How to Cope with the Curse of Dimensionality?

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Curse of Dimensionality

ε	**error demand**
d	**the number of variables**
$n(\varepsilon, d)$	**the minimal cost**

Many problems suffer from the *curse of dimensionality*

$$n(\varepsilon, d) \ \geq c\ (1 + C)^{\,d}$$

for all $d = 1, 2, \ldots$ with $c, C\ >\ 0.$

Multivariate Integration

For $f \in F_d$ we want to approximate

$$I_d(f) := \int_{[0,1]^d} f(t)\,\mathrm{d}t \quad \approx \quad A_n(f)$$

- **Algorithms:**

$$A_n(f) = \phi(f(x_1), f(x_2), \dots, f(x_n)) \quad \text{with} \quad x_j \in [0,1]^d$$

- **Minimal Worst Case Error:**

$$e(n,d) = \inf_{A_n} \sup_{\|f\|_{F_d} \le 1} |I_d(f) - A_n(f)|$$

- **Information Worst Case Complexity:**

$$n(\varepsilon, d) \;=\; \min\{\, n \,|\; e(n,d) \le \varepsilon \,\}$$

Multivariate Integration for Smooth Functions

$$L = \{L_d\} \qquad L_d > 0$$

$$F_d = C_d^r(L) := \{f : [0,1]^d \to \mathbb{R} \,:\, \|f\|_{\max} \le 1, \ \|D^\alpha f\|_{\max} \le L_d \,\forall\, |\alpha| \in [1,r]\}$$

Bakhvalov [1959]

$$n(\varepsilon, d) = \Theta(\varepsilon^{-d/r})$$

but factors in the Θ-notation depend on d and r. Curse?

Sukharev [1979]: **The curse holds for $r = 1$ and $L_d \equiv 1$.**

Otherwise, curse?

Multivariate Integration for Smooth Functions

$$C_d^r(L) := \{f : [0,1]^d \to \mathbb{R} : \ |f(x)| \le 1, \ \ |D^\alpha f(x)| \le L_d \ \forall \ |\alpha| \in [1,r]\}$$

What are necessary and sufficient conditions for $\{L_d\}$ to have the curse of dimensionality for multivariate integration?

Theorem (Hinrichs, Novak, Ullrich, W [2012])

The curse holds for $C_d^r(L)$ iff $\liminf_{d\to\infty} L_d \sqrt{d} > 0$

Multivariate Integration for Korobov Spaces

$$r = \{r_j\} \quad \textbf{with} \quad 1 \le r_1 \le r_2 \le \cdots$$

H_{r_j}**:** **1-periodic** $f : [0,1] \to \mathbb{C}$, $f^{(r_j-1)}$ **abs. cont,** $f^{(r_j)} \in L_2$

$$\|f\|_{H_{r_j}}^2 = \left|\int_0^1 f(t)\,\mathrm{d}t\right|^2 + \int_0^1 \left|f^{(r_j)}(t)\right|^2 \mathrm{d}t$$

For $d \ge 1$**,**

$$F_d = H_{d,r} = H_{r_1} \otimes H_{r_2} \otimes \cdots \otimes H_{r_d}$$

Usually, it is assumed that $r_j \equiv r$

Theorem

Let $r_j \equiv r$. Then there exists $c_r, C_r > 0$ such that

$$n(\varepsilon, d) \, > \, c_r \, (1 + C_r)^{\,d}$$

Based on Hickernell+W [2001] and Novak+W[2001], see also Sloan+W[2001]

Multivariate integration for Korobov space with arbitrarily smooth functions suffers from the curse of dimensionality

How to cope with the curse of dimensionality

- switch to spaces of increased smoothness with respect to successive variables
- switch to weighted spaces, i.e., groups of variables are of varying importance
- switch to a more lenient setting, i.e, from the worst case setting to the randomized or average case setting

Increasing Smoothness

Multivariate integration for Korobov spaces in the worst case setting with $r_1 \le r_2 \le \cdots$.

But we now allow to increase r_j

Let

$$R := \limsup_{k \to \infty} \frac{\ln\, k}{r_k}$$

Theorem

If $R < 2 \ln 2\pi$ then

- **no curse**
- $n(\varepsilon, d) \le C\, \varepsilon^{-p(1+p/2)}$ **with** $p := \max(r_1^{-1}, R/\ln 2\pi) < 2$,
  i.e., strong polynomial tractability

Based on Papageorgiou+W [09], Kuo, Wasilkowski+W[09]

Weighted Spaces

Major research activities in last 10 years...

In particular, for $r_j \equiv r$ and $\gamma = \{\gamma_j\}$, redefine H_{r_j,γ_j} with

$$\|f\|^2_{H_{r_j,\gamma_j}} = \left|\int_0^1 f(t)\,\mathrm{d}t\right|^2 + \frac{1}{\gamma_j}\int_0^1 \left|f^{(r_j)}(t)\right|^2 \mathrm{d}t$$

For $d \geq 1$,

$$H_{d,r} = H_{r_1,\gamma_1} \otimes H_{r_2,\gamma_2} \otimes \cdots \otimes H_{r_d,\gamma_d}$$

Theorem

- **Gnewuch+W[08]**

  $\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_j}{d} = 0$ **iff no curse,**

- **Hickernell+W[01]**

  $\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_j}{\ln\, d} < \infty$ **iff polynomial tractability,**

  i.e., $n(\varepsilon, d) \le C\, d^q\, \varepsilon^{-p}$

- **Hickernell+W[01]**

  $\sum_{j=1}^{\infty} \gamma_j < \infty$ **iff strong polynomial tractability,**

  i.e., $n(\varepsilon, d) \le C\, \varepsilon^{-p}$

More Lenient Settings

From Worst Case Setting to

- **Randomized Setting**
- **Average Case Setting**

Average Case Setting $\leq$ Randomized Setting

Randomized Setting

- **Algorithms:**

$$A_{n,\omega}(f) = \phi_\omega(f(x_{1,\omega}), f(x_{2,\omega}), \dots, f(x_{n(\omega),\omega})) \quad \text{for a random } \omega$$

- **Minimal Randomized Error:**

$$e(n,d) = \inf_{A_n} \sup_{\|f\|_{H_{d,r}} \le 1} \left[\mathbb{E}\, |I_d(f) - A_{n,\omega}(f)|^2\right]^{1/2}$$

- **Information Randomized Complexity:**

$$n(\varepsilon, d) \ = \ \min\{\, n \,|\ \ e(n,d) \le \varepsilon \,\}$$

Monte Carlo Algorithm

$$A_{n,\omega}(f) = \frac{1}{n}\sum_{j=1}^{n} f(x_{j,\omega})$$

with

$x_{j,\omega}$ **iid with uniform distribution over** $[0,1]^d$

Sloan+W[01] for Korobov spaces, obvious for $C_d^r(L)$ **spaces**

- $n(\varepsilon, d) \le \varepsilon^{-2}$
- **no curse and strong polynomial tractability**

Conclusions

- Many multivariate problems suffer from the curse of dimensionality in the worst case setting
- We may sometimes break the curse of dimensionality by
  - switching to spaces of increased smoothness with respect to successive variables
  - switching to weighted spaces, i.e., groups of variables are of varying importance
  - switching to a more lenient setting, i.e, from the worst case setting to the randomized or average case setting

Book

More can be found in

Tractability of Multivariate Problems

Erich Novak and Henryk Woźniakowski

- **Volume I: Linear Information (2008)**
- **Volume II: Standard Information for Functionals (2010)**
- **Volume III: Standard Information for Operators (2012?)**