Double vector bundles in classical mechanics.

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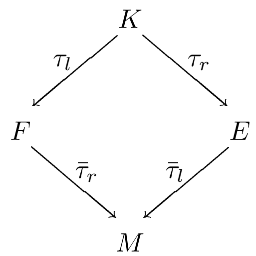
Abstract.
The notion of a double vector bundle and the dual double vector bundle is defined. Theorems on canonical isomorphisms are formulated. Examples related to classical mechanics are given.

1. Introduction.

The existence of two different vector bundle structures on $\mathsf{TT}M$ and TT^*M makes possible the lagrangian formulation of the dynamics in classical mechanics ([3]). $\mathsf{TT}M$ and TT^*M are examples of double vector bundles. The concept of a double vector bundle was first introduced by Pradines ([2]). In this note we present a definition of a double vector bundle following Mackenzie ([1]). In Section 4, we define right and left dual double vector bundles. The main example is the cotangent bundle T^*E of a vector bundle. The most important theorem stated in this note concerns the existence of the canonical isomorphism of a double vector bundle and its third right dual. In Sections 7, 8, and 9, we give examples of situations in classical mechanics which adopt the notion of a double vector bundle (special symplectic manifolds, linear connections, torsion-free connections). In the following we state theorems and provide examples. Proofs will be presented in a separate publication.

2. Double vector bundles.

Let $\mathbf{K}$ be a system $(\mathbf{K}_r, \mathbf{K}_l, \mathbf{E}, \mathbf{F})$ of vector bundles, where $\mathbf{K}_r = (K, \tau_r, E)$, $\mathbf{K}_l = (K, \tau_l, F)$, $\mathbf{E} = (E, \bar{\tau}_l, M)$ and $\mathbf{F} = (F, \bar{\tau}_r, M)$, such that the diagram



(1)

is commutative.

We introduce the following notation:

(1) m_r, m_l, $\bar{m}_r$ and $\bar{m}_l$ will denote the operation of addition in $\mathbf{K}_r$, $\mathbf{K}_l$, $\mathbf{E}$ and $\mathbf{F}$ respectively.
(2) we use also $v +_r w$ for $m_r(v, w)$, $v +_l w$ for $m_l(v, w)$ and simply $+$ for all other additions,
(3) $\mathbf{0}_r$, $\mathbf{0}_l$, $\bar{\mathbf{0}}_r$, $\bar{\mathbf{0}}_l$ will denote the zero sections of τ_r, τ_l, $\bar{\tau}_r$ and $\bar{\tau}_l$ respectively.

Let the pair $(\tau_r, \bar{\tau}_r)$ be a vector bundle morphism $\mathbf{K}_l \to \mathbf{E}$. It follows that $K \times_E K$ is a vector subbundle of $\mathbf{K}_l \oplus \mathbf{K}_l$ with $(\tau_l \times \tau_l)(K \times_E K) = F \times_M F$. We denote this subbundle by $\mathbf{K}_l \oplus_E \mathbf{K}_l$. Moreover, the addition $m_r\colon K \times_E K \to K$ is a fiber bundle morphism which projects to $\bar{m}_r\colon F \times_M F \to F$.

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Definition 1. *A double vector bundle* $\mathbf{K}$ *is a system* $(\mathbf{K}_r, \mathbf{K}_l, \mathbf{E}, \mathbf{F})$ *of vector bundles* $\mathbf{K}_r = (K, \tau_r, E)$, $\mathbf{K}_l = (K, \tau_l, F)$, $\mathbf{E} = (E, \bar\tau_l, M)$ *and* $\mathbf{F} = (F, \bar\tau_r, M)$ *such that the diagram (1) is commutative and the following conditions are satisfied:*

(1) *pairs* $(\tau_l, \bar\tau_l)$, $(\tau_r, \bar\tau_r)$ *are vector bundle morphisms,*
(2) *pairs of additions* $(m_l, \bar m_l)$ *and* $(m_r, \bar m_r)$ *are vector bundle morphisms* $\mathbf{K}_r \times_F \mathbf{K}_r \to \mathbf{K}_r$ *and* $\mathbf{K}_l \times_E \mathbf{K}_l \to \mathbf{K}_l$ *respectively,*
(3) *zero sections* $\mathbf{0}_r\colon \mathbf{E} \to \mathbf{K}_l$, $\mathbf{0}_l\colon \mathbf{F} \to \mathbf{K}_r$ *are vector bundle morphisms.*

In the following we use the diagram (1) to represent the double vector bundle $\mathbf{K}$.

Proposition 2. *The vector bundle structures of* $\mathbf{K}_r$ *and* $\mathbf{K}_l$ *coincide on the intersection* C *of* $\ker\tau_l$ *and* $\ker\tau_r$.

Thus, we have a vector bundle $\mathbf{C} = (C, \tau, M)$, where $\tau = \bar\tau_l\tau_r = \bar\tau_r\tau_l$. This vector bundle is called *the core* of $\mathbf{K}$.

Proposition 3.

(1) $\ker\tau_r$ *with the vector bundle structure induced from* $\mathbf{K}_r$ *is canonically isomorphic to the Whitney sum* $\mathbf{F} \oplus_M \mathbf{C}$.
(2) $\ker\tau_r$ *with the vector bundle structure induced from* $\mathbf{K}_l$ *is canonically isomorphic to the vector bundle* $F \times_M \mathbf{C}$, *i.e., to the pull-back of* $\mathbf{C}$ *by the projection* $\bar\tau_r$.
(3) $\ker\tau_l$ *with the vector bundle structure induced from* $\mathbf{K}_l$ *is canonically isomorphic to the Whitney sum* $\mathbf{E} \oplus_M \mathbf{C}$.
(4) $\ker\tau_l$ *with the vector bundle structure induced from* $\mathbf{K}_r$ *is canonically isomorphic to the vector bundle* $E \times_M \mathbf{C}$, *i.e., to the pull-back of* $\mathbf{C}$ *by the projection* $\bar\tau_l$.

Local coordinates. Let $(x^i)_{i=1}^n$ be a coordinate system on M. In the bundles $\mathbf{E}, \mathbf{F}$ we introduce coordinate systems $((x^i)_{i=1}^n, (e^a)_{a=1}^{n_E})$ and $((x^i)_{i=1}^n, (f^A)_{A=1}^{n_F})$. By x^i, e^a, f^A we denote also their pull-backs to the coordinates on K. We can introduce coordinates $(c^\alpha)_{\alpha=1}^{n_C}$ such that $(x^i, e^a, f^A, c^\alpha)$ is a local coordinate system on K and

$$\begin{aligned}
&c^\alpha(v +_r w) = c^\alpha(v) + c^\alpha(w),\\
&c^\alpha(v +_l w) = c^\alpha(v) + c^\alpha(w),\\
&c^\alpha \circ \mathbf{0}_r = 0,\\
&c^\alpha \circ \mathbf{0}_l = 0,
\end{aligned}$$

It follows that (x^i, c^α) is a vector bundle coordinate system in $\mathbf{C}$. The operation of addition $+_r$ is characterized by the following equalities

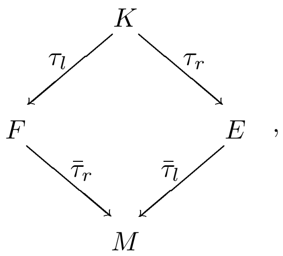
$$\begin{aligned}
c^\alpha(v +_l w) &= c^\alpha(v) + c^\alpha(w)\\
f^A(v +_l w) &= f^A(v) + f^A(w)\\
e^a(v +_l w) &= e^a(v) = e^a(w)\\
x^i(v +_l w) &= x^i(v) = x^i(w).
\end{aligned}$$

The operation of addition $+_l$ is characterized by

$$\begin{aligned}
c^\alpha(v +_r w) &= c^\alpha(v) + c^\alpha(w)\\
f^A(v +_r w) &= f^A(v) = f^A(w)\\
e^a(v +_r w) &= e^a(v) + e^a(w)\\
x^i(v +_r w) &= x^i(v) = x^i(w).
\end{aligned}$$

Examples.

1. Let $\mathbf{E} = (E, \bar{\tau}_l, M)$, $\mathbf{F} = (F, \bar{\tau}_r, M)$, $\mathbf{C} = (C, \tau, M)$ be vector bundles and let $K = F \times_M C \times_M E$. By $\mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$ we denote a double vector bundle represented by the diagram



, (2)

where $\tau_r \colon K = F \times_M C \times_M E \to E$ and $\tau_l \colon K = F \times_M C \times_M E \to F$ are the canonical projections. The right and left vector bundle structures are obvious:

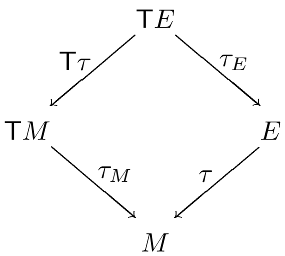
$$(f, k, e) +_r (f', k', e) = (f + f', k + k', e)$$
$$(f, k, e) +_l (f, k', e') = (f, k + k', e + e').$$

The core of $\mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$ can be identified with $\mathbf{C}$.

2. Let $\mathbf{E} = (E, \tau, M)$ be a vector bundle. The tangent manifold $\mathsf{T}E$ has two vector bundle structures ([3]):

the canonical vector bundle structure of the tangent bundle, with respect to the projection $\tau_E \colon \mathsf{T}E \to E$,
the tangent vector bundle structure with respect to the projection $\mathsf{T}\tau \colon \mathsf{T}E \to \mathsf{T}M$.

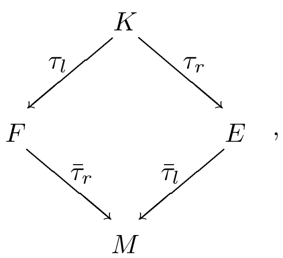
It is easy to verify that the diagram



(3)

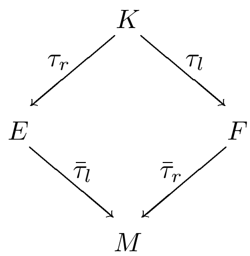
represents a double vector bundle. We denote this double vector bundle by $\mathsf{T}\mathbf{E}$. The core consists of vertical tangent vectors at the zero section of $\mathbf{E}$. Thus, it can be identified, in an obvious way, with $\mathbf{E}$.

3. Let $\mathbf{K}$ be a double vector bundle represented by the diagram



, (4)

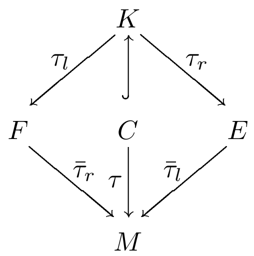
then the diagram



(5)

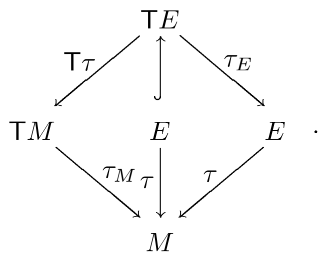
also represents a double vector bundle. We denote it by $\mathsf{J}(\mathbf{K})$.

In examples (1) and (2) we identified canonically the core with a certain vector bundle. In the following we shall write a diagram



(6)

instead of the diagram (1), if we identify the core of (1) with the vector bundle $\mathbf{C}$. In the case of $\mathbf{K} = \mathsf{T}\mathbf{E}$ we write the diagram



(7)

3. Morphisms of double vector bundles.

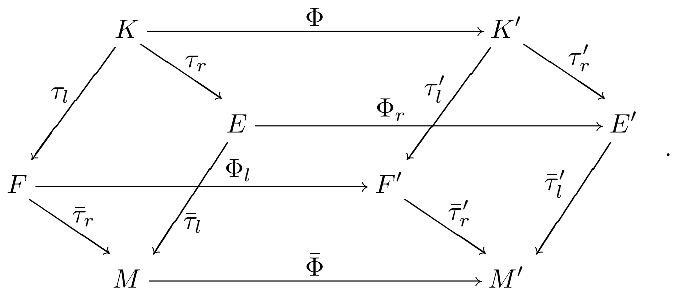
Let $\mathbf{K} = (\mathbf{K}_r, \mathbf{K}_l, \mathbf{E}, \mathbf{F})$ and $\mathbf{K}' = (\mathbf{K}'_r, \mathbf{K}'_l, \mathbf{E}', \mathbf{F}')$ be double vector bundles with cores $\mathbf{C}$ and $\mathbf{C}'$ respectively. A morphism $\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}'$ of double vector bundles is a family $\mathbf{\Phi} = (\Phi, \Phi_r, \Phi_l, \bar{\Phi})$ of morphisms

$$\begin{aligned} \Phi\colon K \to K', \qquad & \Phi_r\colon E \to E', \\ \Phi_l\colon F \to F', \qquad & \bar{\Phi}\colon M \to M', \end{aligned}$$

such that $\mathbf{\Phi}_r = (\Phi, \Phi_r)$, $\mathbf{\Phi}_l = (\Phi, \Phi_l)$, $\bar{\mathbf{\Phi}}_r = (\Phi_r, \bar{\Phi})$ and $\bar{\mathbf{\Phi}}_l = (\Phi_l, \bar{\Phi})$ are morphisms of vector bundles

$$\begin{aligned} \mathbf{\Phi}_r\colon \mathbf{K}_r \to \mathbf{K}'_r, \qquad & \mathbf{\Phi}_l\colon \mathbf{K}_l \to \mathbf{K}'_l, \\ \bar{\mathbf{\Phi}}_r\colon \mathbf{E} \to \mathbf{E}', \qquad & \bar{\mathbf{\Phi}}_l\colon \mathbf{F} \to \mathbf{F}'. \end{aligned}$$

We have thus a commutative diagram



(8)

Proposition 4. *Let $\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}'$ be a morphism of double vector bundles. Then*

(1) $\Phi(\ker \tau_r) \subset \ker \tau'_r$,
(2) $\Phi(\ker \tau_l) \subset \ker \tau'_l$,
(3) $\Phi(C) \subset C'$.

PROOF: Since $\bar{\mathbf{\Phi}}_r\colon \mathbf{E} \to \mathbf{E}'$ is a morphism of vector bundles, it maps the zero section of $\mathbf{E}$ into the zero section of $\mathbf{E}'$. It follows that $\Phi(\ker\tau_r) \subset \ker\tau'_r$. Similarly, $\Phi(\ker\tau_l) \subset \ker\tau'_l$ and, consequently, $\Phi(C) \subset C'$. ■

We denote by $\mathbf{\Phi}_c = (\Phi_c, \bar{\Phi})$ the induced by Φ morphism of vector bundles $\mathbf{C}$ and $\mathbf{C}'$.

Let $(x^i, e^a, f^A, c^\alpha)$ be an adapted local coordinate system on K and let $(\bar{x}^{\bar{i}}, \bar{e}^{\bar{a}}, \bar{f}^{\bar{A}}, \bar{c}^{\bar{\alpha}})$ be an adapted coordinate system in K'. We have, that in local coordinate systems,

$$\begin{aligned}
\bar{x}^{\bar{i}} \circ \Phi &= \Phi^{\bar{i}}, \\
\bar{e}^{\bar{a}} \circ \Phi &= \Phi^{\bar{a}}_b e^b, \\
\bar{f}^{\bar{A}} \circ \Phi &= \Phi^{\bar{A}}_B f^B, \\
\bar{c}^{\bar{\alpha}} \circ \Phi &= \Phi^{\bar{\alpha}}_\beta c^\beta + \Phi^{\bar{\alpha}}_{aA} e^a f^A,
\end{aligned} \tag{9}$$

where $\Phi^{\bar{i}}$, $\Phi^{\bar{a}}_b$, $\Phi^{\bar{A}}_B$, $\Phi^{\bar{\alpha}}_\beta$, $\Phi^{\bar{\alpha}}_{aA}$ are functions on the domain of (x^i) in M.

Examples.

1. Let $\mathbf{E} = (E, \tau, M)$ and $\mathbf{E}' = (E', \tau', M')$ be vector bundles and let $\mathbf{\Phi} = (\Phi, \bar{\Phi})$ be a vector bundle morphism

$$\mathbf{\Phi}\colon \mathbf{E} \to \mathbf{E}'.$$

The quadruple of mappings $\mathsf{T}\mathbf{\Phi} = (\mathsf{T}\Phi, \Phi, \mathsf{T}\bar{\Phi}, \bar{\Phi})$ is a morphism of double vector bundles

$$\mathsf{T}\mathbf{\Phi}\colon \mathsf{T}\mathbf{E} \to \mathsf{T}\mathbf{E}'$$

and, with the identification of cores as in the diagram (7), we have $\mathbf{\Phi}_c\colon \mathbf{E} \to \mathbf{E}'$, $\Phi_c = \Phi$.

2.An essential role in the Lagrangian formulation of a classical mechanical system is played by the isomorphism κ_M, which relates $\mathsf{TT}M$ with $\mathsf{J}(\mathsf{TT}M)$ ([3]). Here, $\mathsf{T}M$ is the vector bundle of tangent vectors. All three vector bundle morphisms $\bar{\mathbf{\Phi}}_r, \bar{\mathbf{\Phi}}_l, \mathbf{\Phi}_c\colon \mathsf{T}M \to \mathsf{T}M$ are the identities.

3. Let be $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$ and $\mathbf{K}' = \mathbf{K}(\mathbf{F}', \mathbf{C}', \mathbf{E}')$. If $\mathbf{\Phi} = (\Phi, \Phi_r, \Phi_l, \bar{\Phi})$ is a morphism of double vector bundles,

$$\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}', \tag{10}$$

then

$$\Phi(f, c, e) = (\Phi_l(f), \Phi_c(c) + \Psi(f, e), \Phi_r(e)),$$

where

$$\Psi\colon \mathbf{F} \times_M \mathbf{E} \to \mathbf{C}'$$

is bilinear.

4. The right dual.

Let $\mathbf{K}^*_r$be the vector bundle, dual to $\mathbf{K}_r$. We denote by K^{*_r} the total fiber bundle space and by π_l the projection

$$\pi_l\colon K^{*_r} \to E.$$

Let $a \in K^{*_r}$ and $k \in C$ satisfy $\tau(k) = \bar{\tau}_l(\pi_l(a))$. We can evaluate a on a vector $(\pi_l(a), k)$ of $\ker\tau_l$. We define a mapping $\pi_r\colon K^{*_r} \to C^*$ by the formula

$$\langle k, \pi_r(a)\rangle = \langle(\pi_l(a), k), a\rangle. \tag{11}$$

It follows directly from this construction that

Proposition 5. *The mapping $\pi_r\colon K^{*_r} \to C^*$ is a morphism of vector bundles*

$$\pi_r\colon \mathbf{K}_r^* \to \mathbf{C}^*.$$

We define a relation

$$P_r(m_l)\colon K^{*_r} \times K^{*_r} \to K^{*_r}$$

in the following way: $c \in P_r(m_l)(a,b)$ if

(1) $\pi_l(c) = \pi_l(a) + \pi_l(b)$,
(2) $\langle w, c\rangle = \langle v, a\rangle + \langle v', b\rangle$ for each $w, v, v' \in K$ such that $\tau_r(w) = \pi_l(c)$, $\tau_r(v) = \pi_l(a)$, $\tau_r(v') = \pi_l(b)$ and $w = m_l(v, v')$.

Local coordinates. Let $(x^i, e^a, f^A, c^\alpha)$ be an adapted coordinate system on K and let $(x^i, e^a, p_A, q_\alpha)$ be the adopted coordinate system on the dual bundle K^{*_r}, i. e., the canonical evaluation is given by the formula

$$\langle v, a\rangle = \sum_A p_A(a) f^A(v) + \sum_\alpha q_\alpha(a) c^\alpha(v). \tag{12}$$

We use (x^i, c^α) as a coordinate system in C and (x^i, q_α) as a coordinate system on C^*. In these coordinate sytems we have

$$\begin{aligned} x_i \circ \pi_r(a) &= x_i(a), \\ q_\alpha \circ \pi_r(a) &= q_\alpha(a) \end{aligned} \tag{13}$$

and

$$\begin{aligned} x^i(a +_r b) &= x^i(a) = x^i(b), \\ e^a(a +_r b) &= e^a(a) + e^a(b), \\ p_A(a +_r b) &= p_A(a) + p_A(b), \\ q_\alpha(a +_r b) &= q_\alpha(a) = q_\alpha(b). \end{aligned} \tag{14}$$

It follows that (K^{*_r}, π_r, C^*) is a vector bundle. We denote it by $\mathbf{K}_r^{*_r}$. The vector bundle (K^{*_r}, π_l, E) we denote by $\mathbf{K}_l^{*_r}$.

Theorem 6. *The system $\mathbf{K}^{*_r} = (\mathbf{K}_r^{*_r}, \mathbf{K}_l^{*_r}, \mathbf{C}^*, \mathbf{E})$ is a double vector bundle.*

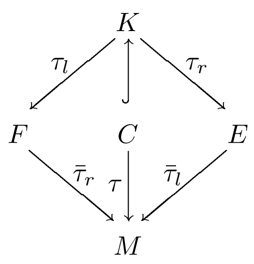
We identify the kernel $\ker \tau_r$ with $\mathbf{F} \oplus_M \mathbf{C}$ and, consequently, the kernel of π_l with $\mathbf{C}^* \oplus_M \mathbf{F}^*$. With this identifications we have

$$\langle (f, c), (c^*, f^*)\rangle = \langle f, f^*\rangle + \langle c, c^*\rangle \tag{15}$$

and that π_r is the canonical projection $\mathbf{C}^* \oplus \mathbf{F}^* \to \mathbf{C}^*$. It follows that the core of $\mathbf{K}^{*_r}$ can be identified with $\mathbf{F}^*$ and that

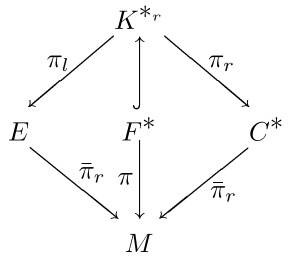
$$\ker \pi_r = \mathbf{E} \oplus \mathbf{F}^*.$$

Now, we can write the diagram for $\mathbf{K}^{*_r}$. If $\mathbf{K}$ is represented by the diagram



(16)

then the right dual is represented by the diagram



(17)

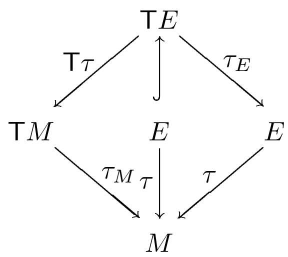
The left dual. A similar construction shows that the left dual K^{*_l} has natural structure of double vector bundle. We denote it by $\mathbf{K}^{*_l} = (\mathbf{K}_r^{*_l}, \mathbf{K}_l^{*_l}, \mathbf{F}, \mathbf{C}^*)$. The core of $\mathbf{K}^{*_l}$ is $\mathbf{E}^*$. There is an obvious identity

$$\mathsf{J}(\mathbf{K}^{*_l}) = (\mathsf{J}(\mathbf{K}))^{*_r}. \tag{18}$$

Examples.

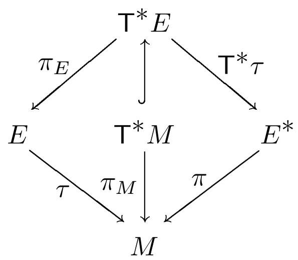
1. Let $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$. The right dual can be canonically identified with $\mathbf{K}(\mathbf{E}, \mathbf{F}^*, \mathbf{C}^*)$ and the left dual with $\mathbf{K}(\mathbf{C}^*, \mathbf{E}^*, \mathbf{F})$.

2. If $\mathbf{E} = (E, \tau, M)$ is a vector bundle, then $\mathsf{T}\mathbf{E}$ is a double vector bundle with the diagram



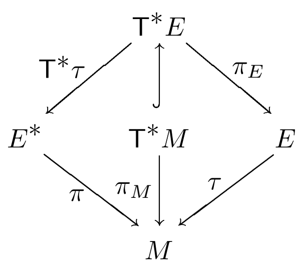
(19)

Its right dual $(\mathsf{T}\mathbf{E})^{*_r}$ is represented by the diagram



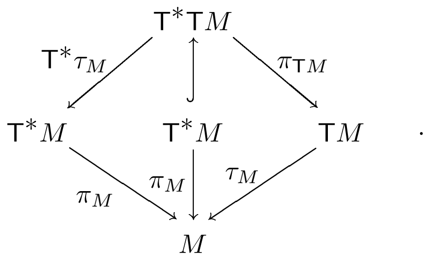
(20)

We see that the manifold of cotangent vectors to a vector bundle has two compatible vector bundle structures. The double vector bundle $\mathsf{J}((\mathsf{T}\mathbf{E})^{*_r})$, represented by the diagram



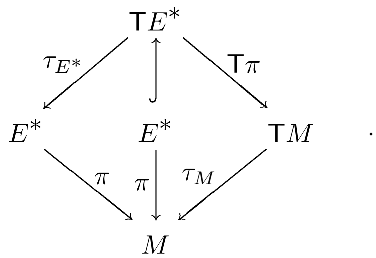
(21)

will be denoted by $\mathsf{T}^*\mathbf{E}$. In particular, for $\mathbf{E} = \mathsf{T}M$, the diagram (21) assumes the form



(22)

3. In order to identify the left dual of $\mathsf{T}\mathbf{E}$ let us recall that the dual to the vector bundle $(\mathsf{T}E, \mathsf{T}\tau, \mathsf{T}M)$ can be canonically identified with $(\mathsf{T}E^*, \mathsf{T}\pi, \mathsf{T}M)$ ([3]). It follows that the left dual to $\mathsf{T}\mathbf{E}$ is canonically isomorphic to $\mathsf{J}(\mathsf{T}\mathbf{E}^*)$ with the diagram



(23)

5. Dual morphisms.

Since the dual to a vector bundle mapping is, in general, not a mapping but a relation, we discuss in this section the case of isomorphisms only. It follows from the formulae (9) that $\mathbf{\Phi}$ is an isomorphism of double vector bundles if and only if $\bar{\mathbf{\Phi}}_r$, $\bar{\mathbf{\Phi}}_l$ and $\bar{\mathbf{\Phi}}_c$ are isomorphisms of vector bundles.

Let $\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}'$ be an isomorphism of double vector bundles. Then the right dual $\mathbf{\Phi}_r^*\colon (\mathbf{K}_r')^* \to (\mathbf{K}_r)^*$ is a morphism of vector bundles. The corresponding mapping of bundle spaces we denote by Φ^{*_r}.

Proposition 7. *Φ^{*_r} defines an isomorphism of vector bundles ${\mathbf{K}'}_r^{*_r}$ and $\mathbf{K}_r^{*_r}$.*

Corollary 8. *We have the following equalities for $\mathbf{\Psi} = \mathbf{\Phi}^{*_r}$*

$$\begin{aligned} \bar{\mathbf{\Psi}}_r &= (\bar{\mathbf{\Phi}}_c)^* \\ \bar{\mathbf{\Psi}}_l &= (\bar{\mathbf{\Phi}}_r)^{-1} \\ \bar{\mathbf{\Psi}}_c &= (\bar{\mathbf{\Phi}}_l)^* \end{aligned} \tag{24}$$

Examples.

1. Let $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$, $\mathbf{K}' = \mathbf{K}(\mathbf{F}', \mathbf{C}', \mathbf{E}')$ and $\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}'$, with

$$\Phi(f, c, e) = (\Phi_l(f), \Phi_c(c) + \Psi(f, e), \Phi_r(e)). \tag{25}$$

Since we identify $\mathbf{K}^{*_r}$ with $\mathbf{K}(\mathbf{E}, \mathbf{F}^*, \mathbf{C}^*)$ and $(\mathbf{K}')^{*_r}$ with $\mathbf{K}(\mathbf{E}', (\mathbf{F}')^*), (\mathbf{C}')^*$, we have

$$\mathbf{\Phi}^{*_r}\colon \mathbf{K}(\mathbf{E}', (\mathbf{F}')^*, (\mathbf{C}')^*) \to \mathbf{K}(\mathbf{E}, \mathbf{F}^*, \mathbf{C}^*).$$

One can easily verify the equalitity

$$\Phi^{*_r} = (\Phi_l^{*_r}, \Phi_c^{*_r} + \Psi^{*_r}, \Phi_r^{*_r}),$$

where

$$\begin{aligned} \Phi_r^{*_r}(q') &= \Phi_c^*(q'), \\ \Phi_l^{*_r}(e') &= \Phi_r^{-1}(e'), \\ \Phi_c^{*_r}(p') &= \Phi_l^*(p'), \\ \Psi^{*_r}(e', q') &= \Psi^*(q', \Phi_r^{-1}(e')), \end{aligned} \tag{26}$$

and Ψ^* is the vector bundle morphism, dual to Ψ with respect to the left argument, i. e., with respect to F.

2.([3]) We have (Section 3.) an isomorphism of double vector bundles

$$\kappa_M \colon \mathsf{T}\mathsf{T}M \to \mathsf{J}(\mathsf{T}\mathsf{T}M).$$

The right dual

$$(\kappa_M)^{*_r} \colon \mathsf{T}\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}M$$

is usually denoted by α_M and plays a crucial role in the Lagrangian formulation of the dynamics of mechanical systems.

6. Canonical isomorphisms.

Proposition 9. *We have*

$$(\mathbf{K}^{*_r})^{*_l} = (\mathbf{K}^{*_l})^{*_r} = \mathbf{K}. \tag{27}$$

Proof: It follows from the construction that we can identify manifolds $(K^{*_r})^{*_l}$ and K. Also the right vector bundle structures coincide. Let $\Phi\colon K \to (K^{*_r})^{*_l}$ be the canonical diffeomorphism and let ϑ_r, ϑ_l be projections in $(\mathbf{K}^{*_r})^{*_l}$. For $f^* \in F^* = \ker \pi_r \cap \ker \pi_l$ we have

$$\langle f^*, \vartheta_l(\Phi(v))\rangle = \langle(\tau_r(v), f^*), \Phi(v)\rangle = \langle v, (e, f^*)\rangle = \langle \tau_l(v), f^*\rangle,$$

hence $\vartheta_l(\Phi(v)) = \tau_l(v)$. Equality of the left vector bundle structures follows from

$$\langle a +_r b, \Phi(v) +_l \Phi(w)\rangle = \langle a, \Phi(v)\rangle + \langle b, \Phi(w)\rangle = \langle v, a\rangle + \langle w, b\rangle = \langle v +_l w, a +_r b\rangle.$$

■

In the following, we show that there is a canonical isomorphism of double vector bundles $\mathbf{K}$ and $((\mathbf{K}^{*_r})^{*_r})^{*_r}$. Through this section we denote by $\tau_r, \pi_r, \xi_r, \vartheta_r$ and by $\tau_l, \pi_l, \xi_l, \vartheta_l$ the right and left projections in $\mathbf{K}$, $\mathbf{K}^{*_r}$, $(\mathbf{K}^{*_r})^{*_r}$, $((\mathbf{K}^{*_r})^{*_r})^{*_r}$ respectively. Identifying vector bundles with their second duals, we have

$$\begin{aligned} \xi_r &\colon (K^{*_r})^{*_r} \to F & \xi_l &\colon (K^{*_r})^{*_r} \to C^* \\ \vartheta_r &\colon ((K^{*_r})^{*_r})^{*_r} \to E & \vartheta_l &\colon ((K^{*_r})^{*_r})^{*_r} \to F. \end{aligned}$$

The core of $(\mathbf{K}^{*_r})^{*_r}$ is $\mathbf{E}^*$ and the core of $((\mathbf{K}^{*_r})^{*_r})^{*_r}$ is $(\mathbf{C}^*)^* = \mathbf{C}$.

We define a relation $\mathcal{R}_K \subset K \times ((K^{*_r})^{*_r})^{*_r}$ in the following way:

Let $v \in K$, $\varphi \in ((K^{*_r})^{*_r})^{*_r}$ be such that $\bar\tau_l(\tau_r(v)) = \bar\vartheta_l(\vartheta_r(\varphi))$. We say that $(v, \varphi) \in \mathcal{R}_K$ if for each $a \in K^{*_r}$, $\alpha \in (K^{*_r})^{*_r}$ such that

$$\tau_r(v) = \pi_l(a),\ \pi_r(a) = \xi_l(\alpha),\ \xi_r(\alpha) = \vartheta_l(\varphi)$$

we have

$$\langle a, \alpha\rangle = \langle v, a\rangle + \langle \alpha, \varphi\rangle. \tag{28}$$

Theorem 10. *The relation $\mathcal{R}_K$ is an isomorphism of double vector bundles.*

If we replace the right-hand side of (28) by a different combination of $\langle v, a\rangle$ and $\langle \alpha, \varphi\rangle$, we obtain another isomorphism. The isomorphism corresponding to $\langle v, a\rangle - \langle \alpha, \varphi\rangle$ we denote by $\mathcal{R}^{\pm}_K$, the isomorphism corresponding to $-\langle v, a\rangle + \langle \alpha, \varphi\rangle$ we denote by $\mathcal{R}^{\mp}_K$ and the isomorphism corresponding to $-\langle v, a\rangle - \langle \alpha, \varphi\rangle$ we denote by $\mathcal{R}^{=}_K$.

Proposition 11.

(1) $\varphi = \mathcal{R}^{\pm}_K(v)$ *iff* $\varphi = \mathcal{R}_K((-1)\cdot_r v)$ *or equivalently,* $(-1)\cdot_l \varphi = \mathcal{R}_K(v)$,
(2) $\varphi = \mathcal{R}^{\mp}_K(v)$ *iff* $\varphi = \mathcal{R}_K((-1)\cdot_r v)$ *or equivalently,* $(-1)\cdot_r \varphi = \mathcal{R}_K(v)$,
(3) $\varphi = \mathcal{R}^{=}_K(v)$ *iff* $(-1)\cdot_l \varphi = \mathcal{R}_K((-1)\cdot_r v)$ *or equivalently,* $(-1)\cdot_l ((-1)\cdot_r \varphi) = \mathcal{R}_K(v)$.

Examples.

1. Let $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$. Using the identifications: $\mathbf{K}^{*_r} = \mathbf{K}(\mathbf{E}, \mathbf{F}^*, \mathbf{C}^*)$, $\mathbf{E}^{**} = \mathbf{E}$, $\mathbf{F}^{**} = \mathbf{F}$, and $\mathbf{C}^{**} = \mathbf{C}$, we get

$$\mathbf{K}^{*_r *_r} = \mathbf{K}(\mathbf{C}^*, \mathbf{E}^*, \mathbf{F})$$

and

$$\mathbf{K}^{*_r *_r *_r} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E}).$$

Thus, we have obtained another identification of $\mathbf{K}$ and $\mathbf{K}^{*_r *_r *_r}$. With this identification the formula (28) looks like follows

$$\begin{aligned} &\langle (e, \varphi, \gamma), (\gamma, \varepsilon, \bar{f}) \rangle = \langle (f, c, e), (e, \varphi, \gamma) \rangle + \langle (\gamma, \varepsilon, \bar{f}), (\bar{f}, \bar{c}, \bar{e}) \rangle, \\ &\langle e, \varepsilon \rangle + \langle \bar{f}, \varphi \rangle = \langle f, \varphi \rangle + \langle c, \gamma \rangle + \langle \bar{c}, \gamma \rangle + \langle \bar{e}, \varepsilon \rangle, \end{aligned} \tag{29}$$

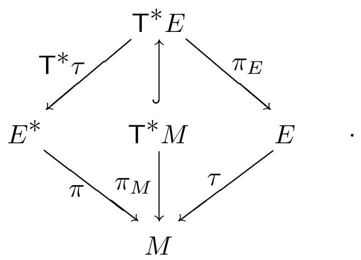
where $e, \bar{e} \in E, f, \bar{f} \in F, c, \bar{c} \in C, \varepsilon \in E^*, \varphi \in F^*, \psi \in C^*$. Hence, $e = \bar{e}, f = \bar{f}, c = -\bar{c}$, and, consequently,

$$\mathcal{R}_K(f, c, e) = (f, -c, e). \tag{30}$$

Analogously,

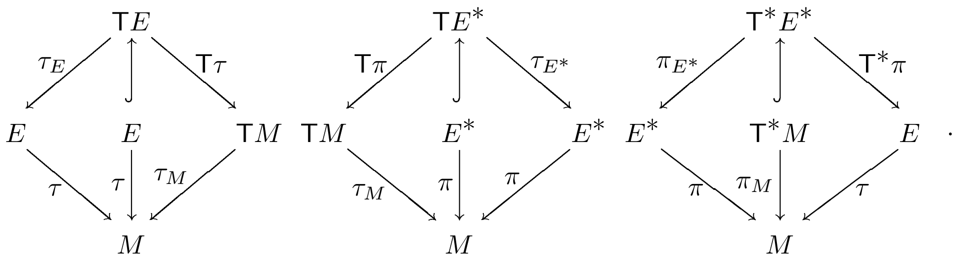
$$\begin{aligned} \mathcal{R}^{\pm}_K(f, c, e) &= (f, c, -e) \\ \mathcal{R}^{\mp}_K(f, c, e) &= (-f, c, e) \\ \mathcal{R}^{=}_K(f, c, e) &= (-f, -c, -e). \end{aligned} \tag{31}$$

2. Let $\mathbf{K} = \mathsf{T}^*\mathbf{E}$:



(32)

Then the first, second and third right duals can be identified with double vector bundles $\mathsf{J}(\mathsf{T}E)$, $\mathsf{T}E^*$ and $\mathsf{J}(\mathsf{T}^*E^*)$, represented by the diagrams



(33)

Thus the canonical isomorphisms $\mathcal{R}_K$, $\mathcal{R}^{\pm}_K$, $\mathcal{R}^{\mp}_K$, $\mathcal{R}^{=}_K$ define diffeomorphisms from T^*E^* to T^*E. For $\mathcal{R}_K$, $\mathcal{R}^{=}_K$ these diffeomorphisms are antisymplectomorphisms with respect to the canonical symplectic structure of the cotangent bundle and for $\mathcal{R}^{\pm}_K$, $\mathcal{R}^{\mp}_K$ we obtain symplectomorphisms.

Identification of isomorphisms. Let $\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}'$ be an isomorphism of double vector bundles. We have

$$\mathbf{\Phi}^{*_r *_r *_r}\colon (\mathbf{K}')^{*_r *_r *_r} \to \mathbf{K}^{*_r *_r *_r}.$$

Using one of the introduced isomorphisms of a double vector bundle and its third right dual, we can compare $\mathbf{\Phi}$ and its third right dual.

Proposition 12. *We have the following equality*

$$\mathcal{R}_K^{-1} \circ \Phi^{*_r *_r *_r} \circ \mathcal{R}_{K'} = \Phi^{-1}. \tag{34}$$

In this formula $\mathcal{R}$ can be replaced by $\mathcal{R}^{\mp}$, $\mathcal{R}^{\pm}$ and $\mathcal{R}^{=}$.

Remark. In the case of $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$ and $\mathbf{K}' = \mathbf{K}(\mathbf{F}', \mathbf{C}', \mathbf{E}')$, we have another isomorphisms of $\mathbf{K}$ and $\mathbf{K}^{*_r *_r *_r}$, $\mathbf{K}'$ and $(\mathbf{K}')^{*_r *_r *_r}$ (see Example 1 of this section). It easy to verify that, with respect to these isomorphisms, $\Phi^{*_r *_r *_r}$ does not correspond to Φ^{-1}.

7. Special symplectic manifolds.

Let $\mathbf{E} = (E, \tau, M)$ be a vector bundle and let ω be a 2-form on E. By $\widetilde{\omega}$ we denote the corresponding vector bundle morphism

$$\widetilde{\omega}: \mathsf{T}E \to \mathsf{T}^*E. \tag{35}$$

We say that ω is linear with respect to the vector bundle structure $\mathbf{E}$ if $\widetilde{\omega}$ is a morphism double vector bundles

$$\widetilde{\omega}: \mathsf{T}\mathbf{E} \to \mathsf{T}^*\mathbf{E}. \tag{36}$$

If ω is linear then there are also three vector bundle morphisms:

$$\begin{aligned} &\widetilde{\omega}_r: E \to E, \\ &\widetilde{\omega}_l: \mathsf{T}M \to E^*, \\ &\widetilde{\omega}_c: E \to \mathsf{T}^*M. \end{aligned}$$

Of course, $\widetilde{\omega}_r = \mathrm{id}_E$ and, because $\widetilde{\omega}$ is skew-symmetric, we have, from (24),

$$\widetilde{\omega}_l = -\widetilde{\omega}_c^*.$$

Proposition 13. *ω is closed if and only if the pull-back of the canonical symplectic form ω_M on T^*M by $\widetilde{\omega}_c$ is equal ω:*

$$\omega = \widetilde{\omega}_c^* \omega_M. \tag{37}$$

If ω is nondegenerate, i.e., $\widetilde{\omega}$ is an isomorphism of vector bundles, then also $\widetilde{\omega}_c$ is an isomorphism. In that case $\widetilde{\omega}_c$ is a symplectomorphism. Thus, we can consider the pair $(\mathbf{E}, \omega)$ as a *special symplectic manifold* ([4], [5]).

8. Linear connections.

A connection on a vector bundle $\mathbf{E}$ can be regarded as a spliting of the tangent bundle $\mathsf{T}E$ into the vertical and horizontal parts. Since the bundle $\mathsf{V}E$ of vertical vectors can be identified with the product $\mathbf{E} \times_M E$, we can look at the splitting map as an isomorphism of vector bundles

$$D: \mathsf{T}E \to (\mathsf{T}M \oplus_M \mathbf{E}) \times_M E \tag{38}$$

over the identity of E.

Proposition 14. *A mapping $D: \mathsf{T}E \to (\mathsf{T}M \oplus_M \mathbf{E}) \times_M E$ is the splitting related to a linear connection if and only if D defines a double vector bundle morphism*

$$\mathbf{D}: \mathsf{T}\mathbf{E} \to \mathbf{K}(\mathsf{T}M, \mathbf{E}, \mathbf{E}) \tag{39}$$

such that the corresponding mappings

$$\begin{aligned} &D_r: E \to E \\ &D_l: \mathsf{T}M \to \mathsf{T}M \\ &D_c: E \to E \end{aligned} \tag{40}$$

are identities.

Let D be the the splitting of a linear connection on $\mathbf{E}$. The transposed left dual to D defines an isomorphism

$$\mathbf{D}^*\colon \mathsf{T}\mathbf{E}^* \to \mathbf{K}(\mathsf{T}M, \mathbf{E}^*, \mathbf{E}^*) \tag{41}$$

and, because of (40) and (24), D^*_r, D^*_l, D^*_c are identities. Thus $\mathbf{D}^*$ is the splitting of a linear connection on $\mathbf{E}^*$. We call it *the dual connection.*

Let $g\colon \mathbf{E} \to \mathbf{E}^*$ be a metric on $\mathbf{E}$ (g is a self-adjoint isomorphism of vector bundles). The splitting D is the splitting of a metric connection if the following diagram is commutative

$$\begin{array}{ccc} \mathsf{T}E & \xrightarrow{\quad D \quad} & \mathbf{K}(\mathsf{T}M, \mathbf{E}, \mathbf{E}) \\ \mathsf{T}g \downarrow & & \downarrow \mathrm{id}_{\mathsf{T}M} \times g \times g \\ \mathsf{T}E^* & \xrightarrow{\quad D^* \quad} & \mathbf{K}(\mathsf{T}M, \mathbf{E}^*, \mathbf{E}^*) \end{array} \tag{42}$$

9. Torsion-free connections.

In this section $\mathbf{E} = \mathsf{T}M$. We have then the canonical isomorphism

$$\kappa_M\colon \mathsf{T}\mathsf{T}M \to \mathsf{J}(\mathsf{T}\mathsf{T}M).$$

We introduce also an isomorphism

$$\kappa\colon \mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M) \to \mathsf{J}(\mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M))$$

by

$$\kappa(v, w, u) = (u, w, v).$$

PROPOSITION 15. *A connection D is torsion-free if and only if*

$$\kappa \circ \mathbf{D} = \mathsf{J}(\mathbf{D}) \circ \kappa_M \tag{43}$$

i. e., if the following diagram is commutative

$$\begin{array}{ccc} \mathsf{T}\mathsf{T}M & \xrightarrow{\quad \mathbf{D} \quad} & \mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M) \\ \kappa_M \downarrow & & \downarrow \kappa \\ \mathsf{J}(\mathsf{T}\mathsf{T}M) & \xrightarrow{\quad \mathsf{J}(\mathbf{D}) \quad} & \mathsf{J}(\mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M)) \end{array} \;. \tag{44}$$

The diagram (44) is commutative if and only if the diagram of left duals is commutative. The commutativity of the diagram of left duals is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} \mathsf{T}\mathsf{T}^*M & \xrightarrow{\quad \mathbf{D}^* \quad} & \mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M) \\ \beta_M \downarrow & & \beta \downarrow \\ \mathsf{T}^*\mathsf{T}^*M & \xleftarrow{\quad \mathsf{J}((\mathbf{D}^*)^{*_r}) \quad} & \mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M) \end{array} \;, \tag{45}$$

where $\beta(v, g, f) = (-v, f, g)$ and β_M is the canonical symplectic structure on T^*M. The evaluation in $\mathbf{K}(\mathsf{T}, \mathsf{T}^*M, \mathsf{T}^*M)$ is given by the formula

$$\langle (v, f, g), (w, h, g) \rangle = \langle v, h \rangle + \langle w, f \rangle.$$

For a subbundle $W \subset \mathbf{K}(\mathsf{T}, \mathsf{T}^*M, \mathsf{T}^*M)$ defined by $W = \{(v, f, g)\colon f = 0\}$, we have

$$W^\circ = \beta(W).$$

One can easily see that the diagram (45) is commutative if and only if

$$\beta_M V = V^\circ \tag{46}$$

for $V = (D^*)^{-1}(W)$. This equality means that V is a lagrangian distribution on T^*M and, since V is the horizontal distribution of the connection D^*, we get

Proposition 16. *A connection D on $\mathsf{T}M$ is symmetric if and only if the horizontal distribution of the dual connection D^* is lagrangian with respect to the canonical symplectic structure on T^*M.*

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