AV-differential geometry and Lagrangian Mechanics

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- Hamiltonian for a time-dependent system is not a function, but a section of an affine bundle over the phase manifold (Mangiarotti, Martínez, Popescu, Sarlet, Sardanashvili, ...).

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- Frame independent Lagrangian in Newtonian mechanics is an affine object (Duval, Tulczyjew, ...).

What is AV-differential geometry?

Differential geometry of affine values (AV-differential geometry) is, roughly speaking, the differential geometry built on the set of sections of one-dimensional affine bundle $\zeta : \mathbf{Z} \longrightarrow M$ modelled on $M \times \mathbb{R}$, instead of just functions on M.

What is AV-differential geometry?

The bundle $\mathbf{Z}$ will be called a bundle of affine values.

$\mathbf{Z}$ is modelled on $M \times \mathbb{R}$, so we can add reals in each fiber of $\mathbf{Z}$.

$\mathbf{Z}$ is an $(\mathbb{R}, +)$-principal bundle.

Affine analog of the cotangent bundle T^*M

We define an equivalence relation in the set of pairs of (m, σ), where $m \in M$ and σ is a section of $\mathbf{Z}$.

(m, σ), (m', σ') are equivalent if $m = m'$ and $\mathsf{d}(\sigma - \sigma')(m) = 0$, where we have identified the difference of sections of $\mathbf{Z}$ with a function on M.

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The equivalence class of (m, σ) is denoted by $\mathsf{d}\sigma(m)$. The set of equivalence classes is denoted by $\mathsf{P}\mathbf{Z}$ and called the phase bundle for $\mathbf{Z}$.

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$$\mathsf{P}\zeta : \mathsf{P}\mathbf{Z} \rightarrow M : \mathsf{d}\sigma(m) \mapsto m$$

is an affine bundle modelled on the cotangent bundle $\mathsf{T}^* M \rightarrow M$.

Other examples of affine constructions

A, B - affine spaces modelled on a vector space V.
$A \times B \ni (a, b), (a', b')$ are equivalent if $a - a' = b' - a'$.
Equivalence class is the affine sum $a \boxplus b$.
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There is full duality between affine spaces and special vector spaces.

Example

$A = \mathsf{P}_m \mathbf{Z}$

The AV-bundle for A can be identified with $\mathsf{T}_z Z$ for any $\zeta(z) = m$ with $\tau = \mathsf{T}\zeta$ restricted to $\mathsf{T}_z Z$.

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For the whole bundle, $(\mathsf{P}\mathbf{Z})^\dagger = \widetilde{\mathsf{T}}\mathbf{Z}$, where $\widetilde{\mathsf{T}}\mathbf{Z} = \mathsf{T}Z/\mathbb{R}$

References

First appearances:

W.M. Tulczyjew, P. Urbański, S. Zakrzewski, *A pseudocategory of principal bundles*, Atti Accad. Sci. Torino, **122** (1988), 66–72

W.M. Tulczyjew, P. Urbański, *An affine framework for the dynamics of charged particles*, Atti Accad. Sci. Torino Suppl. n. 2, **126** 1992, 257–265.

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For more recent references look in

K. Grabowska, J. Grabowski and P. Urbański: *AV-differential geometry: Poisson and Jacobi structures*, J. Geom. Phys. **52** (2004) no. 4, 398-446.

P. Urbański, *Affine framework for analytical mechanics*, in "Classical and Quantum Integrability", Grabowski, J., Marmo, G., Urbański, P. (eds.), Banach Center Publications, vol. **59** (2003), 257–279.

and references there.

Lagrangian and action

Lagrangian is a section λ of the AV-bundle $\widetilde{\mathsf{T}}\zeta\colon \widetilde{\mathsf{T}}\mathbf{Z} \to \mathsf{T}M$

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An example is given by a section φ of the phase bundle $\mathsf{P}\mathbf{Z}$: for each $m \in M$, $\varphi(m)$ corresponds to a linear section of the AV-bundle $\widetilde{\mathsf{T}}\zeta\colon \widetilde{\mathsf{T}}_m\mathbf{Z} \to \mathsf{T}_m M$. We denote this section by $\mathrm{i}_{\mathsf{T}}\varphi(m)$.

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$$\int_a^b \lambda \circ \dot\gamma = \int_a^b (\lambda \circ \dot\gamma - \mathrm{i}_{\mathsf{T}}\varphi \circ \dot\gamma) + \int_{\gamma([a,b])} \varphi \quad .$$

It does not depend on the choice of φ.

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It does not depend on the choice of φ.

$$\int_a^b \lambda \circ \dot{\gamma} \in \mathbf{Z}_{\gamma(b)} \boxminus \mathbf{Z}_{\gamma(a)}$$

Euler-Lagrange equation. Standard case

The basis for the representation of the differential of the action functional is the decomposition

$$\mathsf{d}L = ((\tau_2^1)^* \mathsf{d}L - \mathsf{d}_{\mathsf{T}}(\mathrm{i}_F \mathsf{d}L)) + \mathsf{d}_{\mathsf{T}}(\mathrm{i}_F \mathsf{d}L) \qquad (*)$$

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The first component in $(*)$ is a 1-form on $\mathsf{T}^2 M$, vertical with respect to projection $\mathsf{T}^2 M \to M$.

It can be considered a mapping $\mathsf{T}^2 M \to \mathsf{T}^* M$

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- The AV-bundle for $((\tau_2^1)^*\mathrm{d}\lambda - \mathrm{d}_{\mathsf{T}}(\mathrm{i}_F\mathrm{d}\lambda))$ is trivial.

As in the standard case, the form $((\tau_2^1)^* \mathsf{d}L - \mathsf{d}_{\mathsf{T}}(\mathrm{i}_F \mathsf{d}L))$ is semi-basic and can be interpreted as a mapping $\mathsf{T}^2 M \to \mathsf{T}^* M$.

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Forces are co-vectors on M, momenta are affine co-vectors, elements of $\mathsf{P}\mathbf{Z}$.