_\sigma(\mathcal{B})$ be a graph of a symplectomorphism preserving $\mathscr{H}_\sigma \sim T_{\not{\gamma}}$.

Amidst bi-brane dualities, we find the familiar "geometric" dualities, or **SYMMETRIES**, with

$$\mathcal{B} = \big((\mathrm{id}_M \times F)(M), \iota_\alpha = \mathrm{pr}_\alpha, \omega = 0, \mathcal{G} \xrightarrow{\Phi} F^*\mathcal{G} \big), \qquad F \in \mathrm{Iso}(M, g)\,.$$

V.2. A "stringy" example: the T-dual pair

Let the target space contain, as a disjoint component, a torus bundle

$$\mathbb{T}^N \hookrightarrow M \xrightarrow{\pi_M} B \qquad \text{with} \quad \Theta^A \otimes e_A \in \Omega^1(M) \otimes \mathbb{R}^N\,,$$

with a metric

$$\mathrm{g} = \pi_M^* \gamma + \left(\pi_M^* \mathrm{h}_{AB}\right) \Theta^A \otimes \Theta^B\,, \qquad (\mathrm{h}_{AB}) \ \text{invertible}$$

and a gerbe $\mathcal{G}$ of a $\mathbb{T}^N$-invariant curvature

Observation: $\mathcal{G}$ induces another torus bundle,

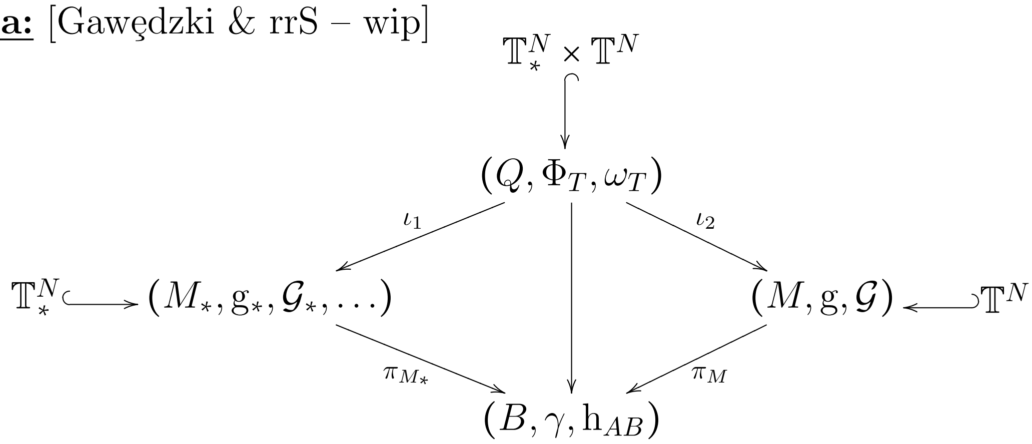
$$\mathbb{T}_*^N \hookrightarrow Q \xrightarrow{\iota_2} M$$

Idea: Equivariantly lift the $\mathbb{T}^N$-action to Q, to obtain

$$\mathbb{T}_*^N \times \mathbb{T}^N \hookrightarrow Q \to B\,,$$

and, subsequently, endow $\iota_2^* \mathcal{G}$ with a $\mathbb{T}^N$-equivariant structure.

Idea: [Gawędzki & rrS – wip]



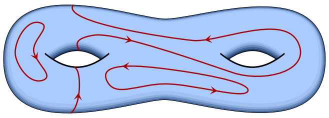
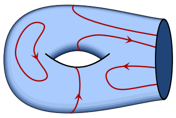
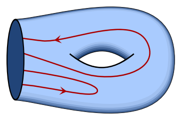
with

$$\Phi_T \ : \ \iota_1^* \mathcal{G}_* \xrightarrow{\cong} \iota_2^* \mathcal{G} \otimes I_{\omega_T}$$

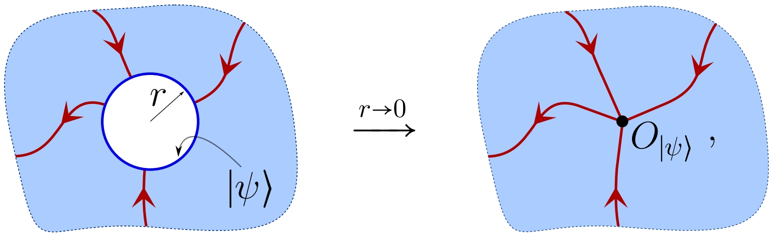
N.B.: For (M, M_*) Calabi–Yau manifolds, we can thus reproduce **MIRROR PAIRS**.

V.3. The 2-category for generic world-sheets

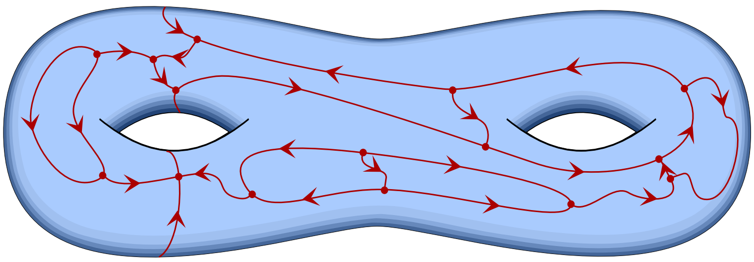
Heuresis: Factorisation of path integrals,

$\int \mathscr{D}X$  $= \sum_\psi \int_{\psi|_{\partial\Sigma_1}=\psi} \mathscr{D}X$  $|\psi\rangle\langle\psi| \int_{\psi|_{\partial\Sigma_2}=\psi} \mathscr{D}X$ 

and state-field correspondence,



lead us to consider world-sheets with arbitrary embedded defect quivers



$$X\;:\;\wp\to M\,,\quad \wp\in\mathfrak{P}_\Sigma\,,\qquad X\;:\;\ell\to Q\,,\quad \ell\in\mathfrak{E}_\Gamma\,,\qquad X\;:\;\{\upsilon_n\}\to T_n\,,\quad \upsilon_n\in\mathfrak{V}_\Gamma^{(n)}\,,$$

for which we find a **FIELD SPACE** $\mathscr{F} = M\sqcup Q\sqcup\bigsqcup_{n\in\mathbb{N}_{\geq 3}} T_n$ and a **BACKGROUND** $\mathfrak{B}$ with components

$$\text{TARGET}\quad \mathcal{M}=(M,\mathrm{g},\mathcal{G})\,,\qquad \mathcal{G}\text{-BI-BRANE}\quad \mathcal{B}=(Q,\iota_\alpha,\omega,\Phi\mid\alpha\in\{1,2\})\,,$$

$$(\mathcal{G},\mathcal{B})\text{-INTER-BI-BRANE}\quad \mathcal{I}=\Big(T_n,\big(\varepsilon_n^{k,k+1},\pi_n^{k,k+1}\mid k\in\overline{1,n}\big),\varphi_n\mid n\in\mathbb{N}_{\geq 3}\Big)\,,$$

where $\varphi_n\;:\;\circ_{k=1}^n\,\pi_n^{k,k+1\,*}\Phi^{\varepsilon_n^{k,k+1}}\xRightarrow{\cong}\mathrm{id}_{\pi_n^{1,2\,*}\iota_1^{\varepsilon_n^{1,2}\,*}\mathcal{G}}$ is a distinguished 2-isomorphism.

Def$^{\underline{n}}$: [Stevenson '00] Let $\Phi^A = (E^A, \alpha^A)$, $A \in \{1,2\}$ be a pair of stable isomorphisms between the two abelian bundle gerbes with connection $(Y^a M, B^a, L^a, \mu^a)$, $a \in \{1,2\}$ over a differentiable manifold M. A **2-ISOMORPHISM** $\varphi = (\beta)$ between Φ^1 and Φ^2, denoted as $\varphi : \Phi^1 \Longrightarrow \Phi^2$, is an isomorphism

$$\beta \;:\; E^1 \xrightarrow{\cong} E^2$$

that intertwines the α^A in the sense of

$$\begin{array}{ccc} L_{1,3} \otimes E^1_{3,4} & \xrightarrow{\alpha^1} & E^1_{1,2} \otimes L^2_{2,4} \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\beta_{3,4}} & \circlearrowright & \Big\downarrow{\scriptstyle \beta_{1,2}\otimes\mathrm{id}} \\ L_{1,3} \otimes E^2_{3,4} & \xrightarrow[\alpha^2]{} & E^2_{1,2} \otimes L^2_{2,4} \end{array} .$$

Statement: [Stevenson '00] One has a natural notion of composition of 1-morphisms and 2-isomorphisms, alongside their respective tensor products. This structure gives rise to the (monoidal) **2-CATEGORY OF ABELIAN BUNDLE GERBES with CONNECTION** over a differentiable manifold $\mathscr{M}$, denoted as $\mathfrak{BGrb}^\nabla(\mathscr{M})$.

Conclusion: Thus, $\mathcal{G}, \Phi$ and the φ_n are distinguished 0-cells, 1-cells and 2-cells, respectively, of the 2-category $\mathfrak{BGrb}^\nabla(\mathscr{F})$.

Implications:

(i) A definition of the σ-model for arbitrary network-field configurations:

$$S_\sigma[(X\,|\,\Gamma);\gamma] = S_{\mathrm{kin}}[X;\gamma] - \mathsf{i}\,\log \mathrm{Hol}_{\mathcal{G},\Phi,\varphi_n}(X\,|\,\Gamma)\,.$$

(ii) $(\mathcal{G},\mathcal{B})$ canonically induce a pre-quantum bundle for both the untwisted and the twisted sector of the σ-model, and $\mathcal{J}$ gives rise to a canonical picture of the (twisted-)loop fusion.

(iii) A classification of inequivalent $\mathcal{G}$-bi-branes ($\sim$ dualities) for given $(Q, \iota_\alpha, \omega)$ by $H^1\big(Q, \mathrm{U}(1)\big)$, and that of $(\mathcal{G},\mathcal{B})$-inter-bi-branes ($\sim$ intertwiners) for given $(T_n, \pi_n^{k,k+1})$ by $H^0\big(T_n, \mathrm{U}(1)\big) \cong \mathrm{U}(1)^{|\pi_0(T_n)|}$.

VI. Rigid symmetries of the σ-model

We want to understand the geometry of those field transformations $X \mapsto f \circ X,\ f \in \mathscr{D}iff(\mathscr{F})$ which preserve S_σ and hence descend the space of classical field configurations.

Consider a flow $\psi_\cdot : [-1,1] \times \mathscr{F} \to \mathscr{F}$ of a vector field $\mathscr{K}$ on $\mathscr{F}$, with restrictions $\mathscr{K}|_{\mathscr{M}} \equiv {}^{\mathscr{M}}\mathscr{K},\ \mathscr{M} \in \{M, Q, T_n\}$ aligned as

$$\pi_{n\,*}^{k,k+1}\big({}^{T_n}\mathscr{K}\big) = {}^{Q}\mathscr{K}\big|_{\pi_n^{k,k+1}(T_n)}\,, \qquad \iota_{\alpha\,*}\big({}^{Q}\mathscr{K}\big) = {}^{M}\mathscr{K}\big|_{\iota_\alpha(Q)}\,.$$

Since $\mathrm{Hol}_{\mathcal{G},\Phi,\varphi_n}$ is a generalised differential character, we find

$$\tfrac{\mathsf{d}}{\mathsf{d}t}\big|_{t=0} S_\sigma[(\psi_t \circ X\,|\,\Gamma);\gamma] = -\tfrac{1}{2}\int_\Sigma \big(\mathscr{L}_{{}^M\mathscr{K}}\,\mathrm{g}\big)_X(\mathsf{d}X \overset{\wedge}{,} \star_\gamma \mathsf{d}X) + \int_\Sigma X^*({}^M\mathscr{K} \lrcorner \mathrm{H}) + \int_\Gamma X_\Gamma^*({}^Q\mathscr{K} \lrcorner \omega)\,,$$

and so symmetries correspond to those globally smooth sections

$$\mathfrak{K} = \big({}^M\mathscr{K} \oplus \kappa, {}^Q\mathscr{K} \oplus k, {}^{T_n}\mathscr{K} \oplus c\big) \in \Gamma_\sigma(\mathcal{E}\mathscr{F})$$

of the **GENERALISED TANGENT SHEAVES**

$$\mathcal{E}\mathscr{F} \equiv \mathcal{E}^{(1,1)}M \sqcup \mathcal{E}^{(1,0)}Q \sqcup \bigsqcup_{n\in\mathbb{N}_{\geq 3}} \mathcal{E}^{(1,-1)}T_n \to M \sqcup Q \sqcup \bigsqcup_{n\in\mathbb{N}_{\geq 3}} T_n\,, \qquad \mathcal{E}^{(1,q)}\mathscr{M} := \mathcal{T}\mathscr{M} \oplus \mathcal{T}_q^*\mathscr{M}\,,$$

written in terms of sheaf components $\mathcal{T}_{-1}^*\mathscr{M} := \underline{\mathbb{R}}$ and $\mathcal{T}_{p\geq 0}^*\mathscr{M} := \underline{\Omega}^p(\mathscr{M})$ of

$$\mathcal{T}_\bullet^*\mathscr{M}\ :\ 0 \to \mathcal{T}_{-1}^*\mathscr{M} \overset{\mathsf{d}^{(-1)}=\mathrm{id}}{\hookrightarrow} \mathcal{T}_0^*\mathscr{M} \xrightarrow{\mathsf{d}^{(0)}=\mathsf{d}} \mathcal{T}_1^*\mathscr{M} \xrightarrow{\mathsf{d}^{(1)}=\mathsf{d}} \cdots\,,$$

that are Killing for g and satisfy the section descent relations

$$\mathsf{d}_{\mathrm{H}}^{(1)}\big({}^M\mathscr{K} \oplus \kappa\big) = 0\,, \qquad \mathsf{d}_\omega^{(0)}\big({}^Q\mathscr{K} \oplus k\big) = -\Delta_Q\kappa\,, \qquad \mathsf{d}_0^{(-1)}\big({}^{T_n}\mathscr{K} \oplus c\big) = -\Delta_{T_n}k$$

wrt. $\Delta_Q := \iota_2^* - \iota_1^*,\ \Delta_{T_n} := \sum_{k=1}^n \varepsilon_n^{k,k+1}\,\pi_n^{k,k+1\,*}$ and $\mathsf{d}_{\mathrm{H}_{(q+2)}}^{(q)}(\mathscr{V} \oplus \upsilon) := \mathsf{d}^{(q)}\upsilon + \mathscr{V} \lrcorner \mathrm{H}_{(q+2)}$.

On $\Gamma(\mathcal{E}\mathscr{F})$, there exists a canonical **ANCHOR (MAP)** $\alpha_{\mathcal{T}\mathscr{F}} : \mathcal{E}\mathscr{F} \to \mathcal{T}\mathscr{F}$, a **CANONICAL CONTRACTION** with restrictions

$$(\cdot,\cdot)_\lrcorner \;:\; \Gamma\big(\mathcal{E}^{(1,1)}\mathscr{M}\big)^{\times 2} \to \Gamma\big(\mathcal{T}_0^*\mathscr{M}\big) \;:\; (\mathscr{V} \oplus \upsilon, \mathscr{W} \oplus \varpi) \mapsto \tfrac{1}{2}\,(\mathscr{V} \lrcorner \varpi + \mathscr{W} \lrcorner \upsilon)\,,$$

$$(\cdot,\cdot)_\lrcorner \;:\; \Gamma\big(\mathcal{E}^{(1,m<1)}\mathscr{M}\big)^{\times 2} \to \Gamma\big(\mathcal{T}_{-1}^*\mathscr{M}\big) \;:\; (\mathscr{V} \oplus \upsilon, \mathscr{W} \oplus \varpi) \mapsto 0\,.$$

and an essentially unique $(\mathrm{H},\omega;\Delta_Q)$**-TWISTED BRACKET** such that

$$[\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)} \;:\; \Gamma_\sigma(\mathcal{E}\mathscr{F})^{\times 2} \to \Gamma_\sigma(\mathcal{E}\mathscr{F})\,, \qquad \alpha_{\mathcal{T}\mathscr{F}} \circ [\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)} = [\cdot,\cdot] \circ \alpha_{\mathcal{T}\mathscr{F}}\,.$$

Given $\mathfrak{V}_i = \big({}^M\mathscr{V}_i \oplus \upsilon_i, {}^Q\mathscr{V}_i \oplus \xi_i, {}^{T_n}\mathscr{V} \oplus c_i\big),\ i \in \{1,2\}$, it restricts as

$$[\![\mathfrak{V}_1,\mathfrak{V}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_M = [{}^M\mathscr{V}_1, {}^M\mathscr{V}_2] \oplus \big(\mathscr{L}_{{}^M\mathscr{V}_1}\upsilon_2 - \mathscr{L}_{{}^M\mathscr{V}_2}\upsilon_1 - \tfrac{1}{2}\,\mathsf{d}({}^M\mathscr{V}_1 \lrcorner \upsilon_2 - {}^M\mathscr{V}_2 \lrcorner \upsilon_1) + {}^M\mathscr{V}_1 \lrcorner {}^M\mathscr{V}_2 \lrcorner \mathrm{H}\big)\,,$$

$$[\![\mathfrak{V}_1,\mathfrak{V}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_Q = [{}^Q\mathscr{V}_1, {}^Q\mathscr{V}_2] \oplus \big({}^Q\mathscr{V}_1 \lrcorner \mathsf{d}\xi_2 - {}^Q\mathscr{V}_2 \lrcorner \mathsf{d}\xi_1 + {}^Q\mathscr{V}_1 \lrcorner {}^Q\mathscr{V}_2 \lrcorner \omega + \tfrac{1}{2}\,({}^Q\mathscr{V}_1 \lrcorner \Delta_Q\upsilon_2 - {}^Q\mathscr{V}_2 \lrcorner \Delta_Q\upsilon_1)\big)\,,$$

$$[\![\mathfrak{V}_1,\mathfrak{V}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_{T_n} = [{}^{T_n}\mathscr{V}_1, {}^{T_n}\mathscr{V}_2] \oplus 0\,.$$

We thus obtain a $(\mathrm{H},\omega;\Delta_Q)$**-TWISTED BRACKET STRUCTURE**

$$\mathfrak{M}^{(\mathrm{H},\omega;\Delta_Q)}(\mathscr{F}) = \big(\mathcal{E}\mathscr{F}, [\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)}, (\cdot,\cdot)_\lrcorner, \alpha_{\mathcal{T}\mathscr{F}}\big)\,.$$

N.B. $\mathfrak{M}^{(\mathrm{H},\omega;\Delta_Q)}(\mathscr{F})|_M$ is the canonical Courant algebroid with the bracket twisted by H à la Ševera–Weinstein. The algebroid is central to the Gaultieri–Hitchin definition of **GENERALISED GEOMETRY**. It can be related, via the Hitchin morphism, to a Courant algebroid with an untwisted bracket but for $\mathcal{E}^{(1,1)}_{\mathcal{G}}M$. A similar phenomenon occurs for $\mathfrak{M}^{(\mathrm{H},\omega;\Delta_Q)}(\mathscr{F})$.

Canonical interpretation: We have a Noether mapping

$$\Gamma_\sigma(\mathcal{E}\mathscr{F}) \to \Gamma\big(\mathcal{E}^{(1,0)}\mathsf{P}_{\sigma,\ldots}\big) \cap \ker\delta^{(1)}_{\Omega_{\sigma,\ldots}} \;:\; \mathfrak{K} \mapsto \widetilde{\mathfrak{K}} \qquad \text{HAMILTONIAN SECTION.}$$

Prop$^{\underline{\text{n}}}$: [rrS '10] $\qquad \big[\widetilde{\mathfrak{K}}_1, \widetilde{\mathfrak{K}}_2\big]^{\Omega_{\sigma,\ldots}}_{\mathrm{V}} = [\![\widetilde{\mathfrak{K}_1, \mathfrak{K}_2}]\!]^{(\mathrm{H},\omega;\Delta_Q)}.$

VII. The Gauge Principle

The next logical step consists in understanding the mechanism of gauging for rigid symmetries G_σ of the σ-model.

Motivation:

(i) **The topography of the theory space**: Working out systematic tools for constructing new σ-models, with field spaces given by G_σ-cosets of the original ones.

(ii) **Stringy dualities**: Obtaining ancillary tools for a rigorous study of *bona fide* dualities of the σ-model (e.g., the mirror symmetry for Calabi–Yau field spaces).

(iii) **"Non-geometry"**: Getting hints as to possible extensions of the smooth category $\mathfrak{Man}$ via stringy-duality quotients.

Challenges:

(i) G_σ**-equivariance**: Lifting the geometric symmetry from $\mathscr{F}$ to $\mathfrak{B}$.

(ii) **A principal extension**: In the case of continuous symmetries, the introduction of the world-sheet G-gauge field and coupling them to $X^*\mathfrak{B}$, in particular in the topologically non-trivial setting.

(iii) **The coset construction**: Understanding the descent $\mathfrak{B} \to \mathfrak{B}/\mathrm{G}_\sigma$ in purely geometric terms.

VII.1. Insights from the study of the next-to-trivial case

Observation: $\mathfrak{g}_\sigma := \alpha_{\mathcal{T}\mathscr{F}}\big(\Gamma_\sigma(\mathcal{E}\mathscr{F})\big)$ is a Lie subalgebra of $\Gamma(\mathcal{T}\mathscr{F})$.

Let $\mathscr{K}_a,\ a \in \overline{1,D}$ be generators of $\mathfrak{g}_\sigma \equiv \mathrm{LieG}_\sigma$, satisfying

$$\textsc{structure relations}\qquad \big[\mathscr{K}_a, \mathscr{K}_b\big] = f_{abc}\,\mathscr{K}_c\,, \qquad f_{abc} \in \mathbb{R}\,.$$

Complete the $\mathscr{K}_a$ to the respective

$$\mathfrak{K}_a = \big({}^M\mathscr{K}_a \oplus \kappa_a\big) \sqcup \big({}^Q\mathscr{K}_a \oplus k_a\big) \sqcup \big({}^{T_n}\mathscr{K}_a \oplus 0\big) \in \Gamma_\sigma(\mathcal{E}\mathscr{F})\,.$$

Gauging G_σ calls for the introduction of

$$\textsc{principal } \mathrm{G}_\sigma\textsc{-bundle}\qquad \mathrm{G}_\sigma \hookrightarrow \mathsf{P} \xrightarrow{\pi_{\mathsf{P}}} \Sigma \qquad \text{with} \qquad r\ :\ \mathsf{P}\times\mathrm{G}_\sigma \to \mathsf{P}\ :\ (p,g) \mapsto p.g$$

$$\textsc{principal } \mathrm{G}_\sigma\textsc{-connection}\qquad \mathcal{A} \in \Omega^1(\mathsf{P}) \otimes \mathrm{g}_\sigma \qquad \text{s.t.} \qquad \begin{cases} {}^{\mathsf{P}}\mathscr{K}_a \lrcorner \mathcal{A} = t_a \\ \mathcal{A}(p.g^{-1}) = \mathrm{Ad}_g \mathcal{A}(p) \end{cases}.$$

Consider, first, a G_σ-invariant top.-trivial background

$$\mathrm{H} = \mathsf{d}B\,, \qquad \Delta_Q B + \omega = \mathsf{d}P\,, \qquad \Delta_{T_n} P = -\mathsf{i}\,\mathsf{d}\log f_n\,,$$

$$\mathscr{L}_{{}^M\mathscr{K}_a} B = 0 = \mathscr{L}_{{}^Q\mathscr{K}_a} P = 0 = \mathscr{L}_{{}^{T_n}\mathscr{K}_a} f_n\,, \qquad \text{with} \qquad \mathfrak{K}_a = \big(\mathrm{e}^{\mathrm{B}} \sqcup \mathrm{e}^{P}\big)\big(\mathscr{K}_a \oplus 0\big)\,,$$

and a top.-trivial principal G_σ-bundle, $\mathsf{P} = \Sigma \times \mathrm{G}_\sigma$, with $\mathrm{A} \in \Omega^1(\Sigma) \otimes \mathfrak{g}_\sigma$.

A particle-physicist's intuition:

$$\textsc{minimal coupling}\qquad \mathsf{d}X^\mu(\sigma) \mapsto \mathrm{e}^{-\mathrm{A}^a(\sigma)\,\mathscr{K}_a(X(\sigma))\lrcorner}\,\mathsf{d}X^\mu(\sigma) \equiv D_{\mathrm{A}}X^\mu(\sigma)\,,$$

$$D_{\mathrm{A}}(g.X)^\mu = \tfrac{\partial(g.X)^\mu}{\partial X^\nu}\,D_{\mathrm{A}}X^\nu\,.$$

Upshot: Upon simple rearrangement, we obtain

EXTENDED FIELD SPACE

$$M \mapsto \Sigma \smallsetminus \Gamma \times M\,,\qquad Q \mapsto \Gamma \smallsetminus \mathfrak{V}_\Gamma \times Q\,,\qquad T_n \mapsto \mathfrak{V}_\Gamma^{(n)} \times T_n$$

EXTENDED BACKGROUND

$$\mathrm{g} \mapsto \mathrm{pr}_2^*\mathrm{g}\,,\qquad \mathcal{G} \mapsto \mathrm{pr}_2^*\mathcal{G} \otimes I_{\rho_{\mathrm{A}}}$$

$$\Phi \mapsto \mathrm{pr}_2^*\Phi \otimes J_{\lambda_{\mathrm{A}}}\,,$$

$$\varphi_n \mapsto \mathrm{pr}_2^*\varphi_n\,,$$

where

$$\rho_{\mathrm{A}} = \mathrm{pr}_2^*\kappa_a \wedge \mathrm{pr}_1^*\mathrm{A}^a - \tfrac{1}{2}\,\mathrm{pr}_2^*\big({}^{\scriptscriptstyle M}\mathscr{K}_a \lrcorner\, \kappa_b\big)\,\mathrm{pr}_1^*\big(\mathrm{A}^a \wedge \mathrm{A}^b\big)\,,\qquad \lambda_{\mathrm{A}} = -\mathrm{pr}_2^*k_a\,\mathrm{pr}_1^*\mathrm{A}^a\,.$$

Ansatz: For $\mathsf{P} = \Sigma \times \mathrm{G}_\sigma$ with $\mathrm{A} \in \Omega^1(\Sigma) \otimes \mathfrak{g}_\sigma$, we take

(i) S_{kin} – minimal coupling;

(ii) S_{top} – decorated-surface holonomy for an extended background

$$\big((\Sigma \smallsetminus \Gamma, \mathrm{pr}_2^*\mathrm{g}, \mathrm{pr}_2^*\mathcal{G} \otimes I_{\varsigma_{\mathrm{A}}}), (\Gamma \smallsetminus \mathfrak{V}_\Gamma, \mathrm{pr}_2^*\Phi \otimes J_{\mu_{\mathrm{A}}}), (\mathfrak{V}_\Gamma^{(n)} \times T_n, \mathrm{pr}_2^*\varphi_n)\big)\,.$$

Upshot: Infinitesimal-invariance analysis yields

$$\varsigma_{\mathrm{A}} \stackrel{!}{=} \rho_{\mathrm{A}}\,,\qquad \mu_{\mathrm{A}} \stackrel{!}{=} \lambda_{\mathrm{A}}\,,\qquad \text{with the } \mathfrak{K}_a \text{ subject to}$$

GAUGEABILITY CONSTRAINTS

$$\begin{cases} \mathscr{L}_{{}^{\scriptscriptstyle M}\mathscr{K}_a}\kappa_b = f_{abc}\kappa_c \quad \wedge \quad \mathscr{L}_{{}^{\scriptscriptstyle Q}\mathscr{K}_a}k_b = f_{abc}k_c\,, \\ {}^{\scriptscriptstyle M}\mathscr{K}_a \lrcorner\, \kappa_b + {}^{\scriptscriptstyle M}\mathscr{K}_b \lrcorner\, \kappa_a = 0\,. \end{cases}$$

VII.2. An algebroidal interpretation of the gaugeability constraints

The action $\ell : \mathrm{G}_\sigma \times \mathscr{F} \to \mathscr{F}$ gives rise to

$$\textbf{ACTION GROUPOID} \qquad \mathrm{G}\ltimes\mathscr{F} \quad : \ \mathrm{G} \times \mathscr{F} \underset{t=\ell}{\overset{s=\mathrm{pr}_2}{\rightrightarrows}} \mathscr{F} \ ,$$

i.e. the small category

$$\mathrm{G}\ltimes\mathscr{F} = (\mathscr{F}, \mathrm{G}_\sigma \times \mathscr{F}, \mathrm{pr}_2, \ell, m \xrightarrow{\mathrm{Id}} (e,m), \circ)$$

with all morphisms invertible, as per

$$\mathrm{Inv} \ : \ \mathrm{G}_\sigma \times \mathscr{F} \to \mathrm{G}_\sigma \times \mathscr{F} \ : \ (g,m) \mapsto (g^{-1}, g.m) \,.$$

As for any Lie groupoid, we define its

$$\textbf{TANGENT (LIE) ALGEBROID} \qquad \mathfrak{g}_\sigma\ltimes\mathscr{F} = \left(\mathrm{Id}^*\ker(\mathsf{d}s), [\cdot,\cdot], \alpha_{\mathsf{T}(\mathrm{Ob}(\mathrm{G}_\sigma\ltimes\mathscr{F}))}\right),$$

with $\alpha_{\mathsf{T}(\mathrm{Ob}\,\mathrm{Gr})}$ inducing the map $\mathsf{d}t \circ i$ between spaces of sections, defined in terms of the canonical vector-space isomorphism

$$i \ : \ \Gamma(\mathrm{Id}^*\ker(\mathsf{d}s)) \xrightarrow{\cong} \mathfrak{X}^s_{\mathrm{R-inv}}(\mathrm{Mor}\,\mathrm{Gr}) \,,$$

and with $[\cdot,\cdot]$ given by the unique bracket on $\Gamma(\mathrm{Id}^*\ker(\mathsf{d}s))$ for which i is an isomorphism of Lie algebras.

In the case in hand,

$$\mathfrak{g}_\sigma\ltimes\mathscr{F} \cong \left(\oplus_{a=1}^D C^\infty(\mathscr{F},\mathbb{R})\,\mathscr{R}_a, [\cdot,\cdot]_{\mathfrak{g}_\sigma\ltimes\mathscr{F}}, \alpha_{\mathsf{T}\mathscr{F}}\right), \qquad \mathscr{R}_a \equiv R_a \circ \mathrm{pr}_1|_{\mathrm{Id}(\mathscr{F})}$$

$$[\,\lambda^a\,\mathscr{R}_a\,,\,\mu^b\,\mathscr{R}_b\,]_{\mathfrak{g}_\sigma\ltimes M} = f_{abc}\,\lambda^a\,\mu^b\,\mathscr{R}_c + \left(\mathscr{L}_{\lambda^a\,\mathscr{K}_a}\mu^b - \mathscr{L}_{\mu^a\,\mathscr{K}_a}\lambda^b\right)\mathscr{R}_b \,.$$

Prop$^{\underline{\mathrm{n}}}$: [rrS '10]

$$\mathfrak{g}_\sigma\ltimes\mathscr{F} \cong \left(\oplus_{a=1}^D C^\infty(\mathscr{F},\mathbb{R})\mathfrak{K}_a, [\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)}, \alpha_{\mathcal{T}\mathscr{F}}\right).$$

VII.3. The global gauge anomaly

Invariance of the gauged σ-model under *large* gauge transformations calls – via a cohomological argument – for the existence of

$$\Upsilon \;:\; \ell^*\mathcal{G} \xrightarrow{\cong} \mathrm{pr}_2^*\mathcal{G} \otimes I_{\rho_{\theta_L}} \qquad \text{over} \qquad \mathrm{Mor}(\mathrm{G}_\sigma \ltimes M)\,,$$

and a consistent 2-extension thereof to Φ and φ_n.

At this stage, we need to comply with the following requirements

(i) Incorporation of topologically non-trivial gauge bundles ($\sim$ G_σ-twisted sectors, or – less evidently – a solution to the field-identification problem).

(ii) Preservation of the original count of the physical degrees of freedom, given by $\dim \mathscr{F}$.

Problem: Goal (i) readily achieved via

PRINCIPAL EXTENSION $\quad \mathscr{F} \mapsto (\mathsf{P}|_{\Sigma\setminus\Gamma} \times M) \sqcup (\mathsf{P}|_{\Gamma\setminus\mathfrak{V}_\Gamma} \times Q) \sqcup \bigsqcup_{n\in\mathbb{N}_{\geq 3}} (\mathsf{P}|_{\mathfrak{V}_\Gamma^{(n)}} \times T_n) \equiv \widetilde{\mathscr{F}}\,,$

with obvious Ansätze:

$$\widetilde{\mathcal{G}}_{\mathcal{A}} = \mathrm{pr}_2^*\mathcal{G} \otimes I_{\rho_{\mathcal{A}}}\,, \qquad \widetilde{\Phi}_{\mathcal{A}} = \mathrm{pr}_2^*\Phi \otimes J_{\lambda_{\mathcal{A}}}\,, \qquad \widetilde{\varphi}_{n,\mathcal{A}} = \mathrm{pr}_2^*\varphi_n\,.$$

However, the typical fibres here are

$$\mathrm{G}_\sigma \times M \hookrightarrow \widetilde{M} \to \Sigma \setminus \Gamma\,, \qquad \mathrm{G}_\sigma \times Q \hookrightarrow \widetilde{Q} \to \Gamma \setminus \mathfrak{V}_\Gamma\,,$$

$$\mathrm{G}_\sigma \times T_n \hookrightarrow \widetilde{T}_n \to \mathfrak{V}_\Gamma^{(n)}\,.$$

Idea: Lift the geometric action of G_σ from $\widetilde{\mathscr{F}}$ to the extended background.

Given a pair $(\widetilde{M},M)$ of manifolds and a *surjective submersion* $\varpi : \widetilde{\mathscr{M}} \to \mathscr{M}$, define over the simplicial manifold

$$\cdots \overset{\mathrm{pr}_{i,j,k}}{\Rrightarrow} \widetilde{M}\times_M \widetilde{M}\times_M \widetilde{M} \overset{\mathrm{pr}_{i,j}}{\Rrightarrow} \widetilde{M}\times_M \widetilde{M} \overset{\mathrm{pr}_i}{\rightrightarrows} \widetilde{M} \xrightarrow{\varpi} M$$

the **DESCENT 2-CATEGORY** $\mathfrak{Desc}(\varpi)$, with objects $(\mathcal{G},\Psi,\chi)$, where

$$\mathrm{pr}_1^*\mathcal{G} \underset{\Psi}{\xrightarrow{\cong}} \mathrm{pr}_2^*\mathcal{G}\,, \qquad \mathrm{pr}_{2,3}^*\Psi \circ \mathrm{pr}_{1,2}^*\Psi \underset{\chi}{\overset{\cong}{\Longrightarrow}} \mathrm{pr}_{1,3}^*\Psi\,,$$

$$\mathrm{pr}_{1,3,4}^*\chi \bullet (\mathrm{id}\circ \mathrm{pr}_{1,2,3}^*\chi) = \mathrm{pr}_{1,2,4}^*\chi \bullet (\mathrm{pr}_{2,3,4}^*\chi \circ \mathrm{id})\,,$$

1-cells $(\Phi,\eta) : (\mathcal{G}_1,\Psi_1,\chi_1) \xrightarrow{\cong} (\mathcal{G}_2,\Psi_2,\chi_2)$, where

$$\mathcal{G}_1 \underset{\Phi}{\xrightarrow{\cong}} \mathcal{G}_2\,, \qquad \mathrm{pr}_2^*\Phi \circ \Psi_1 \underset{\eta}{\overset{\cong}{\Longrightarrow}} \Psi_2 \circ \mathrm{pr}_1^*\Phi\,,$$

$$(\chi_2\circ \mathrm{id}) \bullet (\mathrm{id}\circ \mathrm{pr}_{1,2}^*\eta) \bullet (\mathrm{pr}_{2,3}^*\eta \circ \mathrm{id}) = \mathrm{pr}_{1,3}^*\eta \bullet (\mathrm{id}\circ\chi_1)\,,$$

and 2-cells $\varphi : (\Phi_1,\eta_1) \overset{\cong}{\Longrightarrow} (\Phi_2,\eta_2)$, where

$$\Phi_1 \underset{\varphi}{\overset{\cong}{\Longrightarrow}} \Phi_2\,, \qquad (\mathrm{id}\circ \mathrm{pr}_1^*\varphi)\bullet\eta_1 = \eta_2 \bullet (\mathrm{pr}_2^*\varphi\circ \mathrm{id})$$

<u>Th$^{\mathrm{m}}$:</u> [Stevenson '00]

$$\varpi^* \ :\ \mathfrak{BGrb}^\nabla(M) \xrightarrow{\equiv} \mathfrak{Desc}(\varpi)\ :\ \begin{cases} \mathcal{G} & \mapsto\ (\varpi^*\mathcal{G},\mathrm{id},\mathrm{id}) \\ \Phi & \mapsto\ (\varpi^*\Phi,\mathrm{id}) \\ \varphi & \mapsto\ \varpi^*\varphi \end{cases}.$$

<u>The beautiful:</u>

(i) $\mathfrak{BGrb}^\nabla(\mathscr{M}) \equiv \big(\pi_{Y\mathscr{M}}^*\big)^{-1}\big(\mathfrak{Triv}\text{–}\mathfrak{BGrb}^\nabla(Y\mathscr{M})\big)$, the latter being defined in terms of smooth 2-forms and $\mathfrak{Bun}^\nabla(Y\mathscr{M})$, with $\mathfrak{Bun}^\nabla(Y\mathscr{M}) \equiv \big(\pi_{Y'Y\mathscr{M}}^*\big)^{-1}\big(\mathfrak{Triv}\text{–}\mathfrak{Bun}^\nabla(Y'Y\mathscr{M})\big)$.

(ii) Descent for the action groupoid over $\mathrm{G}_\sigma \hookrightarrow M \overset{\varpi}{\to} M/\mathrm{G}_\sigma$, where $\mathrm{G}_\sigma \subset \mathrm{Iso}(M,\mathrm{g})$ is a group of σ-model symmetries, determines the Gauge Principle (due to a remarkable interplay between Σ and $\mathscr{F}$).

A 2-birds-with-1-stone solution:

(i) Demand of $(\mathcal{G}, \Phi, \varphi_n)$ a full-blown G_σ**-EQUIVARIANT STRUCTURE** (i.e., morally speaking, pass from Čech–Deligne- to Čech–Deligne–G_σ-hypercohomology).

Prop$^{\underline{\mathrm{n}}}$: [Gawędzki, Waldorf & rrS '10] A G_σ-equivariant structure on $\mathfrak{B}$ relative to *arbitrary* $(\rho_., \lambda_.)$ canonically induces a G_σ-equivariant structure on $\widetilde{\mathfrak{B}}_{\mathcal{A}}$ relative to $(\rho_., \lambda_.) = (0,0)$.

(ii) Employ the Principle of Descent, in the guise

$$\mathfrak{BGrb}^{\nabla}_{(\rho_.,\lambda_.)=(0,0)}(\widetilde{\mathscr{F}}) \equiv \mathfrak{BGrb}^{\nabla}(\widetilde{\mathscr{F}}/\mathrm{G}_\sigma)$$

valid for the distinguished surjective submersions $\varpi_{\widetilde{\mathscr{F}}}\widetilde{\mathscr{F}} \to \widetilde{\mathscr{F}}/\mathrm{G} \equiv \mathsf{P} \times_{\mathrm{G}_\sigma} \mathscr{F}$ (engendered by the *free* action $\widetilde{\ell} : \mathrm{G}_\sigma \times \widetilde{\mathscr{F}} \to \mathscr{F}$), to descend

$$(\widetilde{\mathcal{G}}_{\mathcal{A}}, \widetilde{\Phi}_{\mathcal{A}}, \widetilde{\varphi}_{n,\mathcal{A}}) \to \big(\underline{\mathcal{G}}(\mathcal{A}), \underline{\Phi}(\mathcal{A}), \underline{\varphi}_n(\mathcal{A})\big)$$

to the associated bundles.

Upshot: The **GAUGED σ-MODEL**

$$S_\sigma[(\underline{X}\,|\,\Gamma); \gamma, \mathcal{A}] = S_{\mathrm{kin}}^{\mathrm{MC}}[\underline{X}; \gamma, \mathcal{A}] - \mathsf{i}\,\log \mathrm{Hol}_{\underline{\mathcal{G}}(\mathcal{A}), \underline{\Phi}(\mathcal{A}), \underline{\varphi}_n(\mathcal{A})}(\underline{X})\,,$$

manifestly invariant under the action of the **GAUGE GROUP**

$$\Gamma(\mathsf{P} \times_{\mathrm{Ad}} \mathrm{G}_\sigma) \;:\; [(p, g_1)] \cdot [(p, g_2)] := [(p, g_1 \cdot g_2)]\,.$$

The latter is induced by the action

$$\lambda \;:\; (\mathsf{P} \times_{\mathrm{Ad}} \mathrm{G}_\sigma) \times \mathsf{P} \to \mathsf{P} \;:\; \big([(p, g_1)], p.g_2\big) \mapsto p.(g_1 \cdot g_2)$$

and reads

$$(\chi, \underline{X}) \mapsto (\lambda_\chi, \mathrm{id}_M) \circ \underline{X}\,, \qquad\qquad (\chi, \mathcal{A}) \mapsto \lambda^*_{\chi^{-1}}\mathcal{A}\,.$$

VII.4. The coset model

For the topologically trivial gauge field (or locally), we may define the **COSET σ-MODEL** as

$$\mathrm{e}^{-W_{\mathrm{eff}}[(\underline{X}\,|\,\Gamma);\gamma]} := \int_{[\mathrm{A}]} \mathscr{D}\mathrm{A}\,\mathrm{e}^{-S_\sigma[(\underline{X}\,|\,\Gamma);\gamma,\mathrm{A}]}$$

N.B. The above path integral is gaussian, whence

$$W_{\sigma,\mathrm{eff}}[(\underline{X}\,|\,\Gamma);\gamma] \sim S_\sigma[(\underline{X}\,|\,\Gamma);\gamma,\mathrm{A}_{\mathrm{cl.}}]\,.$$

Under certain (mild) technical assumptions regarding $\mathfrak{B}$, the effective field theory is, indeed, a σ-model with a field space $\mathscr{F}$ and

EFFECTIVE BACKGROUND $\qquad \varpi^*_{\widetilde{\mathscr{F}}}\underline{\mathrm{G}}\,, \qquad \mathcal{G}\otimes I_\Delta\,, \qquad \Phi\otimes J_\delta\,, \qquad \varphi_n\,.$

The remarkable, again: The effective background is G_σ-equivariant relative to $(\rho_\cdot,\lambda_\cdot) = (0,0)$ iff the original one is G_σ-equivariant.

Conclusion: $\mathfrak{B}_{\mathrm{eff}}$ descends to a unique equivalence class $\underline{\mathfrak{B}}$ over the coset space $\mathscr{F}/\mathrm{G}_\sigma$ iff $\mathfrak{B}$ is endowed with a G_σ-equivariant structure.

Outlook: Towards “non-geometry” via gauged stringy dualities associated with groupoidal backgrounds. . .

LA FIN,
Pour qu’on n’en ait pas (que) la gerbe. . .