ON EFFECTIVE METHODS IN INVESTIGATION OF QUANTUM OPERATIONS AND PROCESSES

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Outline

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DECOHERENCE-FREE SUBSPACES

Decoherence is a non-unitary dynamics of open quantum systems that is a consequence of system – environment coupling.

Let Φ denote a superoperator on $\mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} = d$, which is completely positive. Then there exist some operators $K_1, \ldots, K_\eta$ on $\dim \mathcal{H} = d$ such that

$$\Phi(X) = \sum_{j=1}^{\eta} K_j X K_j^*,$$

and $\eta \leq d^2$. If $\sum K_j^* K_j \leq \mathbb{I}$, then Φ is called the quantum operation. A decoherence-free subspace (DFS) is a subspace of the space $\mathcal{H}$ that is invariant to non-unitary dynamics.

DECOHERENCE-FREE SUBSPACES

Let $\mathcal{A}$ denote subalgebra of the full algebra $M_d(\mathbb{C})$ generated by operators $K_1, \dots, K_\eta$. Then $\mathcal{A}$ is called the interaction algebra of the superoperator Φ.

In a similar way, evolution of open quantum systems continuous in time may be described by Hamiltonians of the form (closed system formulation)

$$H = H_s \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_I,$$

where H_I denotes the interaction term which can be written in general as

$$H_I = \sum_{\alpha}^{\omega} S_\alpha \otimes E_\alpha.$$

Now, interaction algebra is defined by $S_1, \dots, S_\omega$.

DECOHERENCE-FREE SUBSPACES

In the so-called master equation description of time evolution we assume dynamics of the form

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}[H_S, \rho_S(t)] + L_D[\rho_S(t)],$$

where

$$L_D[\rho_S(t)] = \frac{1}{2} \sum_{\alpha,\beta=1}^{M} A_{\alpha\beta} \left\{ [F_\alpha, \rho_S(t) F_\beta^*] + [F_\alpha \rho_S(t), F_\beta^*] \right\}.$$

General observations.

In all above descriptions we will have a part of the Hilbert space $\mathcal{H}$ decoherence-free (decoherence-free subspace) if and only if the set of operators $\{K_1, \ldots, K_\eta\}$ or $\{S_1, \ldots, S_\omega\}$ or $\{F_1, \ldots, F_M\}$ have invariant subspaces of degree at least 2.

DECOHERENCE-FREE SUBSPACES

Let K_1, K_2 be given $d \times d$ complex matrices. We formulate the following question.

> *Is it possible to verify whether K_1 and K_2 have – or do not have – a common invariant subspace of dimension $1 < m < d$, by an effective procedure?*

For $m = 1$ an answer to this question was given by Shemesh in 1984. For us the case when m is bigger than 1 is interesting.

DECOHERENCE-FREE SUBSPACES

The matrices K_1, K_2 have a common eigenvector if and only if the subspace of $\mathcal{H}$

$$\mathcal{M}_1 := \bigcap_{\alpha,\beta}^{d-1} \mathrm{Ker}[K_1^\alpha, K_2^\beta]$$

is nontrivial: $\mathcal{M}_1 \neq \{\mathbf{0}\}$.

Theorem (Shemesh' criterion)

The above inequality is equivalent to the following geometrical condition. The matrices K_1 and K_2 have a common eigenvector if and only if the $d \times d$ matrix

$$\Omega := \sum_{\alpha,\beta=1}^{d-1} [K_1^\alpha, K_2^\beta]^*[K_1^\alpha, K_2^\beta]$$

is singular, i.e. $\det \Omega = 0$.

POLYNOMIAL IDENTITIES

Now, using the concept of the so-called polynomial identities (PI) and the Amitsur-Levitzki theorem one can generalize the above theorem.

Recall that one says that a polynomial $P(X_1, \ldots, X_r)$ in noncommuting variables defines an identity on an algebra $\mathcal{A}$, if $P(A_1, \ldots, A_r) = 0$ for any $A_1, \ldots, A_r$ that belong to the algebra $\mathcal{A}$.
In particular, the standard polynomial of degree r is the polynomial in noncommutating variables $X_1, \ldots, X_r$ of the form

$$\boxed{S_r(X_1, \ldots, X_r) := \sum \operatorname{sign}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(r)} \qquad (*)}$$

The summation here is assumed over all permutations of $1, \ldots, r$.

POLYNOMIAL IDENTITIES

Let $\mathcal{A}$ be the set of all n by n matrices, $\mathcal{A} = M_n(\mathbb{C})$. By the celebrated Amitsur-Levitzki theorem if $k \geq n$ then the equality

$$\boxed{S_{2k}(N_1, \ldots, N_{2k}) = 0}$$

holds for any $(2k)$-tuple of matrices $N_1, \ldots, N_{2k} \in M_n(\mathbb{C})$. Moreover, for every $n \geq k + 1$, there exists a $(2k)$-tuple of $n \times n$ matrices $P_1, \ldots, P_{2k}$, such that

$$\boxed{S_{2k}(P_1, \ldots, P_{2k}) \neq 0.}$$

In other words, the full matrix algebra $M_n(\mathbb{C})$ satisfies the standard identity (*) with $r = 2n$. The algebra $M_n(\mathbb{C})$ does not satisfy any polynomial identity of degree less than $2n$.

POLYNOMIAL IDENTITIES

Theorem (Alpin,Ikramov)

Let subspaces of $\mathcal{H}$ for k = 1,2,3... be defined by

$$\mathcal{M}_k := \bigcap \text{Ker}\left\{S_{2k}(N_1, \ldots, N_{2k})N_{2k+1}\right\},$$

where the intersection is taken over all $(2k+1)$-tuples of matrices $N_1, \ldots, N_{2k+1} \in \mathcal{A}(K_1, K_2)$.Then $\mathcal{M}_k$ is an invariant subspace for the algebra $\mathcal{A}$ and $\mathcal{A}$ satisfies the identity $S_{2k} = 0$ on this subspace. This means that

$$S_{2k}(N_1, \ldots, N_{2k})x = 0,$$

for all $N_1, \ldots, N_{2k} \in \mathcal{A}$ and all $x \in \mathcal{M}_k$.Moreover , $\mathcal{M}_k$ can be found by an effective way.

Outline

Classical definition

By definition, a matrix A is reducible if there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix},$$

where X and Y are square matrices and 0 denotes a block of zeros. A matrix which is not reducible is called irreducible.

There are three main categories of results in Perron and, respectively, Frobenius approach to linear operators which preserve the nonnegative orthant $\mathbb{R}^n_+$:

POSITIVE MATRICES (PERRON)

C I. If A is strictly positive matrix, $A > 0$, i.e., all entries of A satisfy the inequality $a_{ij} > 0$, then

a) the spectral radius of the matrix A, $r(A)$, is a simple eigenvalue of A, greater than the magnitude of any other eigenvalue;

b) there exists a corresponding eigenvector which is positive (componentwise), $Ax = r(A)x$;

c) if $A \leq B$ and $A \neq B$, then $r(A) < r(B)$.

NONNEGATIVE MATRICES (FROBENIUS)

C II. If A is nonnegative matrix, $A \geq 0$, that is some entries a_{ij} can be equal to zero, then

a) the spectral radius $r(A)$ of the matrix A is an eigenvalue of A;

b) there exists a corresponding eigenvector which is nonnegative;

c) if $A \leq B$, then $r(A) \leq r(B)$.

FROBENIUS IRREDUCIBILITY

IRREDUCIBLE MATRICES (FROBENIUS)

C III. If A is irreducible and nonnegative, $A \geq 0$, then we have

a) $r(A)$ is a simple eigenvalue;

b) there exists a corresponding eigenvector which is positive;

c) if $A \leq B$ and $A \neq B$, then $r(A) < r(B)$.

Outline

Spectral properties of superoperators

K-IRREDUCIBILITY

Now we introduce one of the main ideas of the Perron-Frobenius theory both in classical and quantum case. Let V be a real vector space and K a cone in V. Let $\Pi(K)$ denote the set of all maps such that $\Phi(K) \subseteq K$.

For a fixed K in V a natural generalization of the concept of an irreducible matrix is the following:

Linear map Φ is *K-irreducible* if and only if Φ leaves invariant no face of K except $\{\mathbf{0}\}$ and K itself.

In other words, a linear map in $\Pi(K)$ is K-reducible if and only if it leaves invariant a nontrivial face of K.

Spectral properties of superoperators

K-IRREDUCIBILITY

Another, strictly equivalent, definition of K-irreducibility can be given by the following theorem:

An operator $\Phi \in \Pi(K)$ is K-irreducible if and only if no eigenvector of Φ lies on the boundary of K.

In fact, one can say even more: An operator $\Phi \in \Pi(K)$ is K-irreducible if and only if Φ has exactly one (up to scalar multiples) eigenvector in K and this vector belongs to K^o – the interior of K.

Moreover, for any proper cone K we have

$$\boxed{\Pi^+(K) \subseteq \widetilde{\Pi}(K) \subseteq \Pi(K),}$$

where $\widetilde{\Pi}(K)$ denotes the set of all K-irreducible operators.

Spectral properties of superoperators

Theorem (I)

Let $\Phi \in \Pi^{+}(K)$. Then we have

- **a)** *the spectral radius of the operator Φ is a simple eigenvalue of Φ, greater than the magnitude of any other eigenvalue;*
- **b)** *an eigenvector of Φ corresponding to $r(\Phi)$ belongs to K^{o};*
- **c)** *no other eigenvector of Φ (up to scalar multiples) belongs to K.*

Theorem (II)

Let $\Phi \in \Pi(K)$. Then the following hold

- **a)** *$r(\Phi)$ is an eigenvalue of Φ ;*
- **b)** *K contains an eigenvector of Φ corresponding to $r(\Phi)$;*
- **c)** *if $\Phi \leq \Psi$, then $r(\Phi) \leq r(\Psi)$.*

Spectral properties of superoperators

Theorem (III)

Let $\Phi \in \widetilde{\Pi}(K)$. Then the following hold

a) $r(\Phi)$ *is a simple eigenvalue of* Φ*;*

b) *no eigenvector of* Φ *lies on the boundary of* K*;*

c) Φ *has exactly one (up to scalar multiples) eigenvector in* K *and this vector belongs to* K^o*;*

d) $(I + \Phi)^{n-1} \in \Pi^+(K)$, *where* $n = \dim V$.

Spectral properties of superoperators

Theorem (IV)

The following statements are equivalent for a positive map on PSD.

1.) *There is a nontrivial (that is different from $\{0\}$ and PSD) face of PSD that is invariant under Φ;*

2.) *There is nontrivial projection $P \in \mathcal{P}_n$ and a positive real number $\lambda > 0$ such that $\Phi(P) \leq \lambda P$;*

3.) *There is a nontrivial projection $P \in \mathcal{P}_n$ such that subalgebra $P(\mathcal{B}_*(\mathcal{H}))P$ is invariant under Φ.*

Spectral properties of superoperators

A family of closed subspaces of a given Hilbert space is called trivial if this family contains only $\{0\}$ and $\mathcal{H}$. For a fixed operator $X \in \mathcal{B}(\mathcal{H})$ we will denote by $\mathrm{Inv}\,(X)$ the set of all invariant subspaces of X.

Theorem (V)

Let Φ denote a superoperator on $\mathcal{B}(\mathcal{H})$ which is PSD-positive. If Φ is completely positive, then there exist some operators $A_1, \ldots, A_\eta$ such that

$$\boxed{\Phi(X) = \sum_j A_j X A_j^\star.}$$

Completely positive Φ is irreducible if and only if the Kraus operators A_j do not have a nontrivial common invariant subspace in $\mathcal{H}$.

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REFERENCES

- P. Zanardi, M. Rasetti, Modern Phys. Lett. B **11**, 1085, (1997).
- D. A. Lider, D. Bacon, K. B. Whaley, Phys. Rev. Lett. **82**, 4556, (1999).
- O. Perron, Math. Ann. **64**, 248 (1907).
- G. Frobenius, S. B. Preuss. Akad. Wiss. (Berlin), 256 (1912).
- M. G. Krein, M. A. Rutman, *Linear Operators Leaving Invariant a Cone in a Banach Space*, AMS Translations **26**, (1950).
- J. S. Vandergraft, SIAM J. App. Math. **16**, 1208 (1968).
- D. Shemes, Linear Alg. Appl. **62**, 11, (1984).
- D. R. Farenick, Proc. AMS **124**, 3381, (1996).
- Yu. A. Alpin, Kh. D. Ikramov, J. of Math. Sciences **114**, 1757, (2003).