How to Cope with the Curse of Dimensionality

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Many problems suffer from the curse of dimensionality.

The minimal cost $p(\varepsilon) \geq (p, \varepsilon)u$

For all $p = 1, 2, \ldots$ with $\varepsilon, c > 0$,

$p(\varepsilon, 1 + c) \geq (p, \varepsilon)u$

The number of variables $p$

Error demand $\varepsilon$

Cure of Dimensionality
For $f \in \mathcal{F}$

\[ I_d(f) := \int_{[0,1]} f(t) \, dt \approx A_n(f) \]

\begin{itemize}
  \item **Multivariate Integration**
  \item **Algorithms:**
  \[ A_n(f) = \phi(f(x_1), f(x_2), \ldots, f(x_n)) \text{ with } x_j \in [0,1] \]
  \item **Minimal Worst Case Error:**
  \[ e(n, d) = \inf_{A_n} \sup_{\|f\|_{\mathcal{F}} \leq 1} |I_d(f) - A_n(f)| \]
  \item **Information Worst Case Complexity:**
  \[ n(\varepsilon, d) = \min \{ n \mid e(n, d) \leq \varepsilon \} \]
  \item **Approximation:**
  \[ (f)^{uV} \approx \mathcal{I}^{[1,0]} f \int_{[0,1]} =: (f)^{pI} \]
\end{itemize}

For $f \in \mathcal{F}$ we want to approximate $\int_{[0,1]^d} f(t) \, dt$. 
The curse holds for $\alpha = 1$ and $L_d \equiv 1$.

\[ (\frac{L_d}{\varepsilon})^{\Theta} = (p^\alpha)^n \]

Bakhvalov [1979]

\[ n(\varepsilon, d) = \Theta(\varepsilon - \frac{d}{r}) \]

but factors in the $\Theta$-notation depend on $d$ and $r$.

Sukharev [1979]: The curse holds for $r = 1$ and $L_d \equiv 1$.

Otherwise, curse?

\[ \{\ell, I \} \in \mathcal{V} \quad \text{if} \quad pT \geq \max \| f \| \quad \text{and} \quad p \geq \max \| f \| : \mathbb{R} \leftarrow p[I, 0] : f \} =: (T)^{p\mathcal{C}} = pH \]

\[ 0 < pT \quad \{pT\} = T \]
Multivariate Integration for Smooth Functions

What are necessary and sufficient conditions for \( \{ L_d \} \) to have the curse of dimensionality for multivariate integration?

Theorem (Hinrichs, Novak, Ulrich, W [2012])

The curse holds for \( C^r_d(\mathbb{R}) \) iff
\[
\liminf_{d \to \infty} L_d \sqrt{d} > 0
\]
Usually, it is assumed that \( r_j \equiv r \). For \( d \geq 1 \),

\[
F_d = H_{d,r} = H_{r_1} \otimes H_{r_2} \otimes \ldots \otimes H_{r_d}
\]

\[
\|f\|_{H_{d,r}}^2 = \int_0^1 \left| \int_0^1 f(t) f(t') dt' \right|^2 dt + \int_0^1 \left| f^{(r_1)}(t) \right|^2 dt,
\]

\( H_{r,j} \): 1-periodic \( f : [0,1] \to \mathbb{C}, f^{(r'-1)} \) abs. cont., \( f^{(r_j)} \in L_2 \).

\[
r = \{ r_j \} \text{ with } 1 \leq r_1 \leq r_2 \leq \ldots \}
\]

### Multivariate Integration for Korobov Spaces

How to Cope with the Curse of Dimensionality
Theorem

Let \( r_j \equiv r \). Then there exists \( c_r, C_r > 0 \) such that

\[
\begin{align*}
\epsilon &> c_r (1 + C_r d) \\
n &< p(1 + C_r d)
\end{align*}
\]

Based on Hickernell+W [2001] and Novak+W [2001], see also Sloan+W [2001].

Multivariate integration for Korobov space suffers from the curse of dimensionality with arbitrarily smooth functions.
How to cope with the curse of dimensionality

- switch to spaces of increased smoothness with respect to successive variables
- switch to weighted spaces, i.e., groups of variables are of varying importance
- switch to a more lenient setting, i.e., from the worst case setting to the randomized or average case setting
Based on Papageorgiou+W [09], Kuo, Wasilkowski+W[09]

Theorem

\[
\mathcal{R} = \limsup_{k \to \infty} \frac{\ln k}{r_k}
\]

\[
\text{If } \mathcal{R} < 2 \ln \frac{2}{\pi} \text{ then no curse}
\]

\[
\mathcal{N} = (\varepsilon, d) \leq C \varepsilon^{-(p, q)} + \mathcal{R}
\]

\[
\text{with } p = \max(1, \frac{R}{\ln \frac{2}{\pi}}, 1) + \frac{1}{d+1}
\]

But we now allow to increase \( p \) !

Increasing Smoothness
Weighted Spaces

Major research activities in last 10 years...

For $p \in I$,

\[ \|f\|_p \leq \sum_{i=1}^{p} \|f_i\|_{\mathcal{H}_i} \]

In particular, for $u \equiv \{\mathcal{H}_i\}$, redefine with $\mathcal{H}$.
Theorem

\[ d_{-3} \supseteq (p, \varepsilon) \cup \supseteq \quad \text{iff no curse,} \]

\[ \limsup_{d \to \infty} \sum_{j=1}^{\infty} \gamma_j d_j \leq 1 \quad \text{iff strong polynomial tractability,} \]

\[ \limsup_{d \to \infty} \sum_{j=1}^{\infty} \gamma_j \ln d_j < \infty \quad \text{iff polynomial tractability,} \]

\[ \lim \sup_{d \to \infty} p \cdot \sum_{j=1}^{\infty} \gamma_j < \infty \quad \text{iff strong polynomial tractability,} \]

\[ \lim p \sum_{j=1}^{\infty} \chi_j P_j \leq \infty \quad \text{iff no curse,} \]

\[ \lim p \sum_{j=1}^{\infty} \gamma_j \chi_j P_j \leq \infty \quad \text{iff polynomial tractability,} \]

\[ \lim p \sum_{j=1}^{\infty} \gamma_j \chi_j P_j \leq \infty \quad \text{iff strong polynomial tractability,} \]

\[ \lim p \sum_{j=1}^{\infty} \gamma_j \chi_j P_j \leq \infty \quad \text{iff no curse,} \]
More Lenient Settings

From Worst Case Setting to

\[ \text{Average Case Setting} \leq \text{Randomized Setting} \]
Randomized Setting

Algorithms:

\[ A_{n,\omega}(f(x_1,\omega), f(x_2,\omega), \ldots, f(x_n,\omega)) \]

for a random \( \omega \)

Minimal Randomized Error:

\[ e(n,d) = \inf_{A_n} \sup_{\|f\|_H \leq 1} \left[ E_{|I|d} (f) - A_{n,\omega}(f) \right]^{1/2} \]

Information Randomized Complexity:

\[ n(\varepsilon, d) = \min \{ n | e(n,d) \leq \varepsilon \} \]

Minimal Randomized Error:

\[ (m^u x)^f \cdot \ldots \cdot (m^u x)^f \cdot (m^u I x)^f \phi = (f)^{m^u} \]

Algorithms:

Randomized Setting

\[ \{ \exists \geq (p, u) \exists | u \} \min = (p, \exists) u \]
Monte Carlo Algorithm

\[ (x, \omega) \sum_{j=1}^{n} f(x_j, \omega) \]

with

\[ x_j, \omega \text{ iid with uniform distribution over } [0, 1]^d \]

and with

\[ p[0, 1]^d \text{ iid with uniform distribution over } [0, 1]^d \]

with

\[ f \mathcal{U} \sum_{u \in I} \frac{u}{1} = (f)_{\mathcal{U}} \]

Monte Carlo Algorithm

no curse and strong polynomial tractability

\[ z \geq (p, \epsilon) u \]

\[ (T)_p \text{ for Korobov spaces, obvious for } C_{1}\text{ spaces} \]
Many multivariate problems suffer from the curse of dimensionality. We may sometimes break the curse of dimensionality by switching to a more lenient setting, i.e., from the worst case setting to the randomized or average case setting – switching to weighted spaces, i.e., groups of variables are of varying importance – switching to weighted spaces of increased smoothness.

Conclusions

• Many multivariate problems suffer from the curse of dimensionality.
More can be found in:

- Volume III: Standard Information for Operators (2012?)
- Volume I: Linear Information (2008)

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Tractability of Multivariate Problems