Description of bound states and scattering processes using Hamiltonian renormalization group procedure for effective particles in quantum field theory

Marek Więckowski

June 2005

Doctoral thesis
Advisor: Professor Stanisław D. Głazek

Warsaw University
Department of Physics
Institute of Theoretical Physics
Warsaw 2005

Acknowledgments

I am most grateful to my advisor, Professor Stanisław Głazek, for his invaluable advice and assistance during my work on this thesis. I would also like to thank Tomasz Masłowski for allowing me to use parts of our paper here.

My heartfelt thanks go to Dr. Nick Ukiah, who supported and encouraged me during the writing of this thesis and who helped me in editing the final manuscript in English. Finally, I would like to thank my family, especially my grandmother, who has been a great support to me during the last few years.

Abstract

This thesis presents examples of a perturbative construction of Hamiltonians $\mathcal{H}_\lambda$ for effective particles in quantum field theory (QFT) on the light front. These Hamiltonians (1) have a well-defined (ultraviolet-finite) eigenvalue problem for bound states of relativistic constituent fermions, and (2) lead (in a scalar theory with asymptotic freedom in perturbation theory in third and partly fourth order) to an ultraviolet-finite and covariant scattering matrix, as the Feynman diagrams do. λ is a parameter of the renormalization group for Hamiltonians and qualitatively means the inverse of the size of the effective particles. The same procedure of calculating the operator $\mathcal{H}_\lambda$ applies in description of bound states and scattering. The question of whether this method extends to all orders in QFT is not resolved here.

The relativistic Hamiltonian formulation of QFT is based on a global regularization of all terms in a relativistic operator H^Δ (a canonical Hamiltonian with an ultraviolet cutoff Δ, plus counterterms). The renormalization group procedure for effective particles (RGPEP) makes it possible to find the structure of the counterterms in H^Δ and calculate the effective Hamiltonians $\mathcal{H}_\lambda$ for λ ranging from infinity down to λ on the order of masses of bound states.

I investigate bound states of two relativistic fermions using Yukawa theory as an example. I give an explicit form of the effective Hamiltonian $\mathcal{H}_\lambda$ in the second order, and discuss the reduction of its eigenvalue equation to a Schrödinger equation for the wave function of the constituents. Every interaction term in the Hamiltonian $\mathcal{H}_\lambda$ contains a form factor f_λ generated by RGPEP, which eliminates overlapping divergences in the bound-state eigenvalue problem expressed in terms of effective particles. The overlapping divergences appear in the eigenvalue problem expressed in terms of pointlike particles and without the form factors f_λ, and result from relativistic relative motion of fermions. Such divergences appear in all Hamiltonian theories of pointlike particles with spin, and in particular in quantum chromodynamics (QCD). The advantage of the Yukawa theory is that it allows one to investigate the ultraviolet behavior in bound states of fermions without additional complications of QCD. The form factors f_λ also cause the bound state to be dominated by the lowest sectors in the Fock-space basis built with effective particles. The ultraviolet complications of local QFT are contained in a complex structure that emerges in the effective particles as a result of dressing of the bare particles of the initial canonical theory.

My description of scattering in an asymptotically free scalar field theory of ϕ^3 type in 5+1 dimensions starts from constructing explicitly counterterms in H^Δ by calculating $\mathcal{H}_\lambda$. I then use the Hamiltonian H^Δ to calculate a scattering amplitude for a process analogous to $e^+e^- \to$ *hadrons* in perturbation theory up to the order e^2g^2, i.e., in one loop (e is an analogue of the electric charge of electrons in QED, and g of the color charge of quarks in QCD). I show that counterterms found using RGPEP without referring to the S matrix, remove the divergent regularization dependence from the calculated amplitude. I also give the explicit form of the finite parts of the counterterms in H^Δ that lead to a covariant result for the scattering amplitude. I show that the dependence of the amplitude calculated this way on the momenta of the colliding particles, is the same as the dependence derived from Feynman diagrams (the diagrams are regularized covariantly and without defining a regularized Hamiltonian *ab initio*).

I prove a theorem that states that the scattering amplitude obtained using H^Δ is the same as the scattering amplitude obtained using $\mathcal{H}_\lambda$. Note that in the calculation using H^Δ physical states of colliding particles are expressed in terms of bare particles and one uses the renormalized interaction Hamiltonian H_I^Δ. In the calculation using $\mathcal{H}_\lambda$ physical states of colliding particles are expressed in terms of non-pointlike effective particles and one uses the effective

interaction Hamiltonian $\mathcal{H}_{\lambda I}$ with form factors in all vertices. In the case of the considered amplitude of the type $e^+e^- \to hadrons$ in a one-loop approximation, this theorem implies that the effective Hamiltonian $\mathcal{H}_\lambda$ leads to the same predictions for the scattering matrix as the Feynman diagrams.

I also present an alternative, simplified procedure for deriving the Hamiltonian counterterms needed for the description of the scalar analogue of $e^+e^- \to hadrons$. I illustrate the simplified procedure by giving mass and some vertex counterterms in QCD coupled to QED, in the Appendices.

Streszczenie

Niniejsza praca podaje przykłady perturbacyjnej konstrukcji takich Hamiltonianów $\mathcal{H}_\lambda$ dla cząstek efektywnych w kwantowej teorii pola (KTP) na froncie świetlnym, które (1) mają dobrze zdefiniowany (ultrafioletowo skończony) problem własny dla stanów związanych relatywistycznych fermionów-składników, oraz (2) w skalarnej teorii z asymptotyczną swobodą w rachunku zaburzeń do trzeciego i częściowo czwartego rzędu przewidują ultrafioletowo skończoną i kowariantną macierz rozpraszania, taką jak diagramy Feynmana. λ jest parametrem grupy renormalizacji dla hamiltonianów i jakościowo oznacza odwrotność rozmiaru cząstek efektywnych. Ta sama procedura obliczania operatora $\mathcal{H}_\lambda$ stosuje się w przypadku opisu stanów związanych i rozpraszania. Dowód stosowalności metody do wszystkich rzędów w KTP nie jest rozważany.

Relatywistyczne hamiltonowskie sformułowanie KTP opiera się na globalnej regularyzacji wszystkich wyrazów w relatywistycznym operatorze H^Δ (hamiltonian kanoniczny z ultrafioletowym obcięciem Δ, plus kontrczłony). Procedura grupy renormalizacji dla cząstek efektywnych (RGPEP) pozwala znaleźć strukturę kontrczłonów w H^Δ, a następnie obliczyć hamiltonian efektywny $\mathcal{H}_\lambda$ dla λ z zakresu od nieskończoności do małych wartości rzędu mas stanów związanych.

W przypadku stanów związanych dwóch relatywistycznych fermionów (zbadanym na przykładzie teorii Yukawy) podana jest jawna postać hamiltonianu efektywnego $\mathcal{H}_\lambda$ w drugim rzędzie rachunku oraz redukcja jego równania własnego do równania Schrödingera na funkcję falową efektywnych składników. Hamiltonian $\mathcal{H}_\lambda$ zawiera w oddziaływaniach czynniki kształtu f_λ wygenerowane przez RGPEP, które usuwają nakrywające się rozbieżności (*overlapping divergences*) w problemie własnym stanu związanego. Nakrywające się rozbieżności pojawiają się w problemie własnym zapisanym za pomocą cząstek punktowych bez czynników f_λ, i są spowodowane relatywistycznym ruchem względnym fermionów. Takie rozbieżności pojawiają się we wszystkich hamiltonowskich sformułowaniach teorii cząstek punktowych ze spinem, a w szczególności w chromodynamice kwantowej (QCD). Zaletą teorii Yukawy jest możliwość zbadania ultrafioletowego zachowania fermionów w stanie związanym niezależnie od dodatkowych komplikacji w QCD. Czynniki f_λ powodują również, że w stanie związanym dominują najniższe sektory Focka cząstek efektywnych. Ultrafioletowe komplikacje lokalnej KTP zawarte są w skomplikowanej strukturze, która powstaje wewnątrz cząstek efektywnych na skutek ubierania się cząstek gołych wyjściowej teorii kanonicznej.

W przypadku teorii rozpraszania, jawna konstrukcja kontrczłonów w H^Δ na podstawie obliczeń $\mathcal{H}_\lambda$ jest przeprowadzona w przypadku asymptotycznie swobodnej skalarnej teorii pola typu ϕ^3 w 5+1 wymiarach. Następnie hamiltonian H^Δ jest użyty do obliczenia amplitudy rozpraszania dla procesu analogicznego do $e^+e^- \to$ *hadrony* w rachunku zaburzeń do rzędu e^2g^2, tj. w jednej pętli (e jest analogiem ładunku elektronów w QED a g ładunku kolorowego kwarków w QCD). Kontrczłony znalezione za pomocą RGPEP bez odwoływania się do macierzy rozpraszania, usuwają rozbieżną zależność od regularyzacji w obliczonej amplitudzie. Podane są jawne wzory na skończone części kontrczłonów w H^Δ, prowadzące do kowariantnego wyniku na macierz rozpraszania. Zależność otrzymanej w ten sposób amplitudy od pędów zderzających się cząstek jest taka jak otrzymana z diagramów Feynmana (zregularyzowanych kowariantnie i bez definicji *ab initio* zregularyzowanego hamiltonianu).

Podane jest twierdzenie, które mówi, że amplituda rozpraszania otrzymana za pomocą H^Δ jest taka sama, jak amplituda otrzymana za pomocą $\mathcal{H}_\lambda$. W rachunku z H^Δ stany fizyczne zderzających się cząstek są reprezentowane za pomocą cząstek gołych i rachunek zaburzeń jest prowadzony przy użyciu zrenormalizowanego hamiltonianu oddziaływania H_I^Δ. W rachunku z

$\mathcal{H}_\lambda$ stany fizyczne zderzających się cząstek są reprezentowane za pomocą cząstek efektywnych i rachunek zaburzeń jest prowadzony przy użyciu efektywnego hamiltonianu oddziaływania $\mathcal{H}_{\lambda I}$, zawierającego czynniki kształtu f_λ we wszystkich wierzchołkach. W przypadku rozważanej amplitudy typu $e^+e^- \to$ *hadrony* w przybliżeniu jednej pętli, z tego twierdzenia wynika wniosek, że wyliczony efektywny hamiltonian $\mathcal{H}_\lambda$ przewiduje taki sam wynik na macierz rozpraszania jak diagramy Feynmana.

Podano również alternatywną, uproszczoną procedurę otrzymywania hamiltonowskich kontrczłonów potrzebnych do opisu skalarnego odpowiednika procesu $e^+e^- \to$ *hadrony*. Ilustracją uproszczonej procedury są wzory na kontrczłon masowy i niektóre kontrczłony wierzchołkowe w QCD sprzężonej z QED (podane tylko w dodatkach).

Contents

List of Figures

Chapter 1

Introduction

The problem of the relativistic description of bound states of particles in quantum field theory (QFT) has fascinated physicists for more than a half a century. Yet the theory is still not understood well enough to precisely describe strongly bound particles starting from first principles. The problem is most acute in the case of quantum chromodynamics (QCD), because of the strength of the interactions involved and the inapplicability of perturbation theory. Although high-energy scattering processes can be treated perturbatively in asymptotically free theories, it is not clear how to calculate interactions that govern the formation of bound states at low energies.

The leader in the field of low-energy effects is lattice gauge theory. Its impressive and unparalleled successes elevate lattice gauge theory to the central stage of strong interactions. Still, there is a problem of how the phenomenology of hadrons that is based on the constituent picture could be incorporated using lattice in a single, complete formulation of the theory that could cover both bound and scattering states. Other methods, such as Dyson-Schwinger equations or methods based on the operator product expansion, are advancing at a rapid pace and attempt to compete with, supplement, or take advantage of lattice calculations; however, in principle they also face the problem of how to provide a complete formulation.

The author of this thesis believes that a Hamiltonian approach with a single Hamiltonian defining the whole theory, scattering through an S-matrix calculation, and bound states through an eigenvalue problem, could potentially contribute to the field if one only knew how to calculate the required Hamiltonians. Although we are still far away from a complete Hamiltonian picture, and this approach is much less researched than the currently dominant approaches, this thesis shows that, in some simple cases, recently developed methods for evaluating Hamiltonians for effective particles in QFT, can be employed in construction of the required operators. The main issues I discuss in this thesis are (1) universal regularization of all terms in the entire Hamiltonian operator; (2) construction of Hamiltonian counterterms (these are not the same as in the standard Lagrangian approach to scattering); (3) systematic method for evaluating interaction terms in Hamiltonians for bound states (these are not calculable from perturbative S-matrix considerations); (4) a path to resolution of the problem that an infinite number of bare particles is involved in the bound-state eigenvalue equation; (5) resolution of the overlapping divergences in the bound-state eigenvalue problems that are limited to a small number of constituents; and (6) construction of the finite parts of the Hamiltonian counterterms that guarantee covariance of observables. In addition, an important result of this thesis is that the Hamiltonians for effective particles can produce the same scattering matrix that is obtained from the canonical QFT.

We do not yet know how to overcome all of the conceptual and technical difficulties associated with the so-called small-x singularities in gauge theories (due to interactions that involve massless gauge bosons that carry small longitudinal momentum), but it is possible that these singularities are a benefit in a Hamiltonian formulation and a source of useful effects (especially regarding confinement), rather than artificial singularities that ought to be entirely removed. This thesis does not answer the question of how to seek a solution to the small-x problem in gauge theories, but it does develop a Hamiltonian approach and achieves some success in tackling the six issues listed above.

To place the Hamiltonian approach studied here in the context of the struggle with the description of hadrons using QCD, let us mention that the Feynman-diagram approach, combined with parton models and operator product expansion, is still so far from a direct connection with the constituent classification of hadrons that recent studies of pentaquarks – a flurry of publication activity – involve all kinds of methods and are largely free from the strict theoretical constrains of QCD with exactly three colors and six flavors of quarks. Similar problems plague studies of exotics with constituent gluons – one might even wonder how it is possible that massless gluons are not visible in the hadronic spectrum like photons that correspond to the Coulomb potential with the Balmer series of energy levels. Some QCD-motivated string picture or flux-tube models may correspond to lattice calculations, but how gluons bind and what makes them so greatly inactive in the classification of hadrons is still entirely unclear. This is in a sharp contrast with the success of precise QCD predictions in the high-energy domain, where the widths of hadronic wave functions are small in comparison to the energies and momenta that matter. Therefore, any systematic approach that could offer some hope for constructing a relativistic constituent description of hadrons (to replace a parton model) and a description of scattering processes for the constituents (as Feynman diagrams can for quarks and gluons) is worth investigating. An example of entirely unexpected possibilities that Hamiltonian approaches to QFT may offer is that QCD has an infrared renormalization group limit cycle behavior, whose treatment is perhaps beyond the scope of all other methods. However, this thesis does not deal directly with QCD since the issues (1) to (6) above have to be solved even in much simpler cases before an admissible Hamiltonian approach to QFT can be proclaimed to exist.

I begin my discussion in **Chapter 2**, which is based on a joint publication with Masłowski [1], by presenting an extremely simple Hamiltonian model. This serves the purpose of explaining in concrete terms what type of structure an admissible relativistic Hamiltonian formulation of a theory should have. The model is restricted to two Fock sectors only, but this does not represent a drawback. The simplification allows a relativistic description of bound states and scattering amplitudes to exist with full control over analytic expressions. In particular, it is shown how strict conditions can be placed on Hamiltonian counterterms so that both bound states and scattering satisfy the constraints of special relativity: a physical fermion is described by a Dirac equation and fermion-boson scattering amplitude is fully covariant when proper renormalization conditions are satisfied. One of the critical questions addressed in this thesis is whether similar conditions can be formulated for counterterms in full QFT, which is much more complex than the model Hamiltonian that I use for heuristic purposes.

Chapter 3 presents the general methods that I apply to bound-state and scattering issues in selected QFTs in the chapters that follow. The first part of Chapter 3 contains a brief introduction to the light-front quantization of fields, starting from the more familiar standard approach and proceeding to the light-front Hamiltonian dynamics that forms the core of this thesis. Next, a renormalization group procedure for effective particles (RGPEP), based on the work of Głazek, is reviewed concisely. I use it to develop a theory of effective particles that is

applied in Chapters 4 and 5 to relatively simple QFTs, according to the outline of the heuristic model presented in Chapter 2. This review is also useful at this point because it allows me to present the notation used in this thesis and introduce equations that appear often in the main discussion.

Chapter 4, which is based on a joint publication with Głazek [2], discusses the issues of regularization of the initial Hamiltonian operator, construction of mass counterterms, systematic evaluation of effective Hamiltonians for bound states, and removal of overlapping divergences in the case of fermions. I take the example of Yukawa theory, which is considerably simpler than QCD, as it does not involve the small-x singularity. On the other hand, theories of the Yukawa type are not asymptotically free (which poses problems in perturbation theory), and suffer from the problem of triviality. However, the bound-state problem for effective particles is shown to be free from the ultraviolet divergences that occur in the Tamm-Dancoff approach, and the treatment of bound states I describe is more generic than the actual example I consider: namely, the overlapping divergences are removed by the from factors that emerge from RGPEP and will emerge in a similar way in the treatment of bound states in asymptotically free theories. RGPEP thus allows us to derive from QFTs well-defined equations for bound states.

Chapter 5 is entirely new. Taken together with Chapters 2 and 4 it completes my investigation of whether the heuristic reasoning outlined in a very simple model in Chapter 2 can be applied to QFT, at least in lowest orders.

Chapter 5 applies the Hamiltonian approach developed in this thesis to scattering processes, which are treated in perturbation theory in the simplest asymptotically free theory of which I am aware: scalar theory in 5+1 dimensions. I address the issue of complete regularization of the Hamiltonian for the entire theory (not a loop-by-loop type of approach, but the regularization of an operator in a Fock space). I explicitly construct the counterterms that lead to covariant scattering amplitudes. The most difficult part of the construction is establishing finite parts of the counterterms, that is, the parts of the Hamiltonian operator that are independent of the cutoff parameters but nevertheless depend on regularization. I provide explicit expressions for these finite parts. I also demonstrate in a one-loop example of a model amplitude (analogous to the amplitude $e^+e^- \to hadrons$) the details of cancellations between different terms and factors of a Hamiltonian quantum-mechanical calculation of the scattering process. These cancellations are responsible for obtaining covariant results, despite the fact that the initial Hamiltonian is regularized in a way that is not fully covariant (it explicitly preserves boost invariance, but not rotational invariance). This is possible because of light-front Hamiltonian dynamics. An interesting aspect of the whole calculation, despite the fact that it is effectively only a third-order perturbative calculation, is that it can be carried out using a Hamiltonian for effective particles that appear to have all the properties required of constituents such as constituent quarks and gluons in hadrons. The interactions of the effective particles are highly non-local because they are modulated by form factors whose width is a renormalization-group parameter and can be tuned for optimal economy of calculations. Nevertheless, the physical scattering amplitude obtained in perturbation theory is completely independent of the renormalization-group parameter, and exactly the same as in local asymptotically free theory. This is established by showing that the Hamiltonian theory leads to the same answers as the Feynman diagrams, independently of all details of the explicitly non-covariant elements involved in defining the Hamiltonian operator in the Fock space. Additionally, I consider two procedures, a full one (complete RGPEP in the third order) and a simplified one (limited to dressing by strong interactions only), and I show that they both lead to the same counterterms that contribute to my scalar analog of the amplitude $e^+e^- \to hadrons$. However, the Hamiltonians that I obtain contain interaction terms that are

completely different from the terms that are typically inferred from Born-like approximations for scattering amplitudes. For example, the Hamiltonian terms have well-defined off-energy-shell behavior that cannot be deduced from the on-shell S-matrix calculus and they contain dynamical information reaching far beyond the perturbative aspects of the theory. (This is similar to the situation with the Coulomb potential, which describes an unimaginable richness of bound states of charged particles despite the fact that it is only of order e^2). I provide a few examples of counterterms that are calculated by the same methods in QED and QCD.

Chapter 6 summarizes the key findings of this thesis. It is followed by a series of appendices that contain the details of the material presented in the main part of the thesis. A summary of the notation used in this thesis is given in Appendix A.

Chapter 2

Model of a relativistic Hamiltonian for simultaneous description of bound states and scattering processes in a single formalism

2.1 Introduction

In this chapter I present a simple model that features a bound state and scattering states. The model is based on quantum field theory of fermions and bosons coupled by Yukawa interaction, but so severely simplified, that a number of analytic results can be obtained at the expense of a relatively small effort. The bare model interaction leads to infinities when calculating physical observables. Therefore, one is forced to introduce cutoffs in order to obtain finite results. However, this is done according to principles of the renormalization group procedure for Hamiltonians, and the introduction of cutoffs is accompanied with insertion of Hamiltonian (not Lagrangian, as in the Euclidean path-integral approach) counterterms whose structure is found from well-defined finiteness conditions imposed on an effective Hamiltonian theory and from threshold and covariance conditions on the results for bound states and scattering amplitudes that follow from the effective theory. This chapter explains the meaning of the above statements. The model has attractive properties and serves as a heuristic example to follow in the case of QFT.

In particular, despite the fact that the cutoff is introduced in a non-covariant way, the proper choice of finite parts of the counterterms leads to the scattering matrix which is not only finite but also covariant. The same choice of counterterms guarantees that the structure of a physical fermion agrees with demands of the Dirac equation for the wave function that describes motion of the center of mass of the fermion. Nonetheless, the interaction between bare fermions and bare bosons include a vertex that has an unusual structure, and one has to correlate four constants using three conditions to obtain physical results that obey conditions of symmetry of special relativity and proper threshold behavior.

This chapter is not meant to be exhaustive. It aims to present a general picture and ideas that are developed further in the following chapters, where more complex model theories are considered.

This chapter, and the corresponding Appendix C.4, is based on a paper published by Masło-

wski and myself [1]. In the quotations used here, I have made small changes to simplify the notation and unify it with the rest of this thesis. I have also left out parts of the quotations that are not relevant in this chapter.

In the joined paper with Masłowski, we studied the similarity renormalization scheme for Hamiltonians to the fourth order in perturbation theory using a model Hamiltonian for fermions coupled to bosons. The model consists of only **two sectors in the Fock space**. This great simplification of the space of states allowed for a complete analysis of the renormalization scheme and still included typical factors and divergences that appear in quantum field theory. Therefore, the model could be used as a good testing ground. Our model was based on Yukawa theory.

The Hamiltonian of Yukawa theory truncated to one fermion and one fermion plus one boson Fock sectors leads to infinities in the fermion-boson T matrix. Therefore, we introduced an ultraviolet (high-energy) **cutoff** Δ for the momentum transfer in the interaction part of the Hamiltonian. The similarity transformation [3] allowed us to construct **counterterms** in the initial Hamiltonian. The complete Hamiltonian gives finite, cutoff independent results for the T matrix. We constructed renormalized Hamiltonians using expansion in powers of the effective fermion-boson coupling constant and including terms up to the fourth order.

In the **similarity renormalization scheme**, one constructs effective Hamiltonians H_λ whose form is a function of the width λ. H_λ is obtained by a unitary transformation from the initial Hamiltonian H^Δ with counterterms. The transformation and counterterms are found order by order in perturbation theory using the requirement that matrix elements of H_λ are independent of the cutoff Δ when the cutoff goes to infinity.

To find values of the **unknown finite parts of the counterterms** we calculated the T matrix for fermion-boson scattering. The condition that the T matrix is covariant implies relations between the finite parts of different counterterms. We also required, that the physical fermion is described by the Dirac equation with the fermion mass equal to the fermion mass term in the fermion-boson sector. This requirement also provides a relation between finite parts of counterterms called the threshold condition [4].

In this chapter, the so-called algebraic version of the similarity renormalization scheme is used. Although it may be considered simpler, it is not used in complicated cases because it introduces sharp similarity form factors that complicate numerical calculations. Sharp θ functions are also used here for ultraviolet cutoffs.

The model Hamiltonian we studied was originally considered by Głazek and Perry [5]. They guessed the form of counterterms which remove divergences in T matrix and they obtained covariant results for the T matrix to all orders.

Our main question about the model was if the systematic similarity calculation carried out in perturbation theory would produce the same solution to the Hamiltonian renormalization problem as guessed by Głazek and Perry. The cutoff in the model is limited by the triviality bound [5], but one can assume that the coupling constant is small enough for reliable use of the perturbation theory.

2.2 The model

The initial Hamiltonian is a light-front Hamiltonian for Yukawa theory projected on two Fock-space sectors, namely, one with a fermion and one with a fermion and a boson. The reasons for using light-front formulation are explained in Section 3.3.4.

The model Hamiltonian is:

$$H^\Delta = H_{0f} + H_{0fb} + H_Y^\Delta + H_+^\Delta + X^\Delta \,. \tag{2.1}$$

The free part is

$$H_{0f} = \sum_\sigma \int [p]\, |p\sigma\rangle \langle p\sigma| \, \frac{p^2+m^2}{p^+} \,, \tag{2.2}$$

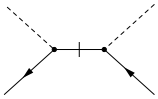
$$H_{0fb} = \sum_\sigma \int [p,k]\, |p\sigma,k\rangle \langle p\sigma,k| \left(\frac{p^2+m^2}{p^+} + \frac{k^2+\mu^2}{k^+} \right) , \tag{2.3}$$

where $|p\sigma\rangle\langle p\sigma|$ is an operator projecting on a Fock state of one fermion of momentum $\vec{p} = (p^+, p^\perp)$ and spin σ, and $|p\sigma,k\rangle\langle p\sigma,k|$ projects on a one-fermion $(\vec{p},\sigma)$ plus one-boson $\left(\vec{k}\right)$ state. This free Hamiltonian assigns the same mass m to fermions in both Fock sectors, and mass μ to the boson (cf. Eq. (3.48)). Boson creation and annihilation vertices, proportional to the coupling constant g, are

$$\begin{aligned} H_Y^\Delta \;=\; & g \sum_{\sigma_1,\sigma_2} \int [p_1,p_2,k] \theta(\Delta^2 - \mathcal{M}^2_{p_2,k}) \widetilde{\delta}(p_1 - p_2 - k) \times \\ & \times \Big[|p_2\sigma_2,k\rangle \langle p_1\sigma_1| \, \bar{u}(p_2,\sigma_2) u(p_1,\sigma_1) + H.c. \Big] = H_{>\!-}^\Delta + H_{-\!<}^\Delta \,, \end{aligned} \tag{2.4}$$

and the seagull term, of order g^2, is

$$\begin{aligned} H_+^\Delta \;=\; & g^2 \sum_{\sigma_1,\sigma_2} \int [p_1,p_2,k_1,k_2] \theta(\Delta^2 - \mathcal{M}_1^2) \theta(\Delta^2 - \mathcal{M}_2^2) \widetilde{\delta}(p_2 + k_2 - p_1 - k_1) \times \\ & \times |p_2\sigma_2,q_2\rangle \langle p_1\sigma_1,k_1| \, \bar{u}(p_2,\sigma_2) \frac{\gamma^+}{2(p_1^+ + k_1^+)} u(p_1,\sigma_1) \,. \end{aligned} \tag{2.5}$$



X^Δ in Eq. (2.1) is an unknown counterterm.

Cutoffs on the free[1] invariant mass $\mathcal{M}^2 = (p+k)^2$ of the two particle sector in the interaction parts of the Hamiltonian, H_Y^Δ and H_+^Δ (see also [6]). For example, theta function in Eq. (2.4) permits creation of only low-energy fermion-boson pairs, that is, only if their $\mathcal{M}^2$ is smaller than Δ^2.

The standard three-dimensional integration measure $[p]$, three-momentum conservation delta function $\widetilde{\delta}$, Fock states $|p\rangle$ and spinors $u_m(p,\sigma)$ are defined in detail in Appendix A. Note, that spinors u depend on the mass of the fermion. In this chapter all spinors correspond to the fermion mass m in H_0 (i.e. $u(p,\sigma) := u_m(p,\sigma)$), unless stated otherwise (cf. Eq. (2.33)).

[1] In the definition of the invariant mass the energy (k^-, a light-front analogue of k^0) enters. Therefore, the value of $\mathcal{M}^2$ assigned to a state depends on how one assigns an energy to this state. "Free invariant mass" means, that to any Fock state a corresponding eigenvalue of the free Hamiltonian H_0 is assigned. $\mathcal{M}^2$ defined this way is invariant under kinematical Lorentz transformations (see Section 3.3).

2.3 Renormalization

The similarity transformation U_λ transforms H^Δ to a new Hamiltonian H_λ :

$$\mathcal{H}_\lambda = U_\lambda^\dagger H^\Delta U_\lambda \,. \tag{2.6}$$

Expressions for U_λ and H_λ are found in perturbation theory. The procedure is tailored in such a way that the effective Hamiltonian H_λ is band diagonal. Namely, one chooses suitable form factors f_λ – functions of momenta in a vertex, which are to limit ranges of possible momenta of particles involved in the effective interaction – and uses renormalization group equations [3] to calculate the transformation U_λ, which leads to a Hamiltonian H_λ with such form factors. The result is that the rotated Hamiltonian H_λ is band diagonal (each term has a factor f_λ), but the matrix elements of H_λ within the band get changed in such a way that both Hamiltonians are unitarily equivalent (see also Figure 3.9 on page 45, and Section 3.4). X^Δ in H^Δ is fitted order by order in g, so that H_λ does not have Δ-dependent (i.e., divergent) matrix elements for $\Delta \to \infty$. This can be guaranteed in any finite order in perturbation theory.[2]

Form factors f_λ (the diagonal proximum operator) for the calculation of this chapter are presented in Appendix C.4.

In the zeroth order, the transformation gives an unchanged free Hamiltonian:

$$\mathcal{H}_{\lambda\,0} = H_0^\Delta \,. \tag{2.7}$$

In the first order (Fig. 2.1), one gets:

$$\begin{aligned} \mathcal{H}_\lambda^{(1)} &= g \sum_{\sigma_1,\sigma_2} \int [p_1,p_2,k]\theta(\lambda^2 - \left|\mathcal{M}^2_{p_2,k} - m^2\right|)\widetilde{\delta}(p_1-p_2-k)\times \\ &\quad \times \Big[|p_2\sigma_2,k\rangle \langle p_1\sigma_1| \bar{u}(p_2,\sigma_2)u(p_1,\sigma_1) + H.c.\Big] = \\ &= f_\lambda H_Y^\Delta \end{aligned} \tag{2.8}$$

– rotation of the basis leads to an interaction term with an induced factor f_λ (also, in the presence of f_λ, one can take the limit $\Delta \to \infty$, and thus there is no initial cutoff factor in the above expression).

In the second order, the transformation gives:

$$\mathcal{H}_\lambda^{(2)} = f_\lambda \left(H_+^\Delta + X^{(2)\Delta} - \frac{1}{2}\left[\left\{ (1-f_\lambda)H_Y^\Delta \right\}_0, (1+f_\lambda)H_Y^\Delta \right] \right) . \tag{2.9}$$

The curly brackets denote the free energy denominator, square brackets a commutator, and f_λ is the form factor (see Appendix C.4).

In the fermion-boson sector, Eq. (2.9) reads:

$$\mathcal{H}^{(2)}_{\lambda\,,fb-fb} = f_\lambda \left(H_+^\Delta - \frac{1}{2}\left\{ (1-f_\lambda)H_{\succ}^\Delta \right\}_0 (1+f_\lambda)H_{\prec}^\Delta + \frac{1}{2}(1+f_\lambda)H_{\succ}^\Delta \left\{ (1-f_\lambda)H_{\prec}^\Delta \right\}_0 \right) . \tag{2.10}$$

[2] The differential equation for renormalization group procedure based on the idea of similarity rotation is discussed in detail in Section 3.4. However, for the model described in the current section it was enough to introduce simple sharp cutoffs (2.4-2.5) and sharp similarity form factors f_λ (C.78). For such factors, the differential equations cannot be easily used, and the model calculation is based on an algebraic similarity renormalization procedure. Although the motivation of Section 3.4 is the same as here, the actual formulae used here are given in the algebraic version of the same procedure, and are the same as those presented in detail in ref. [3].

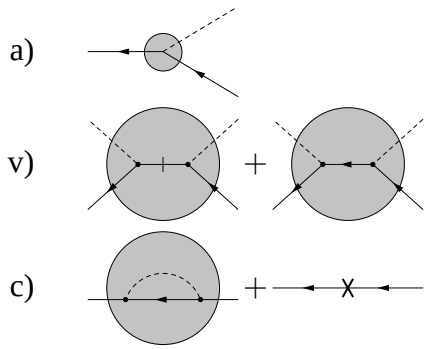


Figure 2.1: Terms in the effective model Hamiltonian in the first and the second orders. (a) Effective vertex $H_\lambda^{(1)}$ is limited by a form factor f_λ. (b) In the fermion-boson – fermion-boson part of $H_\lambda^{(2)}$, apart from a seagull term with a form factor, there is a new interaction term, which makes H_λ unitarily equivalent to H^Δ. (c) The second order fermion – fermion term H_λ is divergent, which enforces existence of a counterterm and determines its form. In all figures, the factors coming from the similarity renormalization group (combinations of the form factors, cf. (2.9)) are marked with grayed circles.

This expression is not divergent for $\Delta \to \infty$. Hence, no counterterm is needed in this sector. However, in the fermion-fermion sector, one obtains:

$$\mathcal{H}^{(2)}_{\lambda,f-f} = -\left\{(1-f_\lambda)H^\Delta_{<}\right\}_0 H^\Delta_{>} + X^{(2)\Delta}\,. \tag{2.11}$$

The loop integration in the first term is quadratically divergent in Δ. The form of this divergence dictates the form of the second-order counterterm. Explicitly, one has to choose:

$$X^{(2)\Delta} = \sum_\sigma \int [p]\,|p\sigma\rangle\,\langle p\sigma|\,\frac{1}{p^+}\frac{g^2}{16\pi^2}\left[\frac{1}{2}\Delta^2 + (3m^2-\mu^2)\log\frac{\Delta^2}{m^2} + A\right]\,. \tag{2.12}$$

This term acts in fermion–fermion sector only and means an infinite fermion mass-shift (cf. Eq. (2.2)), where A is an undetermined finite constant.

Higher-order calculations lead to the following expressions for $X^{(3)\Delta}$ and $X^{(4)\Delta}_{fb-fb}$:

$$\begin{aligned} X^{(3)\Delta} &= X^{(3)}_Y + X^{(3)}_+ = \frac{1}{4}\frac{g^2}{16\pi^2}\log\frac{\Delta^2}{C}H^\Delta_Y + \\ &\quad + \sum_{\sigma_1,\sigma_2}\int [p_1,p_2,k]\theta(\Delta^2-\mathcal{M}^2_{p_2,k})\widetilde{\delta}(p_1-p_2-k)\times \\ &\quad \times\frac{3}{2}\frac{g^3}{16\pi^2}m\log\frac{\Delta^2}{D}\left[|p_2\sigma_2,k\rangle\,\langle p_1\sigma_1|\,\bar{u}(p_2,\sigma_2)\frac{\gamma^+}{2p_1^+}u(p_1,\sigma_1) + H.c.\right]\,, \end{aligned} \tag{2.13}$$

$$X^{(4)\Delta}_{fb-fb} = \frac{1}{2}\frac{g^2}{16\pi^2}\log\frac{\Delta^2}{B}H^\Delta_+\,, \tag{2.14}$$

where B, C, and D are finite unknown constants.

There is also another counterterm of order g^4 in the fermion-fermion part of X^Δ. We did not have to calculate it because our goal was to investigate the possibility of fitting finite parts of

counterterms by requesting the T matrix covariance in the fermion-boson channel (Sec. 2.4.1) and the emergence of the Dirac equation for physical fermions. As $X^{(4)\Delta}_{f-f}$ does not contribute either to $T^{(4)}$ or the second-order Dirac equation, it was irrelevant for our considerations. Also, $X^{(4)\Delta}_{f-f}$ is more complicated to calculate than the terms we need to discuss here, because it involves two correlated loop integrations.

2.4 Relativistic covariance and finite parts of counterterms

The renormalization procedure does not determine values of the finite parts of counterterms. To find them, we need to introduce additional conditions. In principle, the constants should be fitted to the experiment. Still, it is interesting to look for theoretical requirements of symmetries, which may constrain these constants. The T matrix calculated with the general counterterms (2.12)-(2.14) is not automatically covariant. Therefore, the covariance of the T matrix provides useful conditions. Another condition will be provided by the study of the bound-state eigenvalue equation for the full Hamiltonian. In this simple model, the eigenvalue problem reduces to an equation for the physical fermion, and it will be required to reduce to a free Dirac equation.

2.4.1 T matrix

The T matrix which describes fermion-boson scattering can be calculated using the formula [7]:

$$T(E) = H_I + H_I \frac{1}{E - H_0 + i\varepsilon} H_I + \cdots . \tag{2.15}$$

The only non-vanishing terms are those of even order in g.

The second-order T matrix has a covariant form and does not depend on the counterterms – Fig.2.2a. X^{Δ} starts contributing in the fourth order. The explicit Δ-dependence of counterterms cancels divergences in the loop integrations in other terms. This leads to $T^{(4)}$ (Fig. 2.2b) which is finite, but not automatically covariant:

$$\begin{aligned}\langle p_2\sigma_2, k_2 | T^{(4)} | p_1\sigma_1, k_1\rangle = \frac{g^4}{16\pi^2}\theta(\Delta^2 - \mathcal{M}_1^2)\theta(\Delta^2 - \mathcal{M}_2^2)\widetilde{\delta}(p_2 + k_2 - p_1 - k_1)\times \\ \times \bar{u}(p_2,\sigma_2)\left[\Gamma_1(s)\not{P} + \Gamma_2(s) + \Gamma_3(s)\frac{\gamma^+}{2(p_1^+ + k_1^+)}\right] u(p_1,\sigma_1)\,, \end{aligned} \tag{2.15}$$

where $P^+ = p_1^+ + k_1^+$, $P^\perp = p_1^\perp + k_1^\perp$, $P^- = (p^{\perp 2} + m^2)/p^+ + (k^{\perp 2} + \mu^2)/k^+$ and $s = P^\mu P_\mu = (p_1 + k_1)^2 = \mathcal{M}_1^2$. To obtain a covariant result for $T^{(4)}$, we demand that the function $\Gamma_3(s)$ vanishes for arbitrary s. Its explicit form reads:

$$\Gamma_3(s) = \frac{1}{s - m^2}\left[(s - m^2)\frac{1}{2}\log\frac{C}{B} + 3m^2\log\frac{m^2}{D} - A + 16\pi^2\alpha_f(s)(s - m^2) + \gamma_f(s)\right] . \tag{2.16}$$

Functions $\alpha_f(s)$ and $\gamma_f(s)$ are given in Appendix C.4, Eqs.(C.79)-(C.81). The combination $16\pi^2\alpha_f(s)(s-m^2)+\gamma_f(s)$ turns out to be real and independent of s, and the condition $\Gamma_3(s) = 0$ implies two relations:

$$B = C \tag{2.17}$$

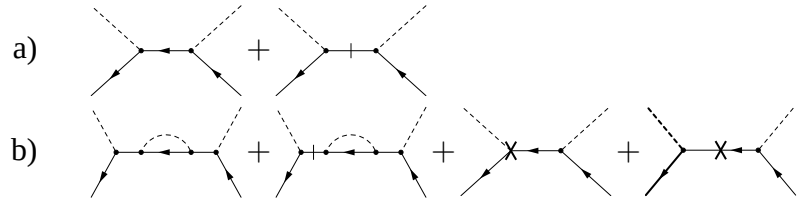


Figure 2.2: Fermion-boson T matrix in the order g^2 (a) and g^4 (b). In the order g^4, there are also terms with sea-gulls and counterterms on the lines on the right-hand side of diagrams (not shown above).

and

$$A = -m^2 + \mu^2 \log \frac{\mu^2}{m^2} + 3m^2 \log \frac{m^2}{D} \, . \tag{2.18}$$

These two equations relate constants A, B, C and D, but there are additional requirements due to the eigenvalue equation.

2.4.2 Eigenvalue equation

To describe a physical state in terms of free Fock states, one considers the eigenvalue equation

$$H^{\Delta} \left| P\sigma \right\rangle_{physical} = \frac{P^{\perp 2} + m^2}{P^+} \left| P\sigma \right\rangle_{physical} \, . \tag{2.19}$$

The physical fermion state is a superposition of the bare fermion and fermion-boson states:

$$\left| P\sigma \right\rangle_{physical} = \sum_{\sigma_2} c^{\sigma}_{\sigma_2} \left| P\sigma_2 \right\rangle + \sum_{\sigma_2} \int [p,k] \widetilde{\delta}(P-p-k) \phi^{\sigma}_{\sigma_2}(x, \mathcal{M}^2) \left| p\sigma_2, k \right\rangle \, . \tag{2.20}$$

By following the steps given in Ref. [5], one can reduce Eq. (2.19) to

$$\left(\rho_1 \not{P}_m - \rho_2 m + \rho_3 \frac{\gamma^+}{2P^+} \right) \psi = 0 \, , \tag{2.21}$$

for the one-body sector wave function ψ. $P^{\mu}_m = \left(P^+, P^-_m, P^{\perp} \right)$ with $P^-_m = (P^{\perp 2} + m^2)/P^+$. Using our Hamiltonian with counterterms restricted by conditions (2.17) and (2.18), one gets:

$$\rho_1 = 1 + \frac{g^2}{16\pi^2} \left[\frac{3}{2} \log \frac{\Delta}{D} - \beta(m^2) \right] + o(g^4) \, , \tag{2.22}$$

$$\rho_2 = 1 + \frac{g^2}{16\pi^2} \alpha(m^2) + o(g^4) \, , \tag{2.23}$$

$$\rho_3 = 0 + o(g^4) \, . \tag{2.24}$$

The earlier demand of the T matrix covariance established the value of the mass counterterm $X^{(2)\Delta}$ in a way that also leads to the vanishing of ρ_3 in order g^2.

In general, one can expand both nonzero terms ρ in a power series in g:

$$\rho = \rho^{(0)} + \rho^{(2)} g^2 + \rho^{(4)} g^4 + \cdots \tag{2.25}$$

and translate the requirement that m is the mass of a physical fermion:

$$\left(\not{p}_m - \frac{\rho_2^{(0)} + \rho_2^{(2)} g^2 + \rho_2^{(4)} g^4 + \cdots}{\rho_1^{(0)} + \rho_1^{(2)} g^2 + \rho_1^{(4)} g^4 + \cdots} m\right) \psi = 0 \tag{2.26}$$

into the condition for all coefficients:

$$\rho_1^{(i)} = \rho_2^{(i)} \ . \tag{2.27}$$

This is the threshold condition [4] that makes the T matrix threshold appear at $s = (m+\mu)^2$, where m is the position of its fermion pole.

Let us now investigate which terms of the Hamiltonian contribute to $\rho^{(i)}$. If one puts $g = 0$, then the only condition one gets is:

$$|P\sigma\rangle_{physical} = |P\sigma\rangle \ . \tag{2.28}$$

Technically, the zeroth order terms $\rho_1^{(0)}$ and $\rho_2^{(0)}$ come from the inversion of $\sum_\sigma u_{P\sigma m}\bar{u}_{P\sigma m} = \not{p}_m + m$, which is a part of $H_{>-}H_{-<}$. Dirac equation results in this order automatically: $\rho_1^{(0)} = \rho_2^{(0)}$.

Likewise, it is clear that the second order terms $\rho_1^{(2)}$ and $\rho_2^{(2)}$ partly come from the term $H_{>-}X_+^{(3)}$. Thus, one needs third order vertex corrections, such as $X_+^{(3)}$, to determine all second order contributions to the Dirac equation. There is an unknown finite parameter D in $X_+^{(3)}$. The condition $\rho_1^{(2)} = \rho_2^{(2)}$ and Eqs. (2.22) and (2.23) lead to the condition:

$$\log\frac{D}{m^2} = \frac{2}{3}\cdot 16\pi^2\left[\alpha_f(m^2) + \beta_f(m^2)\right] \ . \tag{2.29}$$

The functions $\alpha_f(s)$ and $\beta_f(s)$ are given in Appendix C.4, in Eqs. (C.83)-(C.85).

Altogether, the requirement that m is equal to the mass of a physical fermion implies one more condition on the free parts of counterterms.

2.4.3 Discussion

Taking conditions (2.17), (2.18) and (2.29) together and looking at the structure of the counterterms, one can observe the following: $X_Y^{(3)}$ can be accounted for by changing the coupling constant of H_Y^Δ:

$$g \to g + \frac{g^3}{64\pi^2}\log\frac{\Delta^2}{C} \tag{2.30}$$

in the original Hamiltonian, while $X_{fb-fb}^{(4)\Delta}$ shifts g^2 in the seagull term H_+^Δ:

$$g^2 \to g^2 + \frac{g^4}{32\pi^2}\log\frac{\Delta^2}{C} \ . \tag{2.31}$$

Thus, these two counterterms can be absorbed in one Δ-dependent coupling constant (2.30). Note that, in physical results, Δ-dependent logarithms $\log\frac{\Delta^2}{m^2}$ cancel each other, leaving

$$g + \frac{1}{4}\frac{g^3}{16\pi^2}\log\frac{m^2}{C} \ . \tag{2.32}$$

Thus g and C will never appear independently and there is only one parameter, combination (2.32), that can be fixed from experiment.

$X^{(2)\Delta}$ shifts the mass in the one fermion free energy. The sum of H_Y^Δ and one of the third-order counterterms, $X_+^{(3)}$, reproduces the same interaction term, but with a shifted mass of the spinor in the one-particle sector, according to the formula (cf. Appendix B.4):

$$\left(1+\frac{\gamma^+\delta m}{2p^+}\right)u_m(p,\sigma)=u_{m+\delta m}(p,\sigma)\ . \tag{2.33}$$

The same model has been analyzed by other authors – see Ref. [5]. They have shown that, in this model, to get finite and covariant results for the T matrix to all orders of perturbation theory, and to get the mass in the Dirac equation which is required by the threshold condition, it is enough to: (1) add to the bare cutoff Hamiltonian a term that shifts the mass of fermions in the free part H_{0f}; (2) correspondingly, change the spinor mass in the vertex – see Eq. (2.33); and (3) allow the coupling to depend on Δ.

This simple result – renormalization leading just to redefinition of the constants in the canonical Hamiltonian – works only for the severe truncation of the model considered in this section. The authors of Ref. [5] write: "If we were to consider a Tamm-Dancoff truncation that allowed more particles (e.g., two fermions and one boson), we would need terms that are not found in the canonical Hamiltonian to obtain covariant results."

If one rewrites the Hamiltonian of Ref. [5] using the invariant mass cutoff and expands it in powers of $\tilde{g}(m)$ up to the fourth order, one gets the same result as we obtained in our similarity calculation, with:

$$g+\frac{1}{4}\frac{g^3}{16\pi^2}\log\frac{m^2}{C} \tag{2.34}$$

replaced by:

$$\tilde{g}(m)-\frac{1}{2}\tilde{g}^3(m)\alpha_f(m^2)\ . \tag{2.35}$$

Hence, one may choose C to obtain the same result as in Ref. [5].

The results presented in Ref. [5] were obtained before the introduction of the similarity renormalization scheme and were guessed, based on the experience of the authors. Such guessing would be hard, if not impossible, in the case of realistic QFTs. Thus the systematic procedure of this chapter offers a huge advantage.

2.5 Heuristic meaning of the model

The results presented in this chapter, based on [1], provide an example of the application of the similarity renormalization scheme for Hamiltonians in its algebraic version. We have shown how this systematic procedure leads from a divergent Hamiltonian to a finite one. The finite Hamiltonian has well-defined eigenvalue equation for bound states and produces finite and covariant results for the S matrix. The question I address in the remainder of this thesis is whether one can apply a similar procedure in QFT. In Chapter 3, I review the formalism of Renormalization Group Procedure for Effective Particles (RGPEP) which is an extended version of the similarity renormalization scheme used above. RGPEP is formulated in differential form (rather than the algebraic form used above) and in terms of creation and annihilation operators in a Fock space. In Chapter 4, I apply RGPEP to a bound state of fermions in a Yukawa theory. This results in a well-defined bound-state equation, which I contrast with divergent equations based on

the Tamm-Dancoff truncation. In Chapter 5, I show that counterterms found from RGPEP lead to finite S matrix, and their finite parts can be chosen in such a way that the S matrix is covariant. Therefore, both key results of the model calculation presented above appear obtainable in QFT, although the calculations I will discuss are limited to low orders and the required extension to all orders is not yet completed.

Let me remark that even though I have presented the simple Hamiltonian model of this chapter mainly to familiarize the reader with the aim of the thesis, the model itself is interesting because it may find applications in pion-nucleon physics when another Fock sector, with one fermion and two bosons, is included [6].

Chapter 3

Renormalization group procedure for Hamiltonians in quantum field theory

3.1 Introduction

In this chapter I present the formalism which forms the basis of the Chapters 4 and 5.

Section 3.2 summarizes the basic concepts of Quantum Field Theory (QFT) in the language of Hamiltonians. I review the standard formulation, using as an example a scalar theory with ϕ^3 interaction, and I introduce the notation used in this thesis. After I present the standard equal-time quantization and show some of its problems in Section 3.2.6, I introduce the light-front Hamiltonian approach, which avoids similar problems, beginning with Section 3.3.

Section 3.4 summarizes the Hamiltonian renormalization procedure called Renormalization Group Procedure for Effective Particles (RGPEP), whose application is the key focus of this thesis. I present here the motivation behind its use and some basic results; the application of RGPEP to bound states and scattering processes is described in Chapters 4 and 5.

3.2 Canonical Quantum Field Theory

This section reviews the basic construction of Quantum Field Theory (QFT), the basis for the modern description of elementary particles, in the language useful in Hamiltonian dynamics.

One of the best known QFTs is Quantum Electrodynamics (QED) [8-14]. Its formulation is based on two rules. The first of these is the correspondence principle: Maxwell equations of classical electrodynamics described very well most of electromagnetic phenomena known at the end of XIX century; quantum theory had to reproduce these equations in certain limits. The second rule is agreement with the requirements of special relativity. This led first to the formulation of the Dirac equation and then to the conclusion that a complete theory is not consistent within the framework of quantum mechanics of a fixed number of particles.

As the correspondence of quantum mechanics to classical mechanics was already well understood, QFT followed a similar path. The first attempts at formulating relativistic quantum mechanics led to some paradoxes (such as the Klein paradox) and prompted the introduction of the concept of antiparticles and interactions changing the number of particles. Relativistic quantum mechanics, based on a concept of a fixed number of interacting particles, proved untenable and only the introduction of quantum field theory put relativistic quantum theory into a consistent framework.

In this section, I describe the key steps in the Hamiltonian construction of a quantum theory of scalar fields. This example demonstrates the main steps, avoiding many of the complications of gauge theories. QED and QCD are only mentioned briefly here; the canonical quantization procedure for these theories is described in Appendix E.

3.2.1 Classical field theory Lagrangian

The path towards a first guess for the quantum theory of particles begins with a Lagrangian for a classical field. At this point it is easy to introduce into the theory the Lorentz symmetry or other symmetries. Nevertheless, it is not obvious that all such symmetries will hold in the quantum case (and, in fact, for some theories not all the symmetries will). Lagrangian also provides a way to connect the interacting theory, with a certain coupling constant $g \neq 0$, with the free theory in the limit $g \to 0$. The free theories are simpler and better understood, and for weak couplings many effects can be analyzed as perturbations around the free case.

One of the key assumptions of the quantum theories is that there exists a hermitian Hamiltonian operator, which is a generator of time evolution: time derivatives of operators are equal to their commutators with the Hamiltonian (times a universal constant). This assumption leads to conservation of probability. The Hamiltonian, however, is not a Poincaré scalar, and there is no easy way to require or check its Poincaré symmetry. Nevertheless, if the Hamiltonian is build according to the canonical rules, the Heisenberg equations for time evolution are equivalent to the classical Euler-Lagrange equations (which are covariant, provided the Lagrangian is a Poincaré scalar). This not only eases the building of symmetries into the theory, but also allows for correspondence with the classical theory. For example, following this path in the case of Quantum Electrodynamics (QED) leads to the Heisenberg equations which look like the classical Maxwell equations. One may then try to follow the path opened by QED in the case of other theories, even if the classical correspondence principle does not apply to them.

The standard construction consists of the following steps:

1. A Lagrangian density $\mathcal{L}$ is constructed;
2. Classical energy-momentum tensor $T^{\mu\nu}$ is calculated;
3. The fields are required to fulfill canonical commutation relations at equal time;
4. One of the components of $T^{\mu\nu}$ defines a Hamiltonian density.

Certain important points have been excluded from the list above – for example, as the quantum fields do not commute, it is important in which order they are written in $T^{\mu\nu}$. Usually, fields are expressed in terms of their Fourier components (which have a clear interpretation of creation and annihilation operators of particles) and the resulting operators (e.g. H) are normal ordered (corresponding to subtraction of the energy of the vacuum from all energies).

However, quantum theory as defined in this way leads to infinities when one calculates physical observables. To arrive at meaningful results that can be compared to experiments two more steps are required: the introduction of some form of regularization (to parametrize the infinities in terms of an ultraviolet cutoff) and renormalization (to remove artificial cutoff dependence). This can be done at the level of physical observables, e.g. a scattering matrix. However, such an approach cannot easily be extended to other types of predictions: knowing how to calculate a scattering matrix in a meaningful way does not automatically allow us to formulate a well-defined bound-state problem, and there is no clear connection between the

way one describes bound states and scattering when divergences are taken seriously. This is particularly important as there are experiments in which bound states are involved in scattering and the bound states structure matters in the final results.

Therefore, it is preferable to introduce renormalization already at the level of the quantum Hamiltonian (rather than observables, such as the scattering matrix). In principle this allows us to discuss all kinds of phenomena within a single theoretical description. However, as mentioned above, symmetry requirements are hard to maintain in a Hamiltonian approach: the cutoffs break the connection with the initial Poincaré-invariant Lagrangian, and there is the important question of whether it is possible to construct counterterms in the Hamiltonian that remove cutoff dependence from physical results and lead to covariant predictions and, if so, how it may happen. That it is, in fact, possible in the case of a simple model has been shown in Chapter 2; a similar result in the case of divergent QFTs is presented in Chapter 5.

An important fact about classical Lagrangians is that, using this language, symmetry requirements are very simple to formulate. Since the action of the theory is defined as a four-dimensional integral of the Lagrangian density:

$$\int \mathrm{d}t L\left[\phi, \partial^\mu \phi\right] = \int \mathrm{d}^4 x \, \mathcal{L}\left(\phi(x), \partial^\mu \phi(x)\right) \,, \tag{3.1}$$

it is enough to require the classical Lagrangian density to be a Poincaré invariant in order to define a classical theory respecting special relativity. This simple rule is obscured in the quantum theory because of the necessity of regulating divergences. Despite that, there is probably no better starting point for construction of a quantum field theory as a tool for particle physics.

Another advantage of the Lagrangian formulation is that, through the Noether theorem, independence of a Lagrangian from some variables automatically leads to conservation laws. In the case of Eq. (3.1), the fact that the Lagrangian density does not depend directly on the space and time coordinates, $\vec{x}$ and x^0, leads to conservation of momentum and energy.

In the following, I consider the example of a Lagrangian density describing a scalar field $\phi(x^\mu)$:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) + \frac{1}{3!} g \phi^3 \,. \tag{3.2}$$

The terms in parenthesis are kinetic terms with a mass m. These terms would correspond to a free field evolution (i.e. when $g = 0$). The last term is an interaction term; its strength is determined by a coupling constant g.

3.2.2 Classical equations of motion

Classical equations of motion (Euler-Lagrange equations) for the field ϕ are:

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} = \frac{\partial \mathcal{L}}{\partial \phi} \,. \tag{3.3}$$

In the case of Eq. (3.2), there is just one equation:

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi = \frac{1}{2} g \phi^2 \,. \tag{3.4}$$

3.2.3 Energy momentum tensor

The energy-momentum tensor density is:

$$T^{\mu\nu} := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \,. \tag{3.5}$$

The integrals of its components over space represent energy and momenta – quantities which are conserved due to the explicit independence of the Lagrangian density from x^μ.

By defining energy density, one can rewrite equations of motion in a formally equivalent Hamiltonian form. This is a canonical transformation of variables, from fields ϕ and their time derivatives $\dot{\phi}$ in the Lagrangian to fields ϕ and corresponding momenta π in the Hamiltonian.

The energy of the field at a given time x^0 is defined by the integral:

$$H = \int_{x^0=const} \mathrm{d}^3 x \, T^{00} \,, \tag{3.6}$$

and the canonical momentum π is:

$$\pi = \frac{\partial \mathcal{L}}{\partial_0 \phi} \,. \tag{3.7}$$

For the ϕ^3 theory defined by Eq. (3.2), these quantities are:

$$\pi = \partial^0 \phi \tag{3.8}$$

$$T^{00} = \frac{1}{2}\left(\pi^2 + (\vec{\partial}\phi)^2 + m^2\phi^2\right) - \frac{1}{3!} g\phi^3 \,. \tag{3.9}$$

The equations of motion are:

$$\begin{cases} \partial^0 \pi = -\dfrac{\partial T^{00}}{\partial \phi} & (3.10) \\ \partial^0 \phi = \dfrac{\partial T^{00}}{\partial \pi} \,. & (3.11) \end{cases}$$

By substituting the second of these equations into the first one, one can verify that this set of equations reproduces (3.4).

3.2.4 Field quantization, creation and annihilation operators

One of the most important lessons from classical quantum mechanics is that, if the position and momentum variables are replaced by operators, proper commutation relations between them are required. When their evolution is generated by the canonical Hamiltonian, then quantum equations of motion correspond to their classical counterparts. It can be demonstrated that, in the classical Ehrenfest limit, matrix elements of these quantum operators evolve in agreement with the classical equations of motion.

The same applies to field theories. One can require that, at a given time x^0:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x} - \vec{y}) \tag{3.12}$$

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \,. \tag{3.13}$$

In this case, the quantum equations of Heisenberg picture are:

$$\dot{\phi} = i[H,\phi] \tag{3.14}$$
$$\dot{\pi} = i[H,\pi] \ . \tag{3.15}$$

These equations state that H is the generator of a time-translation. Similarly, other space integrals of $T^{\mu\nu}$ have an interpretation of generators of space translations (momenta), rotations (angular momenta), or boost generators.

One can introduce creation and annihilation operators at time $x^0 = 0$ via a spatial Fourier transform of the fields:

$$\phi(\vec{x},x^0=0) = \int \frac{d^3k}{2(2\pi)^3 k^0}\left(a^\dagger_{\vec{k}} e^{-i\vec{k}\vec{x}} + a_{\vec{k}} e^{i\vec{k}\vec{x}}\right) \tag{3.16}$$
$$\pi(\vec{x},x^0=0) = \int \frac{d^3k}{2(2\pi)^3 k^0}\left(k^0_m a^\dagger_{\vec{k}} e^{-i\vec{k}\vec{x}} - k^0_m a_{\vec{k}} e^{i\vec{k}\vec{x}}\right) \ . \tag{3.17}$$

In both equations, there are only three-dimensional space-like integrals. Both equations are needed to express creation and annihilation operators in terms of fields. Note that the appearance of an extra free energy factor, $k^0_m = \sqrt{\vec{k}^2 + m^2}$, introduced in the second of these equations to represent the time derivative, means that some assumption about field evolution was made. In this case, it was assumed that at the time $x^0 = 0$ field ϕ evolves as a free field. The standard momentum integration measures are marked by $[k] := d^3k/\left(2(2\pi)^3 k^0\right)$, and a frequently encountered delta-function $\widetilde{\delta}(k) := (2\pi)^3\delta^3(\vec{k}) = \int d^3x \exp(-\vec{k}\vec{x})$.

Eq. (3.12) implies that the operators a_k fulfill:

$$\left[a_{\vec{k}}, a^\dagger_{\vec{p}}\right] = 2k^0\widetilde{\delta}(\vec{k}-\vec{p}) \ . \tag{3.18}$$

The Hamiltonian, given in Eq. (3.6) in terms of fields ϕ and momenta π, can be expressed in terms of creation and annihilation operators. Further, the space of states on which all operators act is defined in terms of creation and annihilation operators. By definition, there is a unique vacuum state $|0\rangle$ – the state that is annihilated by all annihilation operators:

$$a_{\vec{k}}|0\rangle = 0 \ . \tag{3.19}$$

All states can be written as combinations of the following basis states:

$$|p_1,\ldots,p_n\rangle = a^\dagger_{\vec{p}_1}\ldots a^\dagger_{\vec{p}_n}|0\rangle \ . \tag{3.20}$$

The space of all states with this basis is called the Fock space. Note that the action on any state of the form (3.20) of any operator expressed in terms of creation and annihilation operators, can be calculated using Eqs. (3.18) and (3.19). For example, for the free theory $\big($(Eq. 3.2) with $g=0\big)$, the Hamiltonian is[1]:

$$H_0 = \int [k]\sqrt{\vec{k}^2+m^2}\ a^\dagger_{\vec{k}} a_{\vec{k}} \ , \tag{3.21}$$

and, when acting on $|p_1,\ldots,p_n\rangle$, it gives:

$$H_0|p_1,\ldots,p_n\rangle = \left(p^0_{1,m}+\cdots+p^0_{n,m}\right)|p_1,\ldots,p_n\rangle \ , \tag{3.22}$$

[1] Actually, a constant c-number term was dropped, see discussion at the beginning of Sec.3.2.5.

where $p^0_{i,m} = \sqrt{\vec{p}_i^{\,2} + m^2}$. Thus, the Fock space basis (3.20) is the basis of the eigenstates of the free Hamiltonian H_0. It is a complete basis, and can be used in interacting theory, where eigenstates of H may be complicated combinations of these basis states.

Note also that, with H_0 given by Eq. (3.21), the energy of the vacuum is zero:

$$H_0 |0\rangle = 0 \,. \tag{3.23}$$

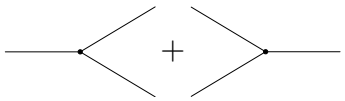
3.2.5 Quantum Hamiltonian of interacting theory

One may express the Hamiltonian operator of quantum field theory (3.6) in terms of the creation and annihilation operators, by substituting the quantum fields ϕ and π with their Fourier expansions (3.16)-(3.17). Using the commutation relations, one can order each expression in such a way that annihilation operators stand to the right of the creation operators. In the process, the δ-functions from commutators contribute additional c-number terms. Thes terms simply add the same – and in fact, infinite – quantity to the energy of each state. Since physically observable quantity is the difference between energies (e.g. of two different states) such terms may be dropped. The procedure of commuting annihilation operators to the right and dropping the δ-functions is called "normal ordering" and it is indicated by double dots (colon) to the left and right of an operator.

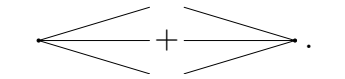
This leads to:

$$H = \int_{x^0=0} d^3x \,:T^{00}: = H_0 + H_Y + H_{\ni+\in} \tag{3.24}$$

$$H_0 = \int [k]\; k^0_m a^\dagger_k a_k \qquad = \tag{3.25}$$



$$H_Y = -\frac{1}{2} g \int [123] \widetilde{\delta}(1-2-3) \left(a^\dagger_1 a_2 a_3 + a^\dagger_3 a^\dagger_2 a_1 \right) = \tag{3.26}$$



$$H_{\ni+\in} = -\frac{1}{3!} g \int [123] \widetilde{\delta}(1+2+3) \left(a_1 a_2 a_3 + a^\dagger_3 a^\dagger_2 a^\dagger_1 \right) = \tag{3.27}$$

(In the above equations, I have used numbers to mark specific momenta: 1 means $\vec{k}_1$, etc.)

The first term, H_0, is the free Hamiltonian. Acting on a bare vacuum $|0\rangle$ it gives zero, and on one particle state $a^\dagger_k |0\rangle$, H_0 gives an eigenvalue factor – the free energy $k^0_m = \sqrt{\vec{k}^2 + m^2}$. This part does not change the structure of the state (the number of particles or their momenta).

The second and third terms, H_Y and $H_{\ni+\in}$ constitute the interaction Hamiltonian. In the limit $g \to 0$, they both vanish. These are two distinct parts. Terms in H_Y change the number of particles by one. For example, when this term acts on four-particle states, it gives a combination of three- and five-particle Fock states. Each term in H_Y has a creation operator on the left-hand-side and an annihilation operator on the right-hand-side, and therefore gives zero, both when acting on the vacuum ket and in matrix elements with the vacuum bra.

The third term, $H_{\ni+\in}$, changes the particle number by three. The part with three creation operators is the only part of H that does not vanish when acting on the vacuum, a fact that has important consequences (see Sec.5.2.5).

3.2.6 Problems

Although the Hamiltonian (3.24) appears quite reasonable, in fact it leads to a number of problems. I list some of them briefly here.

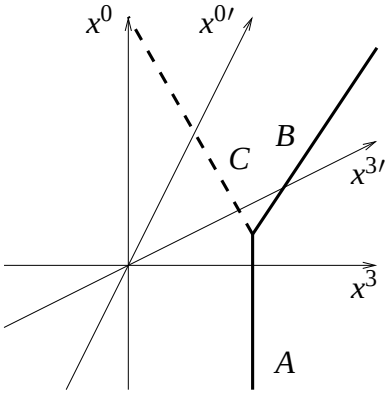


Figure 3.1: In equal-time coordinates (Sec.3.2), boosts are dynamical. For example, for a classical trajectory of a heavy particle (A) decaying into two lighter ones (B and C), it is possible that in one reference frame an observer sees the particle before the decay (at the time $x^0 = 0$) while an observer in another frame sees the result, after the decay (at the time $x^{0\prime} = 0$). Hence, one must know the dynamics in order to be able to describe processes in a boosted reference frame. Contrary to this, in the light-front coordinates (Sec.3.3), boosts are kinematical, and information about the state at $x^+ = 0$ in one reference frame can be immediately translated to the moving reference frame.

The first problem is that in the equal-time formulation of QFT, boost generators depend on interactions and are complicated. This means that, even if one knows a state in one reference frame, one still needs to know all the dynamics in order to predict the corresponding state seen by a moving observer (see Fig. 3.1). The situation is especially unsatisfactory for bound-state calculations. Spectra and wave functions of bound states are usually calculated in their rest frame of reference, and the theory is supposed to automatically take care of describing the states in motion. Yet, it would appear that no such explicit construction has been successfully carried out.

The second problem is, that $H_{\ni+\in}$ term in H does not vanish when acting on the vacuum. Thus the "bare vacuum" $|0\rangle$ is not an eigenstate of the Hamiltonian and a good starting point for solving the theory is lacking. The true vacuum – the ground state of the theory – has to be determined from dynamical equations and may be a very complicated state, involving infinitely many different Fock sectors (i.e., components with different numbers of bare particles).

There are many effects in QFT which are commonly interpreted as a result of the complicated vacuum structure (for example, spontaneous symmetry breaking). Nevertheless, it would be very useful if the theory could be reformulated in such a way that the vacuum was simple, and all "vacuum" effects reproduced by additional interaction terms in the Hamiltonian [15].

The third problem concerns requirements of special relativity in divergent QFTs. Let us imagine, for example, that we want to calculate the energy of a state of one physical particle of momentum $\vec{k}$. If the relativistic dispersion relation,

$$E = \sqrt{\vec{k}^2 + const.}\,, \tag{3.28}$$

is to be preserved, then the only acceptable correction is a change in the value of the particle's mass (*const.* in Eq. (3.28)). Assuming that the mass shift δm^2 is small (compared to $\vec{k}^2$ or the

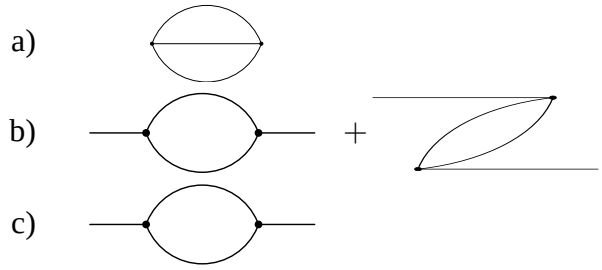


Figure 3.2: Examples of corrections to one-particle energy in the equal-time old fashioned perturbation theory (a-b) and light-front calculation (c). The vacuum term (a) shifts energy of all states and thus has no physical significance. Only two terms (b) (the "O" and the "Z" terms) contribute in the equal-time old-fashioned approach. But in the case (c) of the corresponding old-fashioned light-front Hamiltonian calculation there are no "Z" terms.

initial bare mass m^2) for small coupling constants, the square root can be expanded in a series:

$$E = \sqrt{\vec{k}^2 + m^2 + \delta m^2} \approx E_k^{(0)} + \frac{1}{2}\frac{\delta m^2}{E_k^{(0)}} - \frac{1}{8}\frac{\left(\delta m^2\right)^2}{(E_k^{(0)})^3} + \ldots \tag{3.29}$$

Thus, any small correction to the one particle energy $E_k^{(0)} = \sqrt{\vec{k}^2 + m^2}$ has to be of the form $const./E_k^{(0)}$, or the result will not be relativistic.

Let us now look at the two important second-order corrections to the one-particle energy. Up to the second order in the coupling constant, there are four terms:

$$E_k = \sqrt{\vec{k}^2 + m^2} + \Delta E^{(2)}_{k-O-} + \Delta E^{(2)}_{kZ} + \Delta E^{(2)}_0 \ . \tag{3.30}$$

The first term is the initial, free energy $E_k^{(0)}$. The second term comes from the H_Y acting twice, and corresponds to the first diagram in Figure 3.2(b). Next part comes from $H_{\ni+\in}$ acting twice, and corresponds to the second diagram in Figure 3.2(b) (a "Z" term). The final part comes from $H_{\ni+\in}$ acting twice, but forming a disconnected expression corresponding to the Figure 3.2(a). Since the second-order vacuum energy is exactly $\Delta E_0^{(2)}$, the shift of the physical energy of one particle state (that is, how much more energy than the vacuum this state has) in the second order in g is:

$$\Delta E_k^{(2)} = \Delta E^{(2)}_{k-O-} + \Delta E^{(2)}_{kZ} \ . \tag{3.31}$$

One may calculate it for 1+1 dimensional theory, obtaining a finite result:

$$\Delta E_k^{(2)} = \frac{1}{2}\frac{\delta m^2}{E_k^{(0)}} \tag{3.32}$$

$$\delta m^2 = -\frac{g^2}{4\pi m^2}\cdot f\left(\frac{k}{m}\right) \tag{3.33}$$

$$\begin{aligned} f(u) = & \int_{-\infty}^{\infty} \mathrm{d}p \left(\frac{1}{\sqrt{\left(p+\frac{u}{2}\right)^2+1}}+\frac{1}{\sqrt{\left(p-\frac{u}{2}\right)^2+1}}\right)\times \\ & \times \frac{1}{-(u^2+1)+\left(\sqrt{\left(p-\frac{u}{2}\right)^2+1}+\sqrt{\left(p+\frac{u}{2}\right)^2+1}\right)^2} \end{aligned} \tag{3.34}$$

(Note that in 1+1 dimensions, the coupling constant g has dimension of $mass^2$ – see Appendix A.5.) We can evaluate

$$f(0) = \frac{2\pi}{2\sqrt{3}}\,, \tag{3.35}$$

and the physical mass is:

$$m^2_{phys} = m^2 - \frac{g^2}{6\sqrt{3}m^2}\,. \tag{3.36}$$

(The light-front approach presented in Section 3.3 leads to the same result without relying on the expansion of the square root in Eq. (3.29), cf. Eq. (5.51)-(5.53).)

In the case of scalar theory in 3+1 dimensions, however, the second-order correction to the energy of one particle is divergent. It requires regularization and renormalization. One has to make sure that both diagrams in Figure 3.2(a) are regulated in a consistent way, and that there is no additional non-covariant finite dependence on regularization (for example, as a function of $\vec{k}/m$). Covariant regularizations, such as the dimensional regularization or Pauli-Villars regularization, are not easy to extend to the Hamiltonian formulation in its explicit operator form. And when one cuts off momenta the result of integration will produce a whole function of $\vec{k}$ divided by the cutoff parameter.

A similar problem, discussed in more detail in Chapter 5, occurs due to non-covariant regularization in scattering amplitudes. It is difficult to introduce regularization in standard canonical Hamiltonians that lead to diagrams with different orderings (analogous to the *–O–* and *Z* energy corrections discussed above) in such a way that the whole expression preserves the Poincaré symmetry. These problems with a momentum cutoff regularization seem very unfortunate, because Hamiltonian-based calculations – in particular, the old-fashioned perturbation theory – are very attractive from a conceptual point of view. Among its advantages are a clear physical interpretation of the energy operator and an explicit construction of states in the Fock space. Nevertheless, because of the problems with ultraviolet divergences, regularization and renormalization of Hamiltonians, it was long believed that Hamiltonian-based calculations could not produce relativistic results, and this approach to QFT was neglected for many years. Covariantly regularized Feynman diagrams are beautiful and more effective in perturbation theory for scattering processes. It is very interesting to observe, the the properties of Hamiltonian calculations dramatically change when one switches from the standard time evolution to the light-front of dynamics introduced in the next section.

3.3 Light-front Hamiltonian dynamics

3.3.1 Forms of relativistic dynamics

Different forms of relativistic classical dynamics were analyzed by Dirac in [16]. The dynamical problem is usually formulated as follows: certain arbitrary initial conditions are imposed on a given hyperplane in the space-time. The dynamical equations (Euler-Lagrange equations in Lagrangian dynamics) then allow us to predict what happens with the system elsewhere in the space-time.

Three forms were considered by Dirac:

1) Equal-time ("instant") form : this is the standard form presented in Section 3.2. The initial conditions are specified at a time $x^0 = 0$ (or some other fixed time), and the generator of evolution in the time direction is the energy P^0;

2) Point form : here, the initial conditions are specified at a 4-dimensional hyperboloid $x_0^2 - \vec{x}^2 = a$, and the evolution proceeds in directions perpendicular to this hyper-surface;

3) Light-front form : here, the initial conditions are specified on a plane $x^0 = -x^3$ (which is tangent to a light cone, Fig. 3.3), and dynamical evolution proceeds in x^+, the dynamical generator being P^-.

Choosing one of these three forms looks like a simple choice of parameters for describing one and the same physics. However, from the point of view of special relativity and quantum mechanics, there are some important differences between them.

Special relativity requirements can be formulated in terms of the Poincaré algebra (generators of translations, rotations, and boosts). The same reference frame can be reached by performing different transformations in different orders. Whatever transformations one chooses, and in whatever order they are placed, the description of physical processes must produce the same results. Therefore, there are fixed relations that have to be fulfilled by the Poincaré generators. In a classical theory, these are:

$$\begin{aligned} \{P^\mu, P^\nu\} &= 0 \\ \{M^{\mu\nu}, P^\rho\} &= g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu \\ \{M^{\mu\nu}, M^{\rho\sigma}\} &= g^{\mu\sigma}M^{\nu\rho} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\nu\rho}M^{\mu\sigma}, \end{aligned} \tag{3.37}$$

where P^μ are generators of translations and $M^{\mu\nu}$ are generators of rotations and boosts[2]. In the quantum case, the Poisson brackets $\{\}$ are replaced by commutators.

In the free case, all these generators are easy to construct on the basis of general rules. In interacting theory, the situation is more complicated. For example, let us look at the equal-time theory defined in Section 3.2.5. For the free theory, with $P^0 = H_0$ of Eq. (3.21), all other

[2]More specifically, the angular momentum operators are ($i = 1,2,3$):

$$L^i = \frac{1}{2}\varepsilon^{ijk}M^{jk}\,,$$

and boosts are generated by:

$$K^i = M^{0i}\,.$$

generators are easy to find: for example, the space translation generators are:

$$\vec{P} = \int [k] \vec{k}\, a_k^\dagger a_k \,. \tag{3.38}$$

In the interacting theory (i.e., the one with the coupling constant $g \neq 0$) the conditions defining the Poincaré algebra (3.37) can be split into two parts. Because $g^{\mu\nu}$ is diagonal, the (i, j) components of P and M are not influenced by the change of the interaction part of P^0. They are "kinematical" (i.e., simple, not influenced by the interaction – their form in the interacting theory is exactly the same as in the free theory). However, M^{0i} generating boosts have to fulfill the condition:

$$\left[M^{0i}, P^0\right] \;=\; -P^i \,. \tag{3.39}$$

This means that when P^0 changes (i.e. when there is an additional interaction term), M^{0i} has to change accordingly, such that the commutators still produce the kinematical momentum P^i of the known, simple form.

There does exist a geometrical interpretation of these considerations, albeit a slightly simplified one. In the equal-time form of dynamics, one specifies initial conditions at $x^0 = 0$ hyperplane, and P^0 (the time evolution operator) is responsible for generating the solution of the dynamical problem at all other times. If P^0 is changed, a different solution will be generated. Now, all other Poincaré algebra generators can be split into two groups. The six transformations that do not change the $x^0 = 0$ plane (three translations and three rotations) do not change the system in any way that depends on the interaction in P^0. On the other hand, the three boosts move the system off the $x^0 = 0$ plane:

$$x^0 \to x^{0\prime} = x^0 \cosh\omega + \frac{\vec{v}\cdot\vec{x}}{|v|} \sinh\omega \tag{3.40}$$

(with $\tanh\omega = v$ where $\vec{v}$ is the speed of the new reference frame (x') relative to the old one (x)), and must be modified in the interacting theory (c.f. Fig.3.1).

3.3.2 Light-front coordinates

Dirac's LF form of dynamics [16] contains as many as seven kinematical Poincaré generators. This makes it easier to formulate a relativistic theory in the LF form than in the equal-time form.

Light-front dynamics is defined in terms of new coordinates:

$$\left(x^+, x^-, x^1, x^2\right) \tag{3.41}$$

$$x^+ = x^0 + x^3 \tag{3.42}$$

$$x^- = x^0 - x^3, \tag{3.43}$$

instead of (x^0, x^1, x^2, x^3). One specifies the initial conditions at $x^+ = 0$ (see Fig.3.3). New $+, -$ components of any covariant four-vector are specified in the same way.

The metric tensor is:

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{3.44}$$

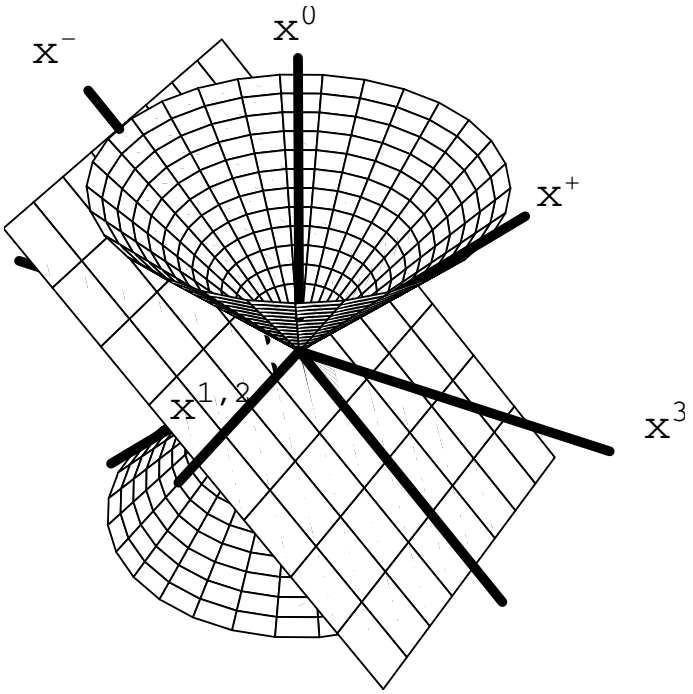


Figure 3.3: Light-front initial conditions are specified on a hyperplane $x^+ = x^0 + x^3 = 0$ (a null-plane tangent to the light cone); dynamical evolution determines the system's state at different light-front times x^+.

For example, a four-dimensional product of two four-vectors, x^μ and p^μ is:

$$x^\mu p_\mu = \frac{1}{2}x^+ p^- + \frac{1}{2}x^- p^+ - x^\perp p^\perp \,, \tag{3.45}$$

where $x^\perp = (x^1, x^2)$. In particular, this means that a momentum conjugate to x^+ (i.e., the evolution operator – the Hamiltonian) is:

$$\frac{1}{2}p^- = p_+ = i\frac{\partial}{\partial x^+} \,, \tag{3.46}$$

and a momentum conjugate to x^- is $p^+ = -i2\partial/\partial x^-$.

Further, square of a four-vector p^μ in the light-front coordinates is:

$$p^\mu p_\mu = \frac{1}{2}p^+ p^- + \frac{1}{2}p^- p^+ - p^{\perp 2} = p^- p^+ - p^{\perp 2} \,, \tag{3.47}$$

and the dispersion relation for a particle on mass shell ($m^2 = p^\mu p_\mu = p^- p^+ - p^{\perp 2}$) becomes:

$$p^-_m \;=\; \frac{p^{\perp 2} + m^2}{p^+} \,. \tag{3.48}$$

This is a light-front analogue of the equal-time dispersion relation:

$$p^0_m = \sqrt{\vec{p}^{\,2} + m^2} \,. \tag{3.49}$$

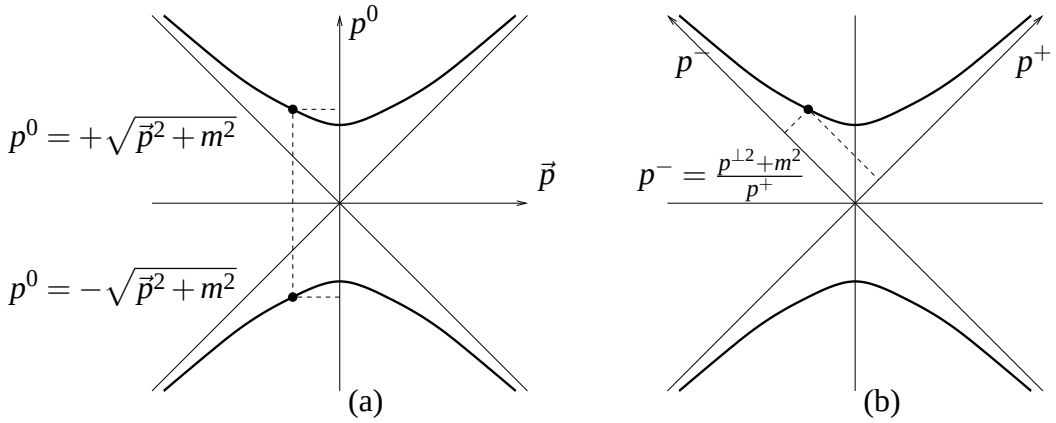


Figure 3.4: (a) In the equal-time dynamics, for each space-like momentum $\vec{p}$ there are two values of p^0 possible on mass shell: $p^0 = \pm\sqrt{\vec{p}^2 + m^2}$; evolution of particles forward in time ($p^0 \geq 0$) is an additional condition which cannot be imposed as a condition on momenta $\vec{p}$. (b) In the light-front dynamics, positiveness of p^+ momentum results in a positive value of the light-front energy $p^- = (p^{\perp 2} + m^2)/p^+$.

There are a number of important differences. First, p^- is not squared in Eq. (3.47) and therefore knowledge of the three-momentum $p^{+,\perp}$ determines the light-front energy p^- without ambiguity in its sign. This has important consequences. For example, as shown in Chapter 5, for physical three-momentum $p^{+,\perp}$, the scalar particle's propagator has only one pole at physical p^- (3.48). This is quite different from the equal-time variables, where for each three-momentum there are two corresponding poles: at p^0 greater than, and less than zero. Thus, in the equal-time form, one has to specify the condition of forward-evolution of particles as an additional condition on their energies $p^0 \geq 0$, which is needed irrespective of particle's three-momentum $\vec{p}$. In the light-front dynamics, the condition $p^- \geq 0$ automatically follows from a kinematical condition $p^+ \geq 0$. In other words, a requirement on the light-front space-like momentum, $p^+ \geq 0$ chooses the forward light-cone for the evolution of particles (Fig.3.4).

In the light-front form, the p^- is the Hamiltonian and $\vec{p} = (p^+, p^\perp)$ constitutes a three-vector of generators of translations in space directions $(x^-, x^\perp)$.

Let us consider a boost in the direction x^3:

$$x^0 \longrightarrow x^0 \cosh\omega + x^3 \sinh\omega \tag{3.50}$$
$$x^3 \longrightarrow x^0 \sinh\omega + x^3 \cosh\omega\,. \tag{3.51}$$

By adding and subtracting these equations, one gets:

$$x^+ \longrightarrow e^{\omega} x^+ \tag{3.52}$$
$$x^- \longrightarrow e^{-\omega} x^-\,. \tag{3.53}$$

This leads to a second difference. In the light-front coordinates, this boost is a simple rescaling operation: it does not mix light-front time (x^+) and space ($x^{-,\perp}$) coordinates, and it preserves the hyperplane $x^+ = 0$.

In total, there are seven kinematical Poincaré generators in the front form (compared to six in the equal-time form): three space-like translations ($P^{+,\perp}$), z-direction boost (M^{+-}), rotation around z-axis (M^{12}), and two mixtures of boosts and rotations (M^{+i}) [17]. Only the other

three Poincaré transformations (M^{-i}) are dynamical and are modified when the Hamiltonian P^- changes.[3]

3.3.3 Quantum Field Theory Hamiltonian

For introduction to light-front quantization see [18-23] and [24]. For a review of modern applications of the light-front formalism in QFTs, see e.g. [25].

For the scalar ϕ^3 Lagrangian (3.2), one can introduce canonical momentum π:

$$\pi \;=\; \partial^+\phi = 2\frac{\partial}{\partial x^-}\phi\,, \tag{3.54}$$

and request commutation relations:

$$[\phi(x),\pi(y)]_{x^+=y^+=0} = i\delta(x^- - y^-)\delta^2(x^\perp - y^\perp)\,. \tag{3.55}$$

Note that the space-like derivative in (3.54) can be formally inverted, leading to:

$$[\phi(x),\phi(y)]_{x^+=y^+=0} = \frac{-i}{4}sgn(x^- - y^-)\delta^2(x^\perp - y^\perp) \tag{3.56}$$

(this can be also verified by differentiating the last equation over y^-).

Fourier expansion of the field at $x^+=0$ introduces creation and annihilation operators:

$$\phi(x^+=0,x^-,x^\perp) = \int [k]\left(a_k^\dagger e^{i(k^+x^-/2-k^\perp x^\perp)} + a_k e^{-i(k^+x^-/2-k^\perp x^\perp)}\right), \tag{3.57}$$

where $[k] = \mathrm{d}^2k^\perp \mathrm{d}k^+\theta(k^+)/2(2\pi)^3$. Eq. (3.55) requires that a_k fulfill:

$$\left[a_k, a_p^\dagger\right] = 2(2\pi)^3 p^+\delta(k^+ - p^+)\delta^2(k^\perp - p^\perp)\,. \tag{3.58}$$

Using (3.5), one may calculate the $+-$ component of the energy momentum tensor:

$$T^{+-} \;=\; \partial^\perp\phi\partial^\perp\phi + m^2\phi^2 - \frac{1}{3}g\phi^3\,, \tag{3.59}$$

and the Hamiltonian:

$$P^- = \int_{x^+=0} \mathrm{d}^2x^\perp \mathrm{d}x_+ : T^{+-} :\,. \tag{3.60}$$

This Hamiltonian can be split into two parts: H_0, H_Y.

$$H \;=\; H_0 + H_Y \tag{3.61}$$

$$H_0 \;=\; \frac{1}{2}\int \mathrm{d}^2x^\perp \mathrm{d}x^- : (\partial^\perp\phi\partial^\perp\phi + m^2\phi^2) : = \int [k]\frac{k^{\perp 2}+m^2}{k^+}a_k^\dagger a_k \tag{3.62}$$

$$H_Y \;=\; \frac{1}{2}\int \mathrm{d}^2x^\perp \mathrm{d}x^- : (-\frac{1}{3}g\phi^3) : = \tag{3.63}$$

$$\;=\; -\frac{g}{2}\int [123]2(2\pi)^3\delta^2(1^\perp + 2^\perp - 3^\perp)\delta(1^+ + 2^+ - 3^+)\left(a_1^\dagger a_2^\dagger a_3 + a_3^\dagger a_1 a_2\right). \tag{3.64}$$

[3] Sometimes in the literature, all the dynamical Poincaré generators are referred to as "Hamiltonians".

In this Hamiltonian, there are no terms similar to (3.27) in the equal-time Hamiltonian, such as:

$$H_{\ni} = -\frac{1}{3!}g\int [123]2(2\pi)^3\delta^2(1^\perp+2^\perp+3^\perp)\delta(1^+ +2^+ +3^+)a_1^\dagger a_2^\dagger a_3^\dagger\,. \qquad (3.65)$$

This term would create three additional particles when acting on any state. Note, however, that p^+ momenta of these particles would have to add up to zero. As all p^+ in this expression are greater than or equal to zero, all three created particles would have p^+ momenta equal exactly to zero. For the massive particles considered here, such momenta mean infinite free energy p^-, and are not possible. Figure 3.4 shows that there is no point on the $p^2 = m^2$ hyperboloid which would corresponds to such situation. Instead, zero p^+ momenta would correspond to a limit of a particle moving with the speed of light in the direction $-x^3$. But as long as the limit is not reached, even for a very fast particle, $p^+ \neq 0$. On the other hand, all $p_i^+ = 0$ are possible if all the created particles are massless. In this thesis, only the bosons in Chapter 4 are massless. Nevertheless, in a term such as the one above, they would be accompanied by a massive fermion and an anti-fermion for which p^+ cannot be equal zero. Thus, terms of the type (3.65) do not matter in this thesis.[4]

Let us turn now to the question of whether results presented here could be extended to other theories. The question arises of whether the terms in a Hamiltonian, such as (3.65), should be taken into account in a massless theory. Particles of zero p^+ are the ones moving exactly in the direction of $-x^3$, see also Fig.5.2. For an external particle scattered on a target, it is enough to choose the x^3 axis not in the direction of motion of the incoming particle to avoid such a problem, and the result should not depend on an artificial choice of coordinates. But for massless particles whose momenta are not fixed by experimental conditions (for example, the final particles in a measurement of a total cross-section), p^+ cannot be made non-zero by a simple choice of coordinates (for example, because one integrates over all possible p^+). Moreover, in the standard calculation of scattering cross sections based on Feynman diagrams, massless particles in the final state may also create complications. These are usually handled by treating the particles as massive and, in the final result, taking a limit $m \to 0$. This is equivalent to including a low-energy (infra-red) cutoff, parametrized by the artificial mass. In Hamiltonian light-front theory, the situation is even more complicated. For gauge theories, choosing the light-front gauge leads to additional powers of p^+-momenta in denominators of certain expressions (see for example a sea-gull term (E.51) and other expressions in Appendix F). This leads to terms divergent for $p^+ \to 0$, irrespective of whether the particles are massive or not. An additional cutoff parameter must be added, and renormalization is more complicated. This additional (and perhaps inevitable) small-p^+ cutoff means that, again, there are no particles of exactly zero p^+, and the delta function in (3.65) cannot be fulfilled.

Within this approach, the canonical light-front Hamiltonian for ϕ^3 scalar theory has only two kinds of terms: (3.62) and (3.64). There are no terms of type (3.65). This simplifies the theory significantly compared to the equal-time Hamiltonian theory. For example, the LF Hamiltonian, when acting on the vacuum state, gives zero: the Fock space vacuum is the physical vacuum, i.e., the eigenstate of the Hamiltonian. As there are no particles of $p^+ = 0$ in this theory (all particles are massive, or a small-p^+ cutoff is introduced), the physical vacuum is simple, having no components with a non-zero number of particles.

This simplification is not possible in the equal-time formulation presented in Section 3.2. The three-dimensional delta function in Eq. (3.27) can be easily fulfilled with the creation of

[4] When I discuss counterterms in QED and QCD, the small-x divergences pose an additional problem but not of the ultraviolet type. See Appendix F.

three particles of finite momenta (and finite energies). In particular, the Fock vacuum state ("free" or "bare" vacuum) is coupled to a huge range of states and it is hard to find the physical vacuum which should be the starting point for any description of physical processes. Nevertheless, the two formulations of the theory may be consistent – one expects that the introduction of the small-p^+ cutoff in P^- operator, which leads to a simple vacuum, will be accompanied by extra counterterms in the Hamiltonian, which may create effects usually attributed to a complicated vacuum structure [15]. See also [26, 27], where issues of discretization of p^+ are considered. Discretized p^+ has also been used by Pauli and Brodsky [28, 29]. One has to be careful when discussing global lower bounds on p^+, because they violate boost invariance and canonical representation of boost symmetry in quantum mechanics does not allow for scaling of the lower bound on p^+ when one boosts bound states [30].

Readers interested in vacuum issues related to small-p^+ singularities can consult Refs. [31-34]. Vacuum studies in Lagrangian approach can be traced starting from [35, 36].

3.3.4 Arguments for and against the light-front Hamiltonian approach

The light-front approach to QFT has a number of pros and cons compared to the equal-time approach. I summarize some of the advantages bellow; the specific examples in Chapters 4 and 5 will make them even more explicit.

First of all, the light-front form of dynamics features kinematical boosts. To describe a process in a boosted frame of reference (or to describe, e.g., a bound state in relativistic motion), one does not have to solve a complicated dynamical problem. This allows one to connect measured or calculated properties of a bound state at rest and in motion with huge energy [37, 38].

The second advantage is that this approach is simpler than the equal-time Hamiltonian theory due to the following facts. The requirement that p^+ of all particles is greater than zero can be imposed even for massless theory, and there are no terms in LF Hamiltonians of type (3.65). This leads to two effects. First, Z diagrams do not appear in perturbation theory (cf. Fig.3.2), and it is possible to introduce cutoffs and counterterms in such a way that the Hamiltonian calculation leads to relativistic results for a scattering amplitude (see Chapter 5). Second, the vacuum structure is simple. Thus, the bare vacuum is a good starting point when building physical states, although additional terms may be needed in Hamiltonians to represent dynamical effects usually associated with the vacuum structure.

The third advantage of the LF form is that the dispersion relation (3.48) is considerably simpler than its equal-time counterpart, since the dependence of energy p^- on the momenta $p^{+,\perp}$ is a rational expression.

However, the light-front approach also introduces a number of problems. Probably the most important is lack of manifest rotational invariance. Formally, one may calculate all components $M^{\alpha\beta}$ of the Poincaré generators:

$$M^{\alpha\beta} = \int_{x^+=0} \mathrm{d}^2x^\perp \mathrm{d}x_+ \left(x^\alpha T^{+\beta} - x^\beta T^{+\alpha} \right) . \tag{3.66}$$

But in the case of the interacting theory not only the Hamiltonian P^-, but also all the dynamical generators M^{-i} lead to divergences. For example, one cannot exponentiate a dynamical generator to get a Poincaré operator, because already a square of any dynamical operator is infinite. One is forced to include regularization to make the operators meaningful, but this in turn leads to violation of the Poincaré algebra, as the regulated generators no longer commute

as expected (cf. Eq. (3.37)). Thus, any divergent quantum theory may lead to violation of Poincaré symmetries, and it is necessary to find counterterms not only in the Hamiltonian P^-, but also in other dynamical operators. Recently, Masłowski and Głazek have shown [39] that the effective particles approach allows one to construct regularized and renormalized effective Poincaré generators which fulfill the algebra conditions order by order in perturbation theory (despite the use of Hamiltonian form factors and non-covariant cutoffs).

3.4 Renormalization Group Procedure for Effective Particles

Below I present key elements of the Renormalization Group Procedure for Effective Particles (RGPEP). I show how this procedure works in the case of a scalar ϕ^3 theory, deriving the effective Hamiltonian up to the second order in perturbation theory in the bare coupling constant. I show the differences between theories in 1+1 and 3+1 dimensions (in 1+1 dimensions, the theory is not divergent and no counterterms are needed); I also review theories other than ϕ^3. The renormalization procedure presented here is applied to the description of bound states and scattering in Chapters 4 and 5.

3.4.1 Regularization

Both the equal-time Hamiltonian (3.24) and the light-front Hamiltonian (3.61) lead to divergent results when physical observables (such as scattering cross-section) are calculated in perturbation theory. Since one cannot even multiply these operators, a statement such as "$e^{-iHx^+/2}$ is an operator of translation in x^+ time" is not mathematically meaningful.

Regularization means that the interaction terms in a Hamiltonian are supplied with factors limiting the range of integration over momenta. These can be sharp θ cutoffs on particles momenta or smooth functions that vanish fast enough for large momenta. They are arbitrary, and are included in the theory only in order to make the Hamiltonian expression meaningful. Each cutoff factor adds a cutoff parameter (Δ), which determines how big a range of momenta is included in the Hamiltonian. The physical results calculated using the regularized Hamiltonian depend on Δ, and diverge for $\Delta \to \infty$. It is assumed that one can add to the Hamiltonian extra terms (counterterms) that explicitly depend on Δ in such a way that physical results have finite, well-defined limits for $\Delta \to \infty$. The procedure of finding counterterms for a given cutoff is called renormalization.

By introducing the cutoffs, the naive connection between the quantum Hamiltonian and the classical Lagrangian is lost. In particular, regulating factors usually violate some of the symmetries of the classical theory. Yet, some form of regulators are necessary, since quantum theory is mathematically meaningful only if the regulators are there. The quantization procedure allows for a reasonable first guess of the quantum theory Hamiltonian, but extra (steps such as the regularization) are needed. The fact that the Hamiltonian is not strictly derived from classical theory does not have to invalidate the procedure: we are trying to build the quantum theory, and the canonical Hamiltonian represents a good starting point.

Of course, this does not mean that it does not matter how one regularizes the Hamiltonian. Finding the proper counterterms may be easier or more difficult depending on the regularization. This brings us to the question of what regularization is the best. Here, above all, one should try to break as few classical symmetries as possible, if they are to be respected by physical results.

If regulators violate them, counterterms should be chosen in such a way that the symmetry is restored; if possible, it is preferable not to break such symmetries in the first place.

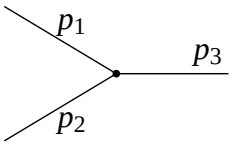
Each term of the Hamiltonian annihilates some particles and creates others. It may act on a state with many particles, some of which do not take part in the interaction. Regulators should only involve momenta of particles taking part in the interaction. Otherwise, the counterterm would have to depend on the spectator particles also, and would have to depend on a Fock sector (and be written in terms of projections on some Fock sectors, rather than in terms of creation and annihilation operators acting in the same way in different sectors). Such counterterms seem very complicated, as they involve independent constants in different Fock sectors. In particular, the results of an experiment involving one set of particles would not be enough to fix counterterms that could be afterwards used for different processes in another set of particles.

Ideally, the regulators depend on relative momenta only. For each particle involved in the interaction, a relative momentum $x, \kappa^\perp$ can be defined:

$$p^+_{child} = x_{c/p} P^+_{parent} \tag{3.67}$$

$$p^\perp_{child} = x_{c/p} P^\perp_{parent} + \kappa^\perp_{c/p} \,. \tag{3.68}$$

For example, in a term on the following picture:



there are two relative momenta, with the first child-particle having ($x_{1/3}$, $\kappa_{1/3}$) and the second child-particle having ($x_{2/3}$, $\kappa_{2/3}$):

$$x_{1/3} = \frac{p^+_1}{p^+_3} = 1 - x_{2/3} \tag{3.69}$$

$$\kappa^\perp_{1/3} = p^\perp_1 - \frac{p^+_1}{p^+_3} p^\perp_3 = -\kappa_{2/3} \,, \tag{3.70}$$

while the parent-particle has p^+_3 and $p^\perp_3$. $x_{1/3}$ will be simply called x, and $\kappa^\perp_{1/3} =: \kappa^\perp$.

The regularized Hamiltonian of the ϕ^3 theory is thus:

$$H^\Delta = H_0 + H^\Delta_Y + X^\Delta \,, \tag{3.71}$$

where H_0 is not changed (see (3.62)), H^Δ_Y is:

$$H^\Delta_Y = -\frac{g}{2} \int [123] \widetilde{\delta}(1+2-3) \left(a^\dagger_1 a^\dagger_2 a_3 + a^\dagger_3 a_1 a_2 \right) r_\Delta(x_{1/3}, \kappa_{1/3}) r_\Delta(x_{2/3}, \kappa_{2/3}) \,, \tag{3.72}$$

and X^Δ are unknown counterterms which are to remove the dependence of the result on regulators r_Δ (the standard three-dimensional delta functions $\widetilde{\delta}$ are defined in Appendix A).

The regulators used throughout the Chapters 4 and 5 are:

$$r_\Delta(x, \kappa^\perp) = \exp\left(-\frac{\kappa^{\perp 2}}{\Delta^2} \right) \,. \tag{3.73}$$

In fact, it is possible use a more general form:

$$r_\Delta(x, \kappa^\perp) = \exp\left(-\eta(x) \frac{\kappa^{\perp 2}}{\Delta^2} \right) \tag{3.74}$$

(cf. [40]). The advantage of (3.73) is that it is simpler, whereas proper choice of the $\eta(x)$ function may preserve more symmetries. For example, choosing $\eta(x) = 1/x$ leads for the full vertex (3.72) to the exponent $\kappa^2/(x(1-x))$, which is very similar to the free invariant mass of the created or annihilated particle pair. Many modifications are possible here, and each will require different counterterms.[5]

The renormalization procedure is described in Section 3.4.2. It is not entirely straightforward to find counterterms X^Δ based on the condition that they remove divergent dependence on regularization from physical observables. This is because observables usually involve some special situation – for example, in an S matrix, the free energy of initial and final particles is the same, while a full operator X^Δ has to be defined also for momenta not respecting this restriction. It is even more difficult to take into account the finite effects of regularization, and these have an impact on whether the results are covariant.

A well-known example of the physically important finite effect of regularization is found in the calculation of electron magnetic moment $(g-2)$ in QED. The result in order e^3 is finite, but it comes from a difference of two divergent terms [43]. If one regularizes them in a non-covariant way, omitting the renormalization, the divergences will cancel each other and the result will be finite, but wrong (in fact, it will be arbitrary, depending on the regularization). Thus, a renormalization procedure has to take into account not only the divergent terms, but also finite dependence on regularization. In the model example presented in Chapter 2, finite parts of counterterms had to fulfill certain relations for the theory to lead to relativistic results. Fixing the finite parts of the counterterms so that they lead to covariant results will be further discussed in Chapter 5.

In discussing the finite effects of regularization, I use the more general form of r_Δ, without specifying the function $\eta(x)$.

In the case of gauge theories such as QED and QCD, there are additional divergences due to small p^+-momenta of particles involved in the interaction. As a result, extra regulators are needed [15]. Such divergences do not appear in the theories considered in detail in this thesis (see also the discussion in the end of Sec. 3.3.3 and Appendix F).

3.4.2 RGPEP — Physical motivation

The divergences in the results obtained from the canonical Hamiltonian are caused by the fact that the Hamiltonian couples each state to other states in a huge range of momenta, and coupling between small-free-energy (or small momentum) and large-free-energy (or large momentum) states is important. The fact that a state is coupled strongly to high-energy states means that it is also strongly coupled to other Fock sectors – Heisenberg's uncertainty principle suggests that if high-energy states are important, so is the creation of additional particles. This adds to the complexities of quantum field theory: states of different momenta and different numbers of particles are coupled and the eigenstates of the Hamiltonian are likely to be complicated, multi-sector and wide-momentum-range states.

In renormalization theory, one assumes that there are counterterms in the regularized Hamiltonian that make low-energy dynamics independent of high-energy details of the theory, and in particular of the ultraviolet cutoff. This is a requirement of existence of an effective theory: one only has access to experiments at limited energies, and if low-energy predictions of the theory

[5] Note that (3.74) is still not the most general form of possible r_Δ. One should also be aware of other possible regularizations, for example, introduction of Pauli-Villars bosons with imaginary couplings [41, 42].

were sensitive to all the details of the high-energy physics, the theory would have no predictive power.

The RGPEP procedure is based on the idea of limiting the range of energies that interactions can couple, rather than only looking at states of limited energies. The latter alternative forms the basis of Wilson's renormalization group [44]. In that formulation, it is hard to make a Hamiltonian theory agree with special relativity, as the possible energy range is arbitrarily limited.

RGPEP avoids the problem of Wilson's theory in the case of Hamiltonians. It also introduces effective particles that allow us to formulate a well-defined bound-state problem with a few effective particles.

The idea of effective degrees of freedom is, in fact, more general. Essentially, it means that, for particular process, a description in terms of specifically defined degrees of freedom (which are superpositions of the original ones) may be significantly simpler (see [2]).

The effective theory should, however, describe the same physics. In particular, all the spectra, S matrices, and other observables, should be the same as in the original theory. For a QFT Hamiltonian, the easiest way to do this is to change the basis of the creation and annihilation operators, i.e., to define new operators $a^\dagger_\lambda$ as combinations of the original a, $a^\dagger$ and re-express the Hamiltonian in terms of new degrees of freedom:

$$\mathcal{H}_\lambda\,(a_\lambda) = H(a)\;. \tag{3.75}$$

RGPEP introduces the simple, effective theory by the requirement that the effective Hamiltonian $\mathcal{H}_\lambda$ contains form factors and its matrix is band-diagonal. This means that $\mathcal{H}_\lambda$ couples each state to only a limited range of states of similar momenta; coupling between different Fock sectors is also limited. Note that this is exactly what is expected of the effective (constituent) quarks. The fact that a simple description of hadrons as bound states of a fixed number of constituents is successful, means that the most important component (e.g., the one with three quarks in the case of baryons), couples weakly with other sectors.

3.4.3 Effective particles and renormalization group equations

RGPEP is defined as a rotation of the basis of Fock-space operators [45, 46]. For each "bare particle" created by an operator $a^\dagger_\infty$, there is a family of "effective particles" (parametrized by a parameter λ), defined as being created by unitarily equivalent operators $a^\dagger_\lambda$:

$$a^\dagger_\lambda = U_\lambda a^\dagger_\infty U^\dagger_\lambda\;. \tag{3.76}$$

The same Hamiltonian operator can be expressed in terms of both operators:

$$\mathcal{H}_\lambda\,(a_\lambda) = H(a_\infty)\;, \tag{3.77}$$

where the functional dependence of the Hamiltonian operator on a_λ (marked $\mathcal{H}_\lambda$) is different from the functional dependence on a_∞ (marked H). Re-expressing the last equation in terms of one operator basis, a_∞, and using the fact that each term of H is a product of some number of creation and annihilation operators, gives the equation:

$$\mathcal{H}_\lambda\,(a_\infty) = U^\dagger_\lambda H(a_\infty) U_\lambda\;. \tag{3.78}$$

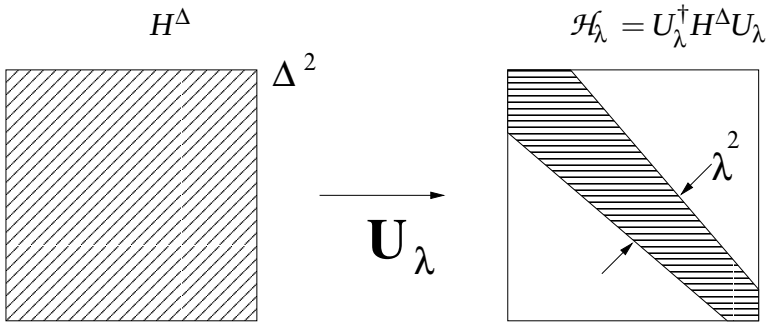


Figure 3.5: By a unitary (similarity) rotation the bare Hamiltonian H^Δ is transformed into an effective Hamiltonian $\mathcal{H}_\lambda$, which is narrow (of width λ), but equivalent to H^Δ (because of an appropriate change of elements within the non-vanishing near-diagonal band).

This means that elements of the operator $\mathcal{H}_\lambda$ come from a unitary rotation of H.[6] The unitary equivalence of H and $\mathcal{H}_\lambda$ means that both operators describe the same physics. For example, they have equal eigenvalues and eigenstates – the energy spectrum of physical states.

I now proceed to defining a specific form of the unitary operator U_λ. For more details see [45, 40].

As argued in Section 3.4.2, $\mathcal{H}_\lambda$ will have interpretation of an effective Hamiltonian and $a_\lambda^\dagger$ of creation operators for effective particles, if the momentum range of non-vanishing elements of $\mathcal{H}_\lambda$ is limited to a band close to the diagonal. RGPEP starts from choosing a form factor f_λ whose purpose is to limit $\mathcal{H}_\lambda$ to a band-diagonal form. Next, using the differential equations, one calculates – order by order in perturbation theory – the transformation U_λ that generates such form factors. The transformation also changes the elements of $\mathcal{H}_\lambda$ within the near-diagonal band such that $\mathcal{H}_\lambda$ is unitarily equivalent to H (Fig.3.5). f_λ may be chosen in any way one likes, and can depend on the difference of energies of particles created and annihilated by a specific term in $\mathcal{H}_\lambda$, or on a difference of invariant masses. Below I show f_λ depending on relative momenta of particles created and annihilated in a Hamiltonian term. The difference of invariant masses of states a and b will be denoted ab (for a detailed definition see Appendix A.3, cf. [40]). f_λ acting on an operator crating particles b and annihilating a is defined as:

$$f_{ba} = \exp\left(-\frac{ab^2}{\lambda^4}\right) . \tag{3.79}$$

This means that each term with f_{ba} will be exponentially weakened when the invariant mass changes from state b to a by more than λ. Note that spectator particles do not influence this expression.

Other choices of f_{ba} are also possible and can influence the complexity of solving the theory significantly. For example, Młynik and Głazek [47] analyzed an asymptotically free model: they find the effective Hamiltonian in perturbation theory by solving RGPEP equations numerically at each order and solved a nonperturbative eigenvalue problem for a small-window Hamiltonian extracted from the effective Hamiltonian. They showed that, for a specific choice of f_λ, the bound-state energy of the initial Hamiltonian is reproduced with 1% accuracy.

For the calculation presented in this thesis, it is sufficient to consider f_λ defined by (3.79); the possible advantages of different forms of f_λ will not be discussed here. Each term of $\mathcal{H}_\lambda$

[6] Henceforth, each operator which does not have its operator basis explicitly indicated is understood to be expressed in terms of a_∞.

can be written as:

$$\mathcal{H}_\lambda = f_\lambda \mathcal{G} \tag{3.80}$$

(note that I use the same symbol f_λ for an operator and a function of momenta; for the definition of f_λ acting on an operator see Appendix A.2).

Differentiating equation (3.80) with respect to λ and using (3.78) leads to the equations:

$$\begin{cases} T &= \left\{ \left((1-f_\lambda)\, \mathcal{G}_I \right)' \right\}_0 & (3.81) \\ \dfrac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{G} &= [f_\lambda \mathcal{G}_I, T]\ , & (3.82) \end{cases}$$

where prime means differentiation with respect to λ, and $\{A\}_0$ marks an operator A with extra energy denominator, see Eq. (A.2) and Appendix C.

From **the first equation**, (3.81), it follows that the operator structure of T is the same as $\mathcal{G}_I$, despite that there are additional factors, i.e., the similarity form factor f_λ, energy denominator $\{\ \}_0$, and factors coming from differentiation of $\mathcal{G}_I$. For example, if $\mathcal{G}_I$ has two creation and one annihilation operator, the same is true for T. The only exceptions are terms proportional to $a_p^\dagger a_p$: as stressed above, the similarity rotation does not change the diagonal Hamiltonian terms.[7] Note also, that (3.81-3.82) are written in terms of $a_\infty^\dagger$ which do not undergo the differentiation, since they do not depend on λ.

The second equation (3.82) expresses $\mathcal{G}$ in terms of a commutator, therefore the resulting operator will not have disconnected parts as long as the operators on the right-hand side do not. If this equation is solved in perturbation theory, the n-th order of $\mathcal{G}$ on the left-hand side depends on the $(n-1)$-th orders of $\mathcal{G}_I$ and T on the right-hand side, since the expansion of both these operators starts at order g and the right-hand side expansion starts at order g^2.

One may thus attempt to solve these equations in perturbation theory. There is a starting point: in the zeroth and the first orders, the right-hand side of the second equation is zero. Thus $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ do not depend on λ and are equal to their values at $\lambda=\infty$, i.e., to the canonical Hamiltonian terms. Using $\mathcal{G}^{(1)}$, one can calculate $T^{(1)}$; from this, $\mathcal{G}^{(2)}$; from this, $T^{(2)}$; and so on.

This procedure also allows us to perform a systematic renormalization. The renormalization is done in the following way. The resulting Hamiltonian is "narrow" in invariant masses and requiring cutoff independence of its matrix elements, or coefficients in front of each combination of creation and annihilation operators, makes it produce only finite results. This requirement allows one to find divergent parts of the counterterms in the initial bare Hamiltonian.

The procedure thus works in two ways: it allows us to construct the effective Hamiltonian $\mathcal{H}_\lambda$ and it also determines the initial H^Δ (Fig.3.6). However, it does not fix the finite parts of the counterterms. In Chapter 2, I show that in a simple model, such finite constants can be used to restore Poincaré covariance of physical observables. In the case of QFT, especially the gauge theories, these finite parts may have to have a more complicated structure [15].

To show how this procedure works in practice, I describe in following sections the terms in the effective Hamiltonian that result from the initial, bare Hamiltonian of Eq.(3.71) using equations (3.81)-(3.82). I also describe how the bare a_∞ and the effective a_λ are related.

[7] Note, however, that the form of f_λ is chosen in such a way that for $ab \to 0$, the numerator of (3.81) goes to zero faster than the denominator. Thus the absence of terms $\sim a^\dagger a$ is not an artificial requirement here, e.g., it does not violate unitarity of U_λ.

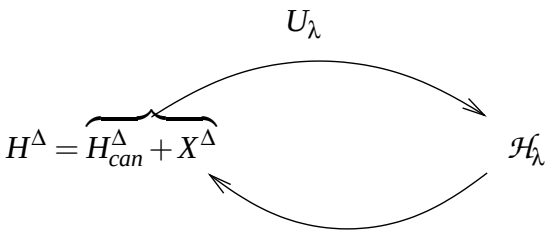


Figure 3.6: The unitary transformation U_λ facilitates the calculation of the effective Hamiltonian $\mathcal{H}_\lambda$. In turn, the condition that the derived $\mathcal{H}_\lambda$ is independent of Δ, determines the form of counterterms X^Δ in the initial, bare Hamiltonian H^Δ.

3.4.4 RGPEP – Order g^0

In the zeroth order, the Hamiltonian does not change:

$$\mathcal{H}_\lambda^{(0)} = f_\lambda \mathcal{G}^{(0)} = H_0 \,. \tag{3.83}$$

For practical reasons, I list here $a_\infty^\dagger$ expressed as a series in $a_\lambda^\dagger$, rather than the other way around. Although it is interesting to see the structure of $a_\lambda^\dagger$ in terms of $a_\infty^\dagger$, $a_\infty^\dagger$ written as a series:

$$a_\infty^\dagger = a_\infty^{\dagger(0)}(a_\lambda^\dagger) + a_\infty^{\dagger(1)}(a_\lambda^\dagger) + a_\infty^{\dagger(2)}(a_\lambda^\dagger) + \dots \,, \tag{3.84}$$

where parenthesis denote functional dependence on the other set of operators, can be used to express any operator which is known as a function of the bare particle creation operators $a_\infty^\dagger$, in terms of the operators $a_\lambda^\dagger$. The series can be inverted to express $a_\lambda^\dagger$ in terms of $a_\infty^\dagger$.

In the zeroth order:

$$a_\infty^{\dagger(0)} = a_\lambda^\dagger \,. \tag{3.85}$$

3.4.5 RGPEP – Order g^1

In order g^1, $\mathcal{G}$ is also independent of λ:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{G}^{(1)} = 0 \tag{3.86}$$

$$\mathcal{G}^{(1)} = H^{(1)} \,. \tag{3.87}$$

The first order effective Hamiltonian is:

$$\mathcal{H}_\lambda^{(1)} = f_\lambda \mathcal{G}^{(1)} = -\frac{g}{2}\int [123]\widetilde{\delta}(1+2-3)\left(a_1^\dagger a_2^\dagger a_3 + a_3^\dagger a_1 a_2\right) f_\lambda(12,3) \,, \tag{3.88}$$

where the regulators r_Δ (cf. (3.72)) can be set to 1 in the limit $\Delta \to \infty$ because f_λ is present. The general form of f_λ defined in Eq. (3.79) reduces in this expression to:

$$f_\lambda = \exp\left[-\frac{1}{\lambda^4}\left(\frac{\kappa^{\perp 2}+m^2}{x(1-x)} - m^2\right)^2\right] \,. \tag{3.89}$$

This means a serious change in the form of the interaction Hamiltonian compared to the bare Hamiltonian H_Y. $\mathcal{H}_\lambda^{(1)}$ has similar terms to the bare interaction Hamiltonian: it can create an

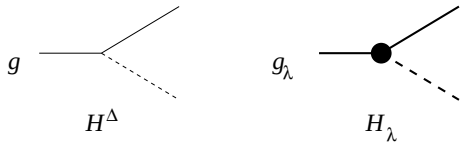


Figure 3.7: The main difference between H^Δ and $\mathcal{H}_\lambda$ is that H^Δ is for pointlike bare particles, whereas $\mathcal{H}_\lambda$ is for effective particles of size λ^{-1}. In $\mathcal{H}_\lambda$, interaction terms are limited by form factors f_λ, marked by a black blob.

additional particle or annihilate a pair and replace it by a single particle. But if the difference of free invariant masses of the one particle and the two particles involved is large, elements of $\mathcal{H}_\lambda^{(1)}$ are weakened by the form factor f_λ. Only states with similar invariant masses are coupled strongly by $\mathcal{H}_\lambda^{(1)}$. In diagrams, the effective Hamiltonian vertices are marked with a black circle, representing the form factor, as in Figure 3.7. Wherever it is important to stress the difference between the bare and effective particles, I use bold lines for the latter.[8]

One can also calculate the first-order operator U_λ and creation operator $a_\lambda^\dagger$.

$$u^{(1)} = \left\{(1-f_\lambda)H^{(1)}\right\}_0 \tag{3.90}$$

As expected from unitarity considerations (see Appendix C.1.2), $u^{(1)}$ is anti-hermitian. This is due to the energy denominators $\{\ \}_0$, which can be written explicitly as:

$$u^{(1)}_{ba} = \frac{P^+_{ba}}{ba}(1-f_{ba})H^{(1)}_{ba}\ . \tag{3.91}$$

It can be easily verified that:

$$(1-u^{(1)}_\lambda)H(1+u^{(1)}_\lambda) = H^{(0)} + f_\lambda H^{(1)}\ , \tag{3.92}$$

if only terms up to order g^1 are retained. This simply confirms that the rotation by $U_\lambda = 1 + u^{(1)}_\lambda + \ldots$ indeed leads to the effective Hamiltonian with a form factor f_λ.

In the first order, the relation between the creation operators of bare and effective particles is:

$$a^{\dagger(1)}_{k\infty} = \left[a^\dagger_{k\lambda}, u^{(1)}\right] = \left[a^\dagger_{k\lambda}, \left\{(1-f_\lambda)H^{(1)}\right\}_0\right]\ . \tag{3.93}$$

As far as the operator structure is concerned, this expression means that, in each term of the interaction Hamiltonian, one annihilation operator and corresponding integration disappear, while the momentum of this operator is replaced by k. Accordingly, in the first order, a bare operator creating a particle of momentum k, $a^{\dagger(1)}_{k,\infty}$, has a term with two creation operators ($a^\dagger_{k1,\lambda}a^\dagger_{k2,\lambda}$ with $k_1+k_2=k$) and a term with one creation and one annihilation operator ($a^\dagger_{k1,\lambda}a_{k2,\lambda}$ with $k_1-k_2=k$).

[8] In the case of coupled QED and QCD, bold lines are also used in this thesis to distinguish between quarks (bold) and electrons (plain lines).

3.4.6 RGPEP – Order g^2

The canonical Hamiltonian H (Eq.(3.61)) for the ϕ^3 theories has only terms of order 0 (the free Hamiltonian) and 1 (terms changing the number of particles by one). Introducing cutoffs requires adding extra counterterms of order 2 and higher. In the theories considered in Chapters 4 and 5 and in the Appendixes, the canonical Hamiltonian will have terms of order g^2. The initial Hamiltonian can be written as a series in the coupling constant:

$$H = H_0 + H^{(1)} + H^{(2)} + \dots \tag{3.94}$$

In this expansion, the second-order effective Hamiltonian is:

$$\mathcal{H}^{(2)}_{\lambda} = \mathcal{H}^{(2)}_{\lambda\, HH} + \mathcal{H}^{(2)}_{\lambda\, H2} \tag{3.95}$$

$$\mathcal{H}^{(2)}_{\lambda\, HH} := f_\lambda \left(\int_\infty^\lambda \mathrm{d}z \left[f_z H^{(1)}, u^{(1)\prime}_z \right] \right) \tag{3.96}$$

$$\mathcal{H}^{(2)}_{\lambda\, H2} := f_\lambda H^{(2)} \,. \tag{3.97}$$

I explicitly show subscripts indicating the scale (λ or z) to which an operator corresponds. The scale z is the scale one integrates over and the scale λ corresponds to the calculated operator $\mathcal{H}_\lambda$.

$\mathcal{H}^{(2)}_{\lambda\, H2}$ is simply a second order counterpart of $\mathcal{H}^{(1)}_\lambda$, Eq. (3.88). The comments in Section 3.4.5 also apply here, i.e., the operator structure of this part of the effective Hamiltonian is identical to the structure of the bare Hamiltonian $H^{(2)}$ apart from the form factor f_λ.

The operator structure of the first part, $\mathcal{H}^{(2)}_{\lambda\, HH}$, is more complicated. It can be written as:

$$\mathcal{H}^{(2)}_{\lambda\, HHac} =: f_{ac} \mathcal{F}^{(2)}_{abc} H^{(1)}_{ab} H^{(1)}_{bc} \Big|_{connected} \,. \tag{3.98}$$

The external similarity factor f_{ac} limits invariant mass change, possible when $\mathcal{H}^{(2)}_{\lambda\, HH}$ acts once. For f_λ defined by Eq.(3.79), $\mathcal{F}^{(2)}_{abc}$ coming from the integral (3.96) is:

$$\mathcal{F}^{(2)}_{abc} = \frac{P^+_{ba} ba + P^+_{bc} bc}{ba^2 + bc^2} \left(f_{ba} f_{bc} - 1 \right) \,. \tag{3.99}$$

The second order part of the rotation operator U_λ is:

$$u^{(2)}_\lambda = \left\{ (1 - f_\lambda) H^{(2)} \right\} + \frac{1}{2} u^{(1)} u^{(1)} + \frac{1}{2} \int_\infty^\lambda \mathrm{d}z \left[u^{(1)}_z, u^{(1)\prime}_z \right] \,. \tag{3.100}$$

The first and the last parts of this expression are anti-hermitian. The middle (hermitian) part, $1/2\, u^{(1)} u^{(1)}$, is exactly what is needed for U_λ to be unitary (cf. (C.14)). Note that this hermitian part is not a commutator, but a product of operators; $u^{(2)}_\lambda$ thus has disconnected parts.

For f_λ defined by Eq. (3.79), the last part of (3.100) is:

$$u^{(2)}_{\lambda HH} =: a^{(2)}_{HHabc} H^{(1)}_{ab} H^{(1)}_{bc} \Big|_{connected} \,, \tag{3.101}$$

where the anti-hermitian factor $a^{(2)}_{HHabc}$ is:

$$\begin{aligned} a^{(2)}_{HHabc} = {} & \frac{P^+_{ac}}{ac} (1 - f_{ac}) \frac{P^+_{bc} bc + P^+_{ba} ba}{bc^2 + ba^2} \left(f_{ba} f_{bc} - 1 \right) + \\ & + \frac{1}{2} \frac{P^+_{ba} P^+_{bc}}{ba\, bc} \left((f_{ba} - f_{bc}) + \frac{bc^2 - ba^2}{(ba^2 + bc^2)} (f_{ba} f_{bc} - 1) \right) \,. \end{aligned} \tag{3.102}$$

The relation between the bare and effective creation operators in the second order is:

$$a_\infty^{\dagger(2)} = \left[a_\lambda^\dagger, \left\{(1-f_\lambda)H^{(2)}\right\}\right] + \frac{1}{2}\left[\left[q_\lambda^\dagger, u_\lambda^{(1)}\right], u_\lambda^{(1)}\right] + \\ + \int_\infty^\lambda \mathrm{d}z \left[a_\lambda^\dagger, \left\{(1-f_z)\left[f_z H^{(1)}, u_z^{(1)\prime}\right]\right\}\right] + \frac{1}{2}\int_\infty^\lambda \mathrm{d}z \left[a_\lambda^\dagger, \left[u_z^{(1)}, u_z^{(1)\prime}\right]\right] \quad , (3.103)$$

see also [39].

3.4.7 Second order effective Hamiltonian for ϕ^3 theory

In Sections 3.4.5 and 3.4.6 I listed perturbative solutions to RGPEP equations. However, I did not analyze the resulting effective Hamiltonian $\mathcal{H}_\lambda$ in detail to find its dependence on the ultraviolet cutoff Δ and the counterterms. In this section, I use the equations presented above to calculate the effective Hamiltonian for the ϕ^3 theory and analyze the details which will be important in Chapters 4 and 5, using similarity form factors of Eq. (3.79).

The starting point for the procedure is the bare, canonical Hamiltonian (3.71) with regulators (3.73). In the zeroth order, the Hamiltonian does not change. The effective free Hamiltonian is still (3.62). However, the effective mass is corrected in the second order. The first-order effective Hamiltonian has already been discussed in detail (cf. Eq.(3.88)). It has a one-additional-particle-creation term and a one-particle-annihilation term. The only difference from the bare Hamiltonian H_Y^Δ, is that each term has a similarity form factor f_λ.

Presence of form factors in the Hamiltonian interaction term is an important feature of the effective theory. Matrix elements of the effective Hamiltonian are small between states whose invariant mass differs greatly. However, if we consider a theory with massive particles – like the ϕ^3 – the invariant mass of a two-particle state must be at least $4m^2$, while the mass of any one particle state is m^2. Therefore, *ba* in the definition of f_λ is at least $3m^2$ and, if λ is of the order of m, the factor f_λ is extremely small, no matter what the momenta of the created particles are. This means that these terms – connecting different Fock sectors of effective particles – are extremely weak for a small λ.

This is part of what is dubbed an effective particle: physics should be described well in the approximation, in which the number of these particles is fixed. And this is, indeed, what we observe in $\mathcal{H}_\lambda$: different Fock sectors are hardly coupled and theory splits into quantum mechanics in each sector separately. The effects of interaction with other sectors are small and can be taken into account in perturbation theory.

Below, I summarize the second-order effective Hamiltonian terms. These are all calculated from Eq. (3.98).

The first type of terms are terms with closed loops (Fig.3.8b). All such terms may be written as:

$$\mathcal{H}^{(2)}_{\lambda,\sim a^\dagger a} = \int [k] \frac{\delta m^2_{\lambda\Delta}}{k^+} a_k^\dagger a_k \, , \quad (3.104)$$

where $\delta m^2_{\lambda\Delta}$ is a constant, dependent on the cutoff Δ and the effective scale λ, but independent of momenta. If one compares this with the form of the free Hamiltonian, it is clear that this term shifts the value of the effective particle mass, without changing the form of the relativistic dispersion relation $p_0^-(p^+, p^\perp)$ of Eq. (3.48).

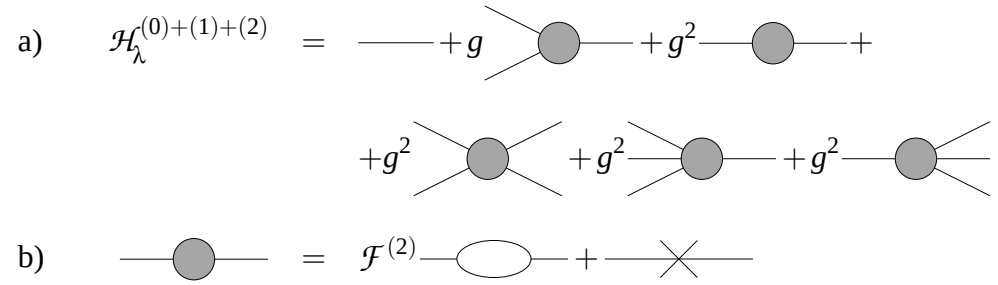


Figure 3.8: The effective Hamiltonian up to the second order has a) a free term, an order-g vertex with a form factor f_λ, and more complicated terms of order g^2. Among the terms of order g^2, there are terms that do not change the number of particles, including a "potential" term $\sim a_\lambda^\dagger a_\lambda^\dagger a_\lambda a_\lambda$ and b) a mass correction $\sim a_\lambda^\dagger a_\lambda$.

One of the requirements of RGPEP is that the effective Hamiltonian matrix elements should be independent of Δ for $\Delta \to \infty$. This is always true for the ϕ^3 theory in 1+1 dimensions. Here, in the limit $\Delta \to \infty$ the effective mass is:

$$m^2_{\lambda,1+1} = \left(\frac{g}{2}\right)^2 \frac{1}{2\pi m^2} \int_0^1 \frac{\mathrm{d}x}{1-x+x^2} \left\{ \exp\left[-2\frac{m^4}{\lambda^4} \frac{\left(1-x+x^2\right)^2}{x^2(1-x)^2} \right] - 1 \right\} , \tag{3.105}$$

a finite, λ-dependent result. But for the theory in 3+1 dimensions described above, the result turns out to be a divergent function of Δ:

$$\begin{aligned} m^2_{\lambda,3+1,\Delta} = \frac{-g^2}{4\cdot 8\pi^2} \log\frac{\Delta^2}{m^2} + \frac{g^2}{4\cdot 8\pi^2}\int_0^1 \mathrm{d}x \log c_\eta + \frac{g^2}{4\cdot 8\pi^2}\int_0^1 \mathrm{d}x \log(1-x+x^2) - \frac{g^2}{4\cdot 8\pi^2} I_\gamma + \\ + \frac{g^2}{4\cdot 8\pi^2}\int_0^1 \mathrm{d}x \int_0^\infty \frac{\mathrm{d}z}{z+1-x+x^2} \exp\left[-2\frac{\left(z+1-x+x^2\right)^2}{x^2(1-x)^2\lambda^4/m^4} \right] , \end{aligned} \tag{3.106}$$

where $c_\eta = 4$ (I_γ is defined in the Appendix (see Eq.(K.3))). This determines the form of the second order counterterm in H_Δ: it must be:

$$X^{\Delta(2)} = \int [k] a_k^\dagger a_k \frac{\delta m^2_\Delta}{k^+} , \tag{3.107}$$

where m^2_Δ is (-1) times the regularization dependent part of (3.106), plus an arbitrary constant C'_X:

$$m^2_\Delta = \frac{g^2}{4\cdot 8\pi^2} \log\frac{\Delta^2}{m^2} - \frac{g^2}{4\cdot 8\pi^2}\int_0^1 \mathrm{d}x \log c_\eta + C'_X . \tag{3.108}$$

$X^{\Delta(2)}$ contributes to the effective Hamiltonian without a change (Eq. (3.97)), leading to the effective kinetic term:

$$\mathcal{H}^{(0)+(2)}_{\lambda\sim a^\dagger a} = \int [k] \frac{k^{\perp 2} + m^2_\lambda}{k^+} a_k^\dagger a_k \tag{3.109}$$

$$m^2_\lambda = \frac{g^2}{4\cdot 8\pi^2}\int_0^1 \mathrm{d}x \int_0^\infty \frac{\mathrm{d}z}{z+1-x+x^2} \exp\left[-2\frac{\left(z+1-x+x^2\right)^2}{x^2(1-x)^2\lambda^4/m^4} \right] + C_X . \tag{3.110}$$

(for convenience, I introduce a new, shifted constant $C_X = C'_X + \frac{g^2}{4\cdot 8\pi^2}\int_0^1 dx \log(1-x+x^2) - \frac{g^2}{4\cdot 8\pi^2} I_\gamma$). In this expression, C_X is a finite constant that is an unknown finite part of the counterterm. To fix its value, one can calculate the physical mass of a particle (the eigenvalue of the Hamiltonian corresponding to a one-particle state) and compare it to the experimental value. Calculating this in perturbation theory to the second order gives the following value:

$$m^2_{phys} = m^2 + C_X \ . \tag{3.111}$$

The same result is found if one starts from the effective Hamiltonian of any scale λ (or even from the initial bare Hamiltonian). The mass term of the effective Hamiltonian depends on the scale λ, Eq. (3.110), but the physical eigenvalue (3.111) does not. This is a consequence of unitarity of the change of basis in RGPEP, and we will observe it also in the theories presented in Chapters 4 and 5.

All other terms of the effective Hamiltonian are finite in the limit $\Delta \to \infty$. Thus, no additional counterterm is needed in order g^2.

The first group of non-divergent effective Hamiltonian terms of order g^2 are the terms with two creation and two annihilation operators. These can be still split into two parts. The first is a potential with three particles in the intermediate state b. The structure of this part is:

$$\mathcal{H}_{\lambda\ \_/\_} = f_{ac}\mathcal{F}^{(2)}_{abc} H_{-<} H_{>-}\big|_{connected,\_/\_} \ . \tag{3.112}$$

Terms like these will be analyzed in more detail in Chapter 4. The second part consists of "s-channel" terms with only one particle in the intermediate b state. The explicit expression is:

$$\mathcal{H}_{\lambda\ >-<} = f_{ac}\mathcal{F}^{(2)}_{abc} H_{>-} H_{-<}\big|_{connected,>-<} \ . \tag{3.113}$$

In the second order, there are also other terms which change the particle number by two (cf. Fig.3.8). For example, the part which creates two additional particles is:

$$\mathcal{H}_{\lambda\ \supset\!\rangle-} = f_{ac}\mathcal{F}^{(2)}_{abc} H_{>-} H_{>-}\big|_{connected,\supset\!\rangle-} \ . \tag{3.114}$$

In each of the terms (3.112)-(3.114), the renormalization group factors $f_\lambda \mathcal{F}$ introduce different dependencies on the momenta of the particles involved. For example, particles a and c may have the same momenta in (3.112) and in (3.113) but the expression ab has a different meaning for each of these terms.

3.4.8 Other theories

Theories other than ϕ^3 in 3+1 dimensions will be discussed in detail in Chapters 4 and 5. Here, I list the key features that differentiate them from the simple scalar ϕ^3 theory described in Section 3.4.7. In the appendix one can find details of the effective Hamiltonians for Yukawa theory, QED and QCD (calculated to different orders in the coupling constants).

One key difference between the scalar theory described above and any theory involving fermions, is that spinor factors introduce additional powers of momenta in the numerator of $H^{(1)}$. For example, if one does not add counterterms, the second-order effective fermion mass term in Yukawa theory would be:

$$\delta m^2_{Yukawa,\Delta} = -\frac{g^2}{16\pi^2}\left[\frac{\Delta^2}{4}\int dx \frac{1}{x} r_\delta^2 + 4m^2 \log\frac{\Delta^2}{m^2}\right] + \text{finite terms} \ , \tag{3.115}$$

(cf. Eq. (D.28) in Appendix D). This expression is linearly divergent in Δ^2, which can be contrasted with only logarithmic divergences in 3+1 scalar theory. Although calculating more divergent expressions is more complicated (and may involve small-p^+ regulators r_δ), they do not present fundamental problems. Simply, extra terms proportional to Δ^2 have to be present in X^Δ (cf. Eq. (4.21)).[9]

Another feature of Yukawa theory is that there are seagull "instantaneous" terms. These do not influence the second order calculations significantly; however, some authors [48] use "self-inertia" (terms that may be obtained from the seagulls when normal-ordering them) to get covariant results, for example for the S matrix. Such an approach cannot easily be extended to a higher order calculation, and is not required for the approach presented here.

Fermion spinors (B.42) introduce into matrix elements additional p^+ factors in denominators. For this reason, low-x regularization (used temporarily also for scalar fields to avoid $H_{\ni+\in}$-like terms) has to be included explicitly. However, in Yukawa theory the only low-x regularization dependence is in the Δ^2-divergent mass term (3.115), and this is removed by the ultraviolet counterterm.

In gauge theories with vector bosons, there are likewise extra boson-line seagulls. Such theories also lead to additional small light-front x divergences that introduce many complications (see Appendix F). QED and QCD are not analyzed in detail in this thesis, although the results presented here are a necessary step towards an analysis of the two gauge theories.

3.4.9 Rotation of operators other than the Hamiltonian

After the basis of creation and annihilation operators for effective particles has been found, one may turn to an examination of the operators other than the Hamiltonian in this basis.

First, it is interesting to describe how effective particles interact in more detail. One way to systematize this description is to determine their relative angular momentum, i.e., a spin of their bound state.

Any operator written in terms of bare creation or annihilation operators a_∞ can be expressed in terms of effective operators, for example by using equations (3.93),(3.103). Masłowski and Głazek have done this for the dynamical Poincaré operators M^{-i}. They found that in the basis of effective particles, the Poincaré algebra relations are fulfilled in perturbation theory. A relativistic angular momentum operator can be defined and used, for example, to classify eigenstates of the effective Hamiltonian P^-_λ or help in the construction of counterterms leading to a theory which agrees with the requirements of special relativity.

There is another situation where expressing certain operators in terms of effective particles may be useful. We may imagine a theory of two kinds of interactions of different strengths, QED and QCD. The strong interactions determine most of the quarks interactions and their electromagnetic interactions may be considered a small correction. One could perform RGPEP taking as H the full Hamiltonian of QED and QCD. This would lead to effective particles that are dressed both strongly and electromagnetically. All the terms of the full effective Hamiltonian are narrow, i.e., have the form factors f_λ, but the theory becomes complicated: there are many terms in $\mathcal{H}_\lambda$ and not only quarks, but also electrons and all other particles, get dressed (cf. Appendix C.2.3).

[9] Note that I have not analyzed here finite dependence on regularization. For the mass term, this would be a finite constant, i.e., momentum-independent shift of the value. Leaving the counterterm in the form of an integral with explicit regulators takes care of both finite and infinite r_Δ dependence, but the degree of divergence is not as explicit as in Eq. (3.115).

One may also consider another way of introducing effective particles. As stated above, the bulk of the interaction is due to the strong interactions. Accordingly, one can base RGPEP on H_{QCD} rather than the full Hamiltonian $H_{QCD}+H_{QED}$. This leads to a description in terms of effective quarks and gluons, dressed up by the strong interaction (i.e., being a combination of a bare quark, a quark and a gluon, a quark and a quark–anti-quark pair, etc., but not a quark and a photon or a quark and an electron-positron pair). In such an approach, bare creation and annihilation operators are used for particles that do not interact strongly. Now, having the definition of effective quarks (from H_{QCD}), one can express the full Hamiltonian ($H_{QCD}+H_{QED}$) in terms of them. This leads to an effective Hamiltonian expressed in terms of effective quarks and gluons interacting with bare electrons and photons. This way one may simplify the calculation of certain S matrices. This is one of the approaches described in Chapter 5.

The simplified form of RGPEP also seems natural from another point of view. If one wants to describe strong bound states of quarks, there is no need to take into account in the first approximation their electromagnetic interactions. One would define effective quarks and their bound states looking at the strong interactions only. Next, one can investigate what the electromagnetic interactions of such bound states look like. It seems natural to leave the bound states expressed by the effective quarks defined by the strong interactions, rather than redefine the degrees of freedom (and define new effective quarks, now dressed both strongly and electromagnetically).

Nevertheless, the electromagnetic interactions of the "strong" effective quarks differ from the interactions of the bare quarks. The effective Hamiltonian in this approach is defined in detail for two coupled scalar theories in Appendix C.3; certain terms of the effective Hamiltonian of QCD coupled to QED are given in Appendix F.

These two approaches – RGPEP based on the full Hamiltonian, and the simplified one based on the strong interaction Hamiltonian only – are used and compared in the description of scattering in Chapter 5.

3.4.10 Other renormalization procedures

The Hamiltonian renormalization group procedure presented in Section 3.4 is similar to the procedure discussed by Wilson [44]. In both procedures, an initial, artificial cutoff Δ is introduced. The Hamiltonian is then transformed to an equivalent form, parametrized by an arbitrary parameter λ. The results would be independent of λ by construction if the transformation was exact. Next, one requires that the resulting Hamiltonian matrix elements do not depend on Δ for $\Delta\to\infty$.

In Wilson's renormalization group, λ limited the space of states by limiting their energies. One could calculate such a limited Hamiltonian matrix using Bloch transformation [49] or the R transformation (presented in Appendix J.2; see also Fig.3.9). Unfortunately, when one calculates this transformation in perturbation theory, small expressions appear in denominators, due to differences of energies of the retained and eliminated states (cf. Appendix J.7). This means that the effective Hamiltonians have to be determined with infinite precision when looking for the counterterms in H^{Δ}, and numerical or any other approximate treatment of renormalization group equations is impossible. The situation is even worse for a theory with degeneracy: if one wanted to eliminate some of the degenerate states, and leave others, the R transformation could not be defined in a plane perturbative way at all.

RGPEP avoids these problems. When calculating the effective Hamiltonians, no small energy denominators are generated. This is because the similarity transformation does not eliminate any of the considered states; instead, it expresses the Hamiltonian in a different basis of

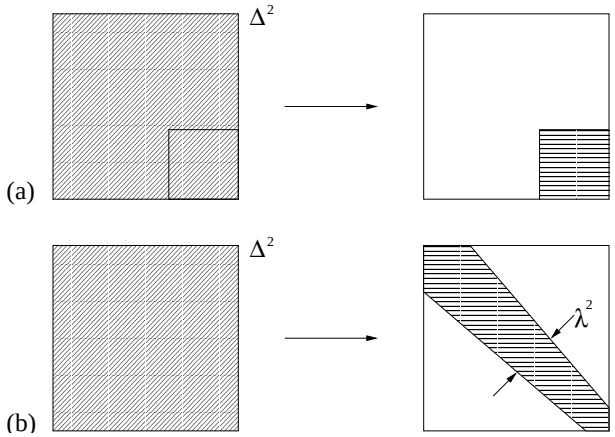


Figure 3.9: (a) Wilson's standard renormalization procedure is based on the reduction of a space of states. This leads to small energy denominators in perturbation theory. (b) The RGPEP is based on a rotation of basis. It avoids small denominators in perturbative derivation of effective dynamics, by integrating out only these energy changes that are larger than λ.

creation and annihilation.

This is a general feature of RGPEP that it shares with the Głazek-Wilson general similarity approach [3, 50]. However, this feature can also be observed explicitly order-by-order in perturbation theory. For example, in the first order rotation (3.91), there is a denominator $1/ab$ that may be small. But when $ab \to 0$, the similarity factor behaves as follows:

$$f_\lambda(ab) = \exp\left(-\frac{ab^2}{\lambda^4}\right) \approx 1 - \frac{ab^2}{\lambda^4}\,, \tag{3.116}$$

and the factor $(1 - f_\lambda)$ goes to zero faster than the denominator. This makes the calculation of $\mathcal{H}_\lambda$ and counterterms in H^Δ possible in practice.

The renormalization procedure for Hamiltonians in QFT described here is different from that of the renormalization of Feynman diagrams. The latter is done by introducing regularization in a specific expression for a physical observable: the scattering amplitude. Until such a regularization is introduced, the expression (including the derivation of the Feynman diagrams) is only formal. When a regularization is introduced, a counterterm to a specific scattering amplitude is constructed. Feynman's approach, although extremely successful, is hard to extend to a description of processes other than the scattering (see Chapter 5).

Unlike the Feynman diagrams, the approach to Quantum Field Theory presented here aims at defining a relativistic and finite Hamiltonian, that may be used to describe both bound states (cf. Chapter 4) and scattering (cf. Chapter 5) within one theory using one set of parameters. Note that the Hamiltonian has matrix elements between states of different free energies. When it contributes to the S matrix, because of free-energy conservation, its contributions simplify greatly. For example, $\mathcal{F}^{(2)}_{abc}$ (cf. (3.99)):

$$\mathcal{F}^{(2)}_{abc} = \frac{P^+_{ba}ba + P^+_{bc}bc}{ba^2 + bc^2}\,(f_{ba}\,f_{bc} - 1)\,, \tag{3.117}$$

simplifies on energy shell to:

$$\frac{P^+_{ba}}{ba}(f^2_{ba} - 1)\,, \tag{3.118}$$

because energy conservation in the S matrix elements means that $ac = 0$. Note that the factor in (3.118) is, in fact:

$$\frac{P^+_{ba}}{ba} = \frac{1}{P^-_b - P^-_a} \,, \tag{3.119}$$

and $\mathcal{H}_\lambda$ contributes to S standard momentum functions. The external form factor $f_{ac} \equiv 1$ for $ac = 0$, and the squared form factor f_λ in (3.118) is canceled by contributions from $\mathcal{H}^{(1)}_\lambda$. Altogether, the complicated effective Hamiltonian $\mathcal{H}_\lambda$ with form factors leads to a simple, standard expression for the scattering matrix S (see Chapter 5). Contrary to this, in bound state calculations, the full form of $\mathcal{H}_\lambda$ (including terms which do not conserve the free energy) is important. In particular, the similarity form factors f_λ are the key to a well-defined bound state equation for two effective fermions (see Chapter 4).

Nevertheless, scattering processes are important and the lessons from the Feynman diagrams are extremely useful. For example, comparing the scattering matrix calculated with the renormalized Hamiltonian with the corresponding relativistic Feynman diagrams may be used to fix finite parts of Hamiltonian counterterms.

Chapter 4

Resolution of overlapping divergence problem in bound-state dynamics of fermions

4.1 Introduction

The development of quantum mechanics at the beginning of the 20^{th} century was mainly focused on trying to understand the structure of matter. The two basic areas of investigation were the description of atoms as bound states of electrons and nuclei, and the description of free radiation which was used to observe this structure. The success of Schrödinger's equation in describing the hydrogen atom as a bound state of one proton and one electron was a basis for further development. Further elements were added to the picture (most importantly, the description of the electron and proton as fermions), which allowed experimental results to be described by nonrelativistic eigenvalue equations with great accuracy.

Today we view quantum mechanics as an approximation to the full description given by QFT. However, it is still not fully understood how a QFT applies to bound states.

The first problem is that the width of the wave functions that solve a bound-state problem depend on the coupling constant. For couplings as small as those in QED, wave functions do not extend to large momenta and a nonrelativistic description may be consistent. This is not true for large coupling constants – one of the reasons why our understanding of bound states in QCD is complicated. The fact that the coupling constant at binding energy scales is large (bigger than 0.3) has two consequences. First of all, standard S-matrix computations based on perturbation theory are no longer precisely valid and may only provide some motivation or starting point for educated guesses. The description of bound states in strongly interacting theories is mainly driven by phenomenology. The second effect of the wave function reaching high momenta is that a bound state cannot be thought of as a state consisting of a fixed number of particles. If large energy components are involved in the motion of bound-state constituents, the Heisenberg uncertainty principle suggests that other Fock sectors with man more constituents will also be important.

The second problem obscuring the connection between QFT and nonrelativistic quantum mechanical models, is a complicated renormalization issue. Most of the developments in renormalization of relativistic quantum field theories has occurred in scattering problems, an it is not clear to what extent similar approaches may be useful in the description of bound-states.

This chapter presents the derivation of the bound-state eigenvalue equation derived using

RGPEP. It focuses on the readily accessible quantitative estimates that show the magnitude of the difference between the convergent bound-state dynamics of two effective fermions and the diverging dynamics of two bare fermions in the approaches based on the Tamm-Dancoff truncation in local QFT [51,52]. I start with the Tamm-Dancoff-like approach to local theory, because it is more familiar. I then introduce the effective particle approach, with its new features exposed through the contrast with the Tamm-Dancoff approach. Section 4.3 provides definitions of the renormalized Tamm-Dancoff scheme in Subsection 4.3.1, and the effective particle scheme in Subsection 4.3.2. Both approaches involve a universal procedure for obtaining a two-fermion eigenvalue equation that can be compared with the nonrelativistic Schrödinger equation. This universal procedure is called *reduction* for brevity. It is denoted by the symbol R and described in Section 4.3.3. Section 4.4 introduces a bare light-front Hamiltonian in Yukawa quantum field theory that serves as a starting point for subsequent sections. The canonical Hamiltonian is supplied with some regularization factors r_Δ and counterterms. Section 4.5 describes details of the approach that treats bound states of fermions as if they could be viewed as made of two bare fermions. This approach is called *approach 1* and encounters conceptual and calculational difficulties in the large-momentum region that are removed by switching over to the approach discussed in detail in Section 4.6. In this section, I present the effective fermion approach, called *approach 2*, comparing and contrasting it with approach 1. In approach 2, bound states of fermions are treated as built from effective fermions of size λ^{-1}. Conclusions are drawn in Section 4.7.

This chapter follows closely the description presented in an article co-authored with Głazek [2], which described the example of a bound state of two effective fermions, coupled to massless scalar bosons by Yukawa coupling. A number of details have been added (see especially Section 4.4 and Appendix D). Quotations from the published material have been reworked in the notation that is consistent with the other parts of this dissertation. I have often decided to use verbatim quotations from Ref. [2], because I need to explain its content and I found it impossible to shorten the original text without loosing clarity.

4.2 Critical aspects of bound-state dynamics of fermions

Nonrelativistic bound states

The notion of a bound state of fermions is based mainly on the examples of atoms and nuclei. The common feature of these systems is that they are **nonrelativistic**. This means three things: (1) Kinetic energies of the fermions are small in comparison to their rest mass energy; (2) dominant interactions are not able to produce fast-moving fermions from the slow ones and hence no significant large-momentum spin effects are generated; and (3) creation of additional particles can be neglected and one can describe the bound states as built from a fixed number of fermion constituents. These features are all related to the fact that the domain of large relative momenta is not important in a bound states of two fermions, such as positronium or deuteron. Their wave functions are self-consistent solutions to the nonrelativistic Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$ with Hamiltonian $H = H_0 + H_I$, where H_0 denotes the kinetic energy operator and H_I stands for the interaction operator. The matrix element $\langle 12|H_I|1'2'\rangle$ is the quantum Coulomb or Yukawa potential with a repulsive core, respectively. The ket $|12\rangle$ denotes a state of two fermions labeled 1 and 2, with all their quantum numbers collectively denoted by these labels. The self-consistency of this well-known picture means that the **wave function** $\psi(1,2) = \langle 12|\Psi\rangle$ **quickly vanishes** when the relative momentum of fermions, $\vec{p} = \vec{p}_1 - \vec{p}_2$, becomes comparable

with masses.

Description of hadrons

The success of the Schrödinger picture for bound states of fermions extends also to quarks. This is reflected in the *constituent quark model (CQM)* (see e.g. [53-60][1]) used in classification of hadrons in the particle data tables [61], and providing a benchmark for more advanced approaches. However, the self-consistency of the nonrelativistic picture is much harder to maintain for bound states of up, down, and strange quarks, than for systems such as positronium and deuteron. This is because the hadronic wave functions tend to have considerable components with $|\vec{p}\,|$ comparable to or exceeding the reduced mass μ for the quarks, and the domain of **large relative momenta** begins to **play a significant role** in the binding mechanism. One is also interested in the description of hadrons moving with speeds very close to the speed of light. Since the **fast-moving hadrons** and their interactions cannot be consistently described within a purely nonrelativistic Schrödinger framework, theorists use the Feynman *parton model* in that case [62, 63, 64]. Unfortunately, the binding mechanism of partons is not explained the parton model. As alternative to these models, one can approach the issue of bound states of fermions using *quantum field theory*, where the corresponding operator H appears to contain all the relevant information about relativistic effects in the domain of large relative momenta of the constituents.

Problems with applying QFT to hadrons

The relativistic description of bound states of fermions in QFT, especially in QCD in the case of light quarks, makes the conceptual difficulties with the constituent picture even greater than in the simple models. In the equation $H_{QCD}|\Psi\rangle = E|\Psi\rangle$, all factors remain unknown. This status of the theory partly originates in the large-relative-momentum domain in the motion of virtual particles. This is illustrated by the following examples. The first example is that local QFTs lead to canonical interaction Hamiltonians H_I that change individual bare particle energies by unlimited amounts (spin-dependent factors grow with momentum transfers). The large-momentum range is enhanced and leads to divergences, invalidating the concept of a nonrelativistic picture entirely unless special conditions, such as an extraordinarily small coupling, are met. Another example is that the interactions create new bare particles and this effect contributes to the boosting of bound states, which implies that the motion of bound states is associated with multiparticle components and the *ad hoc* limitation to a fixed number of bare constituents is no longer consistent in relativistic QFTs. A third example is that even the state with no constituent particles, i.e., the ground state of a theory, or vacuum, proves to be so complicated that no approximate solution of verifiable accuracy has been conceived yet, although many *Ansätze* have to be and are employed in practical attempts. In these circumstances, the main theoretical approach to bound states of quarks (mainly heavy ones that move slowly) is based on the *lattice version of QCD*, and great progress has been achieved in numerical studies of the theory that way [65-69]. Nevertheless, it appears that a quantitative explanation of how the constituent picture with a simple Hamiltonian could be an approximate solution remains a conceptual and quantitative mystery. No detailed constituent wave function picture for relativistic field quanta in Minkowsky space has been theoretically identified or is expected to readily follow from the

[1] These references are provided as examples of most cited literature on the subject.

lattice approach alone. The question of convergence of the binding mechanism in the domain of large relative momenta of constituent particles remains open.

This chapter provides some numerical arguments that the required constituent picture with well-controlled large relative momentum domain may become in principle identifiable if one provides a precise definition of the constituents as effective particles, in distinction from the bare quanta of the local theory. Thus, the process of solving a theory is arranged in two steps, which is typical in *lattice approach* [65] or *sum rules* [70]. In the first step, one derives an effective dynamics, and, in the second step, one attempts to solve the effective theory instead of dealing directly with the original degrees of freedom. Here, one derives the effective fermions of size λ^{-1} using a suitable boost-invariant perturbative renormalization group procedure for their Hamiltonians. The procedure is carried out up to second-order perturbation theory, and the resulting dynamics is compared with the standard picture, where the finite scale λ is absent. In distinction from the diverging bare dynamics, the effective one comes out limited to the momentum range given by λ, and this scale is reduced using differential equations to the scale most suitable for description of bound-states properties in terms of a fixed number of the corresponding constituents. In the renormalized Hamiltonian picture, the pointlike bare particles of the local theory correspond to $\lambda = \infty$ and their dynamics heavily involves large momenta, and multiparticle states, for any finite value of the coupling constant. However, the situation is completely changed when λ is lowered to values comparable to the bound-state masses. The binding is described by a new Schrödinger equation, $\mathcal{H}_\lambda |\Psi\rangle = E|\Psi\rangle$, where the Hamiltonian, $\mathcal{H}_\lambda = \mathcal{H}_{\lambda\,0} + \mathcal{H}_{\lambda\,I}$, is written in terms of creation and annihilation operators for the effective particles, $b_\lambda^\dagger$ and b_λ for fermions, $d_\lambda^\dagger$ and d_λ for anti-fermions, and $a_\lambda^\dagger$ and a_λ for bosons. This effective particle picture is discussed in this thesis, and in this chapter in the context of bound states.

Bound states in QED (small coupling constant), nuclear theory (form factors), and QCD

The key physical reasons for the hope that the **effective constituent picture** does emerge from QFT can be understood by recalling what happens in the well-known cases of atoms (or positronium) and nuclei (or deuteron). These systems can be understood in terms of constituents for quite different reasons. The explanation of the difference is limited below to the positronium and deuteron, but the two examples are sufficient to make the point that concerns all bound systems of fermions in QFT. In the Schrödinger quantum-mechanical picture of positronium, the coupling constant that appears in the Coulomb potential is very small in comparison to 1, i.e., $e^2/4\pi = \alpha \sim 1/137$. Therefore, the interaction produces quite small binding energy, of order $\alpha^2\mu/2$, and the e^+e^- bound-state mass is dominated by $2m_e$. The relative-motion wave function is proportional to $(\alpha^2\mu^2 + |\vec{p}\,|^2)^{-2}$, independently of the fermion spins. When one extends this picture by embedding it in QED, one finds out that the initial wave function is so small for large momenta $\vec{p}$, that no significant correction is able to emerge from that region and alter the original picture with the Coulomb potential. This is found by expanding the theory term by term in powers of α around the initial Schrödinger picture. The interaction linear in α (Coulomb force) is sufficient to describe the main features of fermionic bound states in QED, and higher powers of α are not important for theoretical understanding of the bulk of the bound state structure. Although the integrals in the corrections run over the momentum range that formally extends far beyond μ, the coupling constant is too small for the relativistic fermion spin factors and particle creation processes to produce any major modifications of the leading approximation.

In the meson-exchange models of the deuteron binding mechanism, the analogous **coupling**

constant is three orders of magnitude **larger than in QED**. If one attempted to use QFT to derive the Yukawa potential using the same strategy as in QED, and to calculate corrections, the perturbative procedure would fail. The interactions would accelerate nucleons to the speed of light almost immediately on the bound-state formation time-scale, many new particles would be created, and the large momentum relativistic "corrections" would dominate the "leading" nonrelativistic terms. One could then ask why the famous one-boson-exchange (OBE) potentials, such as the Yukawa potential with a core, could still be used in phenomenology of relativistic nuclear physics and work self-consistently in the nuclear bound-state equations anyway. What saves the picture of a fixed number of relatively slow nucleons interacting through exchange of mesons from serious inconsistency when one includes the elements of QFT, is that the interactions responsible for emission and absorption of mesons by nucleons contain form factors that limit momentum transfers to values so small that the interactions are effectively weak, much weaker than a change of α in QED by the factor 1000 would imply. Consequently, the binding energy is much smaller than the sum of two nucleon masses, e.g., about 2.2 MeV for deuteron. The wave functions of such relativistic nuclear physics picture are not overwhelmingly extending into the large-relative-momentum domain because the form factors eliminate coupling to that region, and the nonrelativistic Yukawa potential with a repulsive core is not invalidated by huge corrections. It could not be so with bare pointlike fermions in local QFT, but it does work in the phenomenological picture of effective particles with the form factors. Incidentally, this example is not intended to suggest that nucleon dynamics should be completely derivable directly from a local QFT that ignores the existence of quarks. A scenario for how to derive the effective nuclear physics picture from QCD is discussed in [2]. Nevertheless, the nuclear physics picture does indicate that an effective particle dynamics may involve large coupling constants in potentials that resemble perturbative second-order interactions with form factors.

In QCD, neither of the schemes can apply separately. On the one hand, the effective coupling constant in the constituent QCD picture cannot be as small as in QED, because QCD is characterized by asymptotic freedom, or infrared slavery. This means that **the effective coupling strength is expected to grow** when the scale of relevant momentum transfers decreases. The coupling constant is already on the order of 0.1 at transfers on the order of 100 GeV and it may be much larger for transfers on the order of nucleon masses. Therefore, the effects that have marginal size in the eigenvalue problem for approximate H_{QED}, such as spin splittings, are expected to be much larger and more important for understanding eigenstates of H_{QCD}, and the initial approximation is not as simple as in QED. On the other hand, **one cannot freely insert form factors** into the local Lagrangian for quark and gluon fields because it would spoil the local gauge symmetry structure. The contact with QCD would be irreversibly lost.

Similarity renormalization group procedure and RGPEP

The situation is changed when QFT is approached using the idea of *similarity renormalization group procedure* for Hamiltonians [3], especially in the case of QCD [15], and when one combines the similarity idea with the concept of form factors in the Hamiltonian interaction vertices for effective particles [45, 46]. Initially, the coupling constant is small due to asymptotic freedom and one can think of using the small coupling constant in canonical QFT to solve the renormalization-group equations for $\mathcal{H}_\lambda$ using a perturbative expansion. The method avoids small energy denominators in the perturbative calculation entirely and the nonperturbative part of the dynamics remains untouched in the perturbative calculation of $\mathcal{H}_\lambda$. That way one derives effective-particle Hamiltonians that involve vertex form factors of small width λ in the

interaction terms. One can have sizable couplings in $\mathcal{H}_\lambda$ with small λ, as required by infrared slavery, without losing control over the size of corrections to the leading constituent picture in diagonalizing $\mathcal{H}_\lambda$. The spectrum of such Hamiltonians can be sought numerically because the form factors limit the range of momenta strongly enough for a discrete computer code to cover the pertinent region, as in the nuclear physics case discussed above. This idea has already been studied qualitatively in a simple numerical matrix model [71] using Wegner's flow equation [72, 73] (and its generalizations) and in [47] using RGPEP. The more detailed effective particle calculus used in the present work in the case of Yukawa theory, is already known to produce asymptotic freedom in $\mathcal{H}_\lambda$ for QCD [74]. The new approach has been also extended to the whole Poincaré algebra [39] in QFT.

Scope of discussion

This chapter describes numerical estimates of the orders of magnitude of the interactions that appear in QFT in the bound-state dynamics of two effective fermions of size λ^{-1}. Our discussion is based on and quotes [2]. Yukawa theory is used to avoid complications related to gauge symmetry (see, for example, problems described in [75, 76] and their resolution [77]) while one still preserves some of the singular large-momentum components in the spinor factors that characterize fermions. The well-known issue of triviality in Yukawa theory is irrelevant here since our goal is to estimate the size of corrections in the bound-state dynamics in an effective theory, rather than in the ultraviolet (UV) dynamics of the initial QFT. The Yukawa example serves only as a source of typical UV factors that QFTs provide anyway, no matter if the theory is trivial, asymptotically free, or otherwise.

The key qualitative question is by how much the Hamiltonian $\mathcal{H}_\lambda$ derived in QFT might differ from familiar models, especially from the nonrelativistic Schrödinger picture with a Yukawa potential (or a Coulomb potential in the case of exchange of massless mesons), for given values of α and λ. Another question is related to the fact that the exact solution of renormalization group equations for $\mathcal{H}_\lambda$ and subsequent exact diagonalization of $\mathcal{H}_\lambda$ should lead to spectra that are independent of λ. However, when one calculates $\mathcal{H}_\lambda$ in perturbation theory of low order, such as the second order that characterizes the Coulomb and Yukawa potentials, the dependence on λ must appear. Bound-state energies may depend on λ when λ is made too small or α is made too large. The question is how large is the residual λ dependence of second-order corrections to the Coulomb-like picture. Another question is how large is a range of values of α and λ that can be self-consistently (i.e., without significant λ dependence) reached in lowest orders of perturbation theory. The second question concerns two cases of perturbation theory. One perturbation theory is for $\mathcal{H}_\lambda$ itself in the renormalization-group part of the calculation. The other perturbation theory is for the eigenvalues and wave functions in the bound-state problem expanded around the Coulomb-like *Ansatz*. Both questions are addressed in the following sections by describing estimates (found in [2]) of the size of those corrections which are most important for large momenta, and which would lead to divergences in the absence of λ. The results imply that the most dangerous corrections that might diverge in the absence of λ turn out to be quite small even for sizable coupling constants.

4.3 Distinction between the Tamm-Dancoff approach and effective-particle approach

This section describes two light-front Hamiltonian approaches to the bound-state dynamics that were considered in [2], the renormalized Tamm-Dancoff approach (approach 1), and the effective-particle approach (approach 2).

4.3.1 Renormalized Tamm-Dancoff approach

This approach is represented by the following diagram,

$$\mathcal{L} \xrightarrow{0} :H_{can}: \xrightarrow{i} :H^{\Delta}_{can}: +X^{\Delta} \xleftrightarrow{ii} \text{solve} \ . \tag{4.1}$$

(0) The initial step on the left denotes a canonical derivation of a field theory Hamiltonian from its Lagrangian, quantization, and normal ordering with respect to the bare vacuum state, and dropping all diverging terms on the basis of hindsight that the normal ordered Hamiltonian will eventually contain counterterms of the same structure.

(i) The next step is regularization. To remove the artificial dependence of observables on regularization, one has to add new terms to the Hamiltonian (called counterterms and denoted X^{Δ}) that also depend on the regularization. The regularized Hamiltonian $:H^{\Delta}_{can}: +X^{\Delta}$ is denoted by H^{Δ}.

(ii) The last arrow indicates solving of the eigenvalue equation for H^{Δ}. A two-step procedure is used.

Step (a). First one finds eigenstates of H^{Δ} whose dominant component for vanishingly small coupling constants is equal to one bare fermion. These states represent what one could call a physical fermion. The solution is found from the eigenvalue equation for the whole H^{Δ} by reducing this equation with the help of operation R to an equivalent equation for the Fock component with one bare fermion. If one requires the eigenvalue to be finite, one has to include in X^{Δ} a mass counterterm of a calculable form.

Step (b). Then, one makes a reduction R of the eigenvalue problem for H^{Δ} to a two-bare-fermion subspace, to find an eigenstate of the Hamiltonian H^{Δ} that is dominated by a pair of bare fermions for infinitesimally small coupling constants. The parameters in the resulting two-fermion eigenvalue problem are expressed in terms of the physical fermion mass found in step (a) above. It turns out that the calculated eigenvalues still depend on the cutoff (some diverge if $\Delta \to \infty$), although the individual matrix elements of the reduced two body Hamiltonian do not depend on the cutoffs once one includes mass counterterms calculated in step (a). Therefore, there is a problem of how to construct counterterms that would remove the Δ-dependence from physical results [78].

4.3.2 Renormalized effective particle approach

The procedure consists of three steps that can be represented by the diagram:

$$\mathcal{L} \xrightarrow{0} :H_{can}: \xrightarrow{i} :H^{\Delta}_{can}: +X^{\Delta} \xleftrightarrow{U_{\lambda}} \mathcal{H}_{\lambda} \xrightarrow{iii} \text{solve} \tag{4.2}$$

The steps (0) and (i) are the same as before except that one works with the bare creation and annihilation operators for efficient bookkeeping for Hamiltonian terms at all times, instead of storing a huge number of selected matrix elements of H^{Δ}.

(ii) This step, marked with U_λ in the diagram, is made using RGPEP [45, 46, 74]. Hamiltonians $\mathcal{H}_\lambda$ are expressed in terms of the effective-particle creation and annihilation operators that depend on the "width" parameter λ. λ ranges from ∞ in H^Δ to a finite value on the order of bound-state masses in the effective constituent dynamics. The Hamiltonian $\mathcal{H}_\lambda$ cannot change invariant masses of effective-particle Fock states by more than about λ in a single interaction. Thus, the emission of effective bosons by effective fermions is possible only if the associated kinetic energy change of relative motion of the particles does not exceed λ. Consequently, when is λ small, Fock sectors with different numbers of effective particles are coupled weakly even for sizable coupling constants, as in the nuclear physics example discussed on page 50.

(iii) This step is analogous to step (ii) in approach 1 and amounts to solving the eigenvalue problem for effective Hamiltonian $\mathcal{H}_\lambda$. The key difference, however, is that when one works using the basis of effective particles in the Fock space, states with two effective fermions couple only to states with similar relative momenta. Therefore, the large relative momentum remains suppressed, and it can be described using perturbation theory without assuming that the coupling constant is very small. Thus, when one solves the eigenvalue equation for $\mathcal{H}_\lambda$, one can introduce a new perturbation theory for the reduction operator R, expanded in powers of $\mathcal{H}_{\lambda I}$. This gives an equivalent Hamiltonian that acts only in the dominant Fock space sectors. There are two steps to do, as in approach 1.

In step (a), one first considers eigenstates dominated by one effective fermion, which defines a physical mass of a physical fermion in approach 2. Next, in step (b) , one finds an equation describing bound states of two effective particles. **The parameter λ is the key to the procedure**. Its value determines whether derivation of the effective Hamiltonian $\mathcal{H}_\lambda$ and its reduction by the operation R to a model subspace Hamiltonian, denoted by H_R , is possible in perturbation theory. The smaller λ, the simpler the approximate solutions for bound states of effective fermions, in the sense that they tend to reduce to the dominant effective Fock sector. But if λ is too small, step (ii) of the derivation of $\mathcal{H}_\lambda$ in perturbation theory loses accuracy (the perturbative integration of renormalization-group equations begins to significantly cut into the bound-state dynamics). Therefore, λ cannot be lowered too far using perturbation theory for $\mathcal{H}_\lambda$. The optimal choice of λ is the one that combines the simplest perturbative expansion for $\mathcal{H}_\lambda$ with the least complicated computer diagonalization of $\mathcal{H}_\lambda$. The main criterion for choosing the right range for λ is that the calculated observables are not sensitive to variation of λ over that range (see also [47]).

The final comment concerns Refs. [79-82], where a different a bound-state calculus has been developed using coupling coherence in second-order perturbation theory for Hamiltonian matrix elements, also in the similarity scheme but without the constraint to a boost-invariant unitary rotation of creation and annihilation operators (see also [83-85]). In distinction from these works, approach 2 is not based on the coupling coherence because no coherent structure is known *a priori* in the region of small λ, far from the initial canonical structure. Instead, one uses a perturbative expansion for the effective-particle renormalization-group flow in terms of a suitably defined coupling constant and tries to find out the relevant structures in a prescribed basis, in which the expansion in powers of the coupling constant may be extrapolated to its physical values. One should stress, however, that the renormalized Tamm-Dancoff approach in Ref. [4], enriched with similarity and coupling coherence [86,80,81], still remains an alternative for nonperturbative studies.

4.3.3 Reduction procedure

The following scenario occurs several times in the next sections. There is an eigenvalue equation for a Hamiltonian $H = H_0 + H_I$,

$$H\,|\psi\rangle = E\,|\psi\rangle \ , \tag{4.3}$$

which is too large to solve exactly on a computer in the sense that the number of *a priori* important basis states is infinite. One looks then for an equivalent Hamiltonian that acts only in a limited subspace of states. One way of constructing the model subspace dynamics is to use the transformation R [49, 44] (details of the transformation are given in Appendix J.2).

The general idea is that one denotes the projection operator on the chosen subspace of the whole Fock space F by $\hat{P}$, and the projector on the complementary space, $1-\hat{P}$, by $\hat{Q}$. If the interaction Hamiltonian H_I is small (in the sense that it only weakly couples states from the subspace $\hat{P}F$ to states in the subspace $\hat{Q}F$), then one can calculate an operator R that produces eigenstates of H from eigenstates of a new Hamiltonian H_R that has eigenstates contained in the subspace $\hat{P}F$. The transformation R leads to the following expression for the Hamiltonian H_R acting in the subspace $\hat{P}F$, expanded in powers of the interaction Hamiltonian H_I (cf. Eq. (J.7)).

$$\langle i|\,H_R\,|j\rangle \;=\; \langle i|\left(\hat{P}H\hat{P} + \frac{1}{2}\hat{P}H_I\frac{\hat{Q}}{E_j - H_0}H_I\hat{P} + \frac{1}{2}\hat{P}H_I\frac{\hat{Q}}{E_i - H_0}H_I\hat{P} + \ldots\right)|j\rangle \ . \tag{4.4}$$

Note that the Hamiltonian H_R does not depend on the eigenvalues of H, but only on the eigenvalues E_i of H_0, $H_0\,|i\rangle = E_i\,|i\rangle$. In particular, one can define H_0 in conjunction with the subspace $\hat{P}F$ so that $H_I = H - H_0$ is as weak as one can get, whilst simultaneously retaining control over the spectrum of H_0.

4.4 Canonical light-front Yukawa theory

In this section, I present the derivation of the canonical light-front Hamiltonian of fermions of two kinds interacting through a Yukawa coupling with scalar bosons. This is the common starting point for both approaches (see Sec. 4.3.1 and 4.3.2) to a two-fermion bound-state dynamics. I present here only key steps, stressing the places where this theory differs from the scalar ϕ^3 theory introduced in Chapter 3; details are given in Appendix D.

4.4.1 Classical Lagrangian density

The starting point is the classical field theory with Lagrangian density:

$$\mathcal{L} = \overline{\psi}_f\left(i\partial\!\!\!/ - m - g\phi_f\right)\psi_f + \frac{1}{2}\left(\partial_\mu\phi_f\partial^\mu\phi_f\right) \ , \tag{4.5}$$

where ψ_f is a doublet of fields $\overline{\psi}_f = (\overline{\psi}_1, \overline{\psi}_2)$. My aim in Sections 4.5 and 4.6 is to analyze the bound state of two different fermions. I introduce the doublet of fermion fields to avoid anti-symmetrization of expressions, and thus to simplify the model.

The Lagrangian consists of the kinetic term for two fermion fields of mass m, the kinetic term for massless scalar boson field ϕ_f, and a pointlike Yukawa interaction of fermions with bosons. Note that this interaction has a Yukawa form, in the sense that it involves the scalar field ϕ and a product of fermion fields $\overline{\psi}_f\psi_f$. But, because the scalar field is massless, one can expect that, if interaction of fermions can be described by some kind of nonrelativistic

potential, it would be a Coulomb rather than a Yukawa potential. The first parenthesis in (4.5) is diagonal in fermion indexes $1,2$. This means that this Lagrangian does not couple two families of fermions directly: in perturbation theory they are coupled only in orders equal to or higher than 2, for example, through boson exchanges.

Euler-Lagrange equations are:

$$\left(i\partial\!\!\!/ - m - g\phi_f\right)\psi_f = 0\,, \tag{4.6}$$

$$\partial^\mu\partial_\mu\phi_f = -g\overline{\psi}_f\psi_f\,. \tag{4.7}$$

Using projection matrices $\Lambda_\pm = \frac{1}{2}\gamma^0\gamma^\pm$, one can obtain fields $\psi_{f\pm} = \Lambda_\pm\psi_f$, and:

$$i\partial^+\psi_{f-} = \left(i\partial^\perp\alpha^\perp + m\beta\right)\psi_+ + \beta g\phi_p\psi_+\,. \tag{4.8}$$

This is a constraint equation and has to be explicitly fulfilled. Accordingly, the full field ψ_f does not have a simple, unconstrained Fourier expansion.

One can introduce free fields ψ. These fields consist of arbitrary ψ_+ components and of ψ_- components fulfilling the free constraint condition:

$$\psi_- = \psi_{f-}(g=0) = \frac{1}{i\partial^+}\left(i\partial^\perp\alpha^\perp + m\beta\right)\psi_+\,. \tag{4.9}$$

The full free field ψ is[2]:

$$\psi = \psi_- + \psi_+\,. \tag{4.10}$$

The energy momentum tensor can be re-expressed using the relation

$$\psi_f = \psi + \frac{1}{i\partial^+}\beta g\phi\psi_+\,. \tag{4.11}$$

The part of this expression with the inverse of the longitudinal derivative ∂^+, leads to the so called "sea-gull" terms below, Eq. (4.15).

One can now state the canonical Yukawa Hamiltonian in terms of the free fields:

$$H = H_0 + H_Y + H_+\,, \tag{4.12}$$

$$H_0 = \frac{1}{2}\int \mathrm{d}^3x\, :\phi\left(-\partial^{\perp 2}\right)\phi: + :\overline{\psi}\gamma^+\frac{-\partial^{\perp 2}+m^2}{i\partial^+}\psi:\,, \tag{4.13}$$

$$H_Y = g\int \mathrm{d}^3x\, :\overline{\psi}\psi\phi:\,, \tag{4.14}$$

$$H_+ = g^2\int \mathrm{d}^3x\, :\overline{\psi}\phi\frac{\gamma^+}{2i\partial^+}\phi\psi:\,. \tag{4.15}$$

:: indicates normal-ordering.

4.4.2 Bare Hamiltonian; regularization.

One can now substitute for each of the fields its Fourier expansion, thereby introducing bare creation and annihilation operators (cf. Appendix B). I list below only the terms that re-occur in later sections.

[2] Note, however, that the mass parameter enters the free Euler-Lagrange equations, and so the fields ϕ and ψ depend on the mass.

All the interaction terms require an ultraviolet cutoff Δ. In normal ordering, the three-particle creation and three-particle annihilation terms (similar to those in Eq. (3.65)) are dropped. This can be seen as introducing a small cutoff on the p^+ momentum of each particle. However, for the Yukawa theory, a small-x cutoff is also needed after these terms have been dropped, irrespective of whether the fermions are massive or not. This is due to two reasons. First, H_Y involves spinor factors $\bar{u}u$. On the light-front these involve p^+ momentum of a fermion in the denominator (cf. Eq. (B.60) in Appendix B.4), and this leads to divergences in integrations over the fermion p^+. Second, the seagull term involves the exchanged p^+ momentum in the denominator, because of the ∂^+ in Eq. (4.15).

The full canonical regulated Hamiltonian for this theory is thus:

$$H^\Delta = H_0 + H_Y^\Delta + H_+^\Delta + X^\Delta \,, \tag{4.16}$$

where X^Δ is an unknown counterterm to be calculated from RGPEP equations.

The free Hamiltonian is

$$H_0 \;=\; \int [k] \frac{k^{\perp 2}}{k^+} a_k^\dagger a_k + \sum_{\sigma i} \int [p] \frac{p^{\perp 2}+m^2}{p^+} \left(b_{p\sigma}^{(i)\dagger} b_{p\sigma}^{(i)} + d_{p\sigma}^{(i)\dagger} d_{p\sigma}^{(i)} \right) \tag{4.17}$$

Here, i indicates the kind (or "flavor") of fermion (1 or 2); σ is the fermion polarization; and $a^\dagger$, $b^\dagger$ and $d^\dagger$ are boson, fermion and anti-fermion creation operators, respectively.

The Yukawa interaction term (4.14) is

$$\begin{aligned} H_Y \;=\; & g \sum_{\sigma\eta i} \int [pkl] 2(2\pi)^3 \delta^3(p_{created} - p_{annihilated}) \times \\ & \times \left[a_k^\dagger b_{p\sigma}^{(i)\dagger} b_{l\eta}^{(i)} \bar{u}_{p\sigma} u_{l\eta} + a_k^\dagger d_{p\sigma}^{(i)} b_{l\eta}^{(i)} \bar{v}_{p\sigma} u_{l\eta} - a_k^\dagger d_{l\eta}^{(i)\dagger} d_{p\sigma}^{(i)} \bar{v}_{p\sigma} v_{l\eta} + \right. \\ & \left. + b_{p\sigma}^{(i)\dagger} b_{l\eta}^{(i)} a_k \bar{u}_{p\sigma} u_{l\eta} + b_{p\sigma}^{(i)\dagger} d_{l\eta}^{(i)\dagger} a_k \bar{u}_{p\sigma} v_{l\eta} - d_{l\eta}^{(i)\dagger} d_{p\sigma}^{(i)} a_k \bar{v}_{p\sigma} v_{l\eta} \right] r_{\Delta\delta} \,. \end{aligned} \tag{4.18}$$

This expression contains terms that can cause the following transitions: fermion into fermion+boson (boson emission from a fermion), fermion+boson into fermion (boson absorption on a fermion), analogous transitions with for anti-fermions, fermion-anti-fermion annihilation into a boson and boson decay into fermion-anti-fermion pair.

$r_{\Delta\delta}$ is an ultraviolet and small-x regulator:

$$r_{\Delta\delta} = \exp\left(\frac{-\kappa^2}{\Delta^2} \right) r_\delta(x_1)\, r_\delta(x_2) \,. \tag{4.19}$$

In this expression, $\kappa^\perp$ indicates the relative transverse momentum of the two particles created or annihilated, and x indicates the relative longitudinal momentum. I leave the x regulator unspecified, requiring only that it cuts off smoothly any x smaller than δ. The seagull term does not contribute to the bound-state calculation described below.

4.5 Tamm-Dancoff approach – bound states of two bare fermions

This section reviews the renormalized Tamm-Dancoff procedure for two-fermion bound states. I begin by examining the single-fermion eigenvalue problem, and then proceed to the two-fermion case.

4.5.1 One-fermion eigenstates

The one-fermion eigenvalue equation is obtained by assuming that the coupling constant in the theory is infinitesimally small and the dominant part of the eigenstate is provided by a single bare-fermion Fock state. The quantum numbers of the lowest-mass eigenstate correspond by definition to one physical fermion associated with the fermion field in the initial Lagrangian. When the coupling constant is no longer infinitesimally small – it is made finite and grows – the eigenvalue equation cannot be solved exactly with currently known mathematical methods, and one has to investigate results that follow from various attempts to find approximate solutions. One such attempt is made by reducing a cutoff dynamics to the one-bare fermion Fock sector for finite coupling constants also. Here, the projection operator in the operator R (see Sec. 4.3.3 and Appendix J.2) has the form $\hat{P} = \sum_\sigma \int [p] b^{(1)\dagger}_{p,\sigma} |0\rangle \langle 0| b^{(1)}_{p,\sigma}$. For finite cutoffs and sufficiently small coupling constants g, one uses expansion in powers of g to evaluate the corresponding Hamiltonian H_R. Up to order g^2, this leads to an equation $H_R |k\rangle = P^- |k\rangle$, with

$$P^- = \frac{k^{\perp 2} + m_\Delta^2}{k^+} + X_{ff} =: \frac{k^{\perp 2} + m_f^2}{k^+} \,, \tag{4.20}$$

where m_Δ^2 results from emission and re-absorption of bosons and X_{ff} is contributed by the counterterm proportional to $b^\dagger b$. Since m_Δ^2 is a diverging function of Δ, X_{ff} has to be adjusted to remove this effect. Note that m_f^2 should not, and does not, depend on the fermion momentum components k^+ and $k^\perp$ even in the presence of regularization (cf. discussion in Sec. 3.2.6).

The result (4.20) for m_Δ^2 requires a counterterm of the form:[3]

$$X_2 \;=\; \sum_{i=1}^{2} \sum_\sigma \int [p] b^{(i)\dagger}_{p\sigma} b^{(i)}_{p\sigma} \frac{1}{p^+} \frac{g^2}{16\pi^2} \left[\frac{\Delta^2}{4} \int dx \frac{1}{x} r_\delta^2 + 4m^2 \log \frac{\Delta^2}{m^2} + C \right], \tag{4.21}$$

where the constant C is a finite part of dimension m^2. This condition removes Δ-dependence from the physical fermion mass m_f in the limit $\Delta \to \infty$.

4.5.2 Two-fermion bound states

The reduction procedure described in Section 4.3.3 can be used to reduce the eigenvalue equation for H^Δ to a two-bare-fermion Fock sector. This employs a projection operator:

$$\hat{P} = \sum_{\sigma_1 \sigma_2} \int [p_1 p_2] b^{(1)\dagger}_1 b^{(2)\dagger}_2 |0\rangle \langle 0| b^{(2)}_2 b^{(1)}_1 \,, \tag{4.22}$$

Note that the two fermions are selected to be of different kinds. This leads to an equation for an eigenstate of H_R, which can be written as:

$$|P\sigma\rangle = \int [p_1 p_2] P^+ (2\pi)^3 \delta^3(P - p_1 - p_2) \phi_\sigma(p_1, p_2) b^{(1)\dagger}_{p_1} b^{(2)\dagger}_{p_2} |0\rangle \,. \tag{4.23}$$

[3] In this thesis I include a regulating factor for each created particle. This introduces regularization in terms with different number of particles in a unified way. For example, in each Yukawa vertex there is a factor $\exp(-2\kappa^2/\Delta^2)$. However, in the original paper [2] on which this chapter is based, corresponding regularization factors were chosen to be $\exp(-\kappa^2/\Delta^2)$. Therefore, for example, the quadratically divergent part of (4.21) differs by a factor 2 from the one shown in [2].

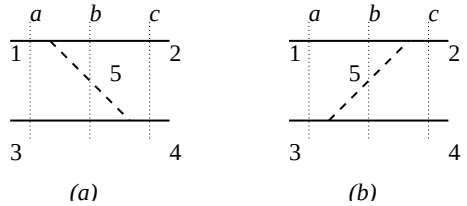


Figure 4.1: Two kinds of terms in one-boson exchange potentials. Following [74], the initial (rightmost), intermediate, and final (leftmost), states are denoted by c, b, and a, respectively. For example, in diagram (a), $p^+_{ba} = p^+_1$, $p^+_{bc} = p^+_4$, $ba = M^2_{2+5} - m^2$, $bc = M^2_{3+5} - m^2$, and $q^+ = p^+_5$.

The eigenvalue equation can be then written in terms of the two-body wave function $\phi_\sigma(p_1, p_2) \equiv \phi_\sigma(\vec{k})$ (where σ denotes spin quantum numbers of both fermions $\sigma = \{\sigma_1, \sigma_2\}$; for a definition of k_3 that forms $\vec{k}$ together with $k^\perp$, see Appendix B.2.1):

$$\frac{\vec{k}^2}{m_f}\phi_\sigma(\vec{k}) + \sum_{\sigma'} \frac{m_f}{(2\pi)^3} \int \frac{d^3k'}{\sqrt{E_k E_{k'}}} v_{OBE}(\sigma, \sigma', \vec{k}, \vec{k}')\phi_{\sigma'}(\vec{k}') = \frac{M^2_{full} - 4m_f^2}{4m_f}\phi_\sigma(\vec{k}) \,, \tag{4.24}$$

where $E_k = (\vec{k}^2 + m^2)^{1/2}$ and the potential kernel v_{OBE} corresponds to the terms shown in Figure 4.1 (discussed below).[4]

Note the mass m_f in Eq. (4.24) is the physical fermion mass obtained from the earlier reduction to one-bare-fermion space, Eq. (4.20). Expanding the bare mass m in the integration measure factor $(E_1 E_2)^{-1/2}$ and potential v_{OBE} in a series of powers of g around m_f, leads to an equation featuring only the physical mass m_f. In this way the bound-state dynamics for two bare fermions is related to a physical fermion mass parameter. This step connects the bare fermions in the two-body problem with the physical fermion obtained in the one-body reduction discussed in Section 4.5.1.

In the discussion of the two-fermion eigenvalue equation below, I denote the single-fermion mass eigenvalue m_f by m (i.e. I drop the subscript f) for the purpose of simplification. The two-fermion bound-state mass M_{full} can be rewritten as $M_{full} = 2m - E_B$. When $E_B \ll m$, the eigenvalue takes the form $(M^2_{full} - 4m^2)/4m = -E_B + E_B^2/4m \approx -E_B$. Thus, the eigenvalue on the right-hand side of Eq.(4.24) can be thought of as the binding energy E_B.

Since the regulator function r_Δ respects the kinematical boost invariance of the light-front scheme, this equation is independent of the total momentum of the two fermions. There is also no explicit Δ-dependence in the matrix elements of the potential v_{OBE} in the limit $\Delta \to \infty$:

$$\begin{aligned} \lim_{\Delta\to\infty} v_{OBE}(\sigma_1\sigma_2\sigma_3\sigma_4 \vec{k}\vec{k}') \;\; = \;\; & -\frac{\pi\alpha}{2m^2}\frac{\bar{u}_1 u_2 \bar{u}_3 u_4}{q^+}\left(\frac{p^+_{ba}}{ba} + \frac{p^+_{bc}}{bc}\right)_{Fig.4.1a} + \\ & + \text{the same}|_{Fig.4.1b} \,, \end{aligned} \tag{4.25}$$

where $\alpha = g^2/4\pi$ (see caption to Fig. 4.1 and Appendix A.3). The spinor matrix elements are:

[4] Choosing two different fermions eliminates s-channel terms, that could occur in the bound state equation for a fermion and an anti-fermion of the same kind. However, even here, the  term would be less important, because of a denominator of order of fermion masses.

$$\bar{u}_1 u_2 = \frac{1}{\sqrt{x_1 x_2}} \chi_1^\dagger [m(x_1+x_2) - \sigma^3 \sigma^\perp (x_1 k_2^\perp - x_2 k_1^\perp)] \chi_2, \tag{4.26}$$

where $\chi^\dagger = [1,0]$ or $[0,1]$, depending on the fermion-spin projection on the z axis (see Appendix B.4).

The potential (4.25) is a complicated, nonlocal function of fermion momenta. However, in the region with both momenta $k, k' \ll m$, it simplifies to the well-known Coulomb potential (see Appendix J.1),

$$v_{Coulomb} = -4\pi\alpha \frac{1}{(\vec{k}-\vec{k}')^2} \,. \tag{4.27}$$

Unfortunately, this heuristic result is not meaningful as the region of the large relative momenta of the fermions introduces important corrections. In Eq. (4.25) one can see that, when one of the relative momenta (k or k') is much bigger than the other, and much bigger than the fermion mass, the spinor factors become proportional to the larger of the two momenta. For example, if $k' \gg k, m$, one obtains $\bar{u}u \sim k'$, and two such factors compensate the denominator that grows as k'^2 [87]. The potential becomes a function of x_1 and x_2, being a constant in the transverse momentum directions. A constant potential in the transverse momentum space is a two-dimensional δ-function potential in configuration space. Such potentials with a negative coefficient lead to bound states of infinite negative energies in the nonrelativistic Schrödinger equation; the light-front transverse dynamics is of this type. One could try to rely on the regulators r_Δ with a finite Δ to resolve the problem: this would correspond to smearing of the δ potential in position space. The eigenvalues of the equation would then depend on Δ. One could naturally try to make the coupling constant g depend on Δ. However, the interaction is specific to the Fock sector under consideration, and is much more complicated than a δ-function itself, due to the presence of the additional x-dependent factors. It is unlikely that a change of g to a function of Δ can remove the cutoff dependence from all eigenvalues. Seeking subtractions cannot be based on exact solutions, because we do not know them.

The two-body equation as it stands is not convergent in the large-relative-momentum domain, and the cutoff dependence invalidates the nonrelativistic approximation as a means for seeking a conceptually satisfying solution of the divergence problem, especially for sizable coupling constants.

The calculation described below illustrates how the overlapping divergence problem arises in approach 1 in a quantitative way. The potential in the nonrelativistic region of k and k' small in comparison to m has the Coulombic form. One can ask therefore with what accuracy Eq. (4.24) can be approximated by a Schrödinger equation with the Coulomb potential (given in Appendix J.1). The potential can be rewritten in the form:

$$v_{OBE} = v_{Coulomb} + \Delta v \tag{4.28}$$

and corrections induced by Δv estimated in perturbation theory. The Coulomb potential does not depend on spins of the interacting fermions. Thus, in zeroth order of the bound-state perturbation theory, there are four degenerate states with the lowest mass M and identical momentum-space wave functions: a triplet of spin-1 states, and a singlet of spin 0. For details of the degenerate perturbation theory for this case see Appendix D.

To estimate the first-order energy correction, one has to find eigenvalues of the 4×4 matrix of matrix elements $\langle \psi_{0i} | \Delta V | \psi_{0j} \rangle$, where i and j refer to the different spin configurations. The eigenstates of this matrix have the spin structure: $(\uparrow\downarrow + \downarrow\uparrow)$, $(\uparrow\downarrow - \downarrow\uparrow)$, $\uparrow\uparrow$ and $\downarrow\downarrow$. The lowest mass eigenstate is $(\uparrow\downarrow - \downarrow\uparrow)$. The fist-order correction to the Coulomb energy for this state

varies between numbers of the order of $-6\times10^{-5}E_0$ for $\alpha=0.01$ to $-0.1E_0$ for $\alpha=0.6$[5]. Note that α is also present in the wave function ϕ_0 and the results are not connected by a straightforward multiplication by the ratio of the coupling constants, although both results are, indeed, small. For an α greater then 0.6, the first-order correction would remain relatively small, but the second-order corrections would become unacceptably large. Accordingly, an α significantly larger then 0.6 is not discussed here.

In the second order, a convergence problem in the domain of large relative momenta of fermions destroys consistency of the naive perturbative procedure around a nonrelativistic approximation. To make it transparent, one can introduce a number of simplifications and isolate the origin of corrections that grow with Δ without worrying about details of secondary importance. Importantly, the Coulomb basis functions have quickly falling-off tails in momentum space. The tails are still small but greatly enhanced by first-order corrections, and the second-order correction already involves matrix elements that diverge with the cutoff Δ.

To see the origin of the overlapping large-relative-momentum divergence in the second-order energy correction, one needs to analyze matrix elements of the type:

$$\Delta E^{(2)}=\langle\phi_0|\,\Delta v\frac{1}{E_0-H_0-V_{Coulomb}}\Delta v\,|\phi_0\rangle\ . \tag{4.29}$$

Such elements involve integration over four relative momenta of fermions: the leftmost wave function argument denoted by k_l; the momentum of states between the left Δv and $1/(E_0-H_0-V_{Coulomb})$, denoted by p_l; the momentum between the operator $1/(E_0-H_0-V_{Coulomb})$ and right Δv, denoted by p_r; and the argument of the right wave function, denoted by k_r. The matrix element can be split into a sum of 2^4 parts, with each part distinguished by indicating whether each of the four integrated momenta is greater or smaller than the fermion mass, m.

Since the Coulomb wave functions strongly limit their arguments, a part with k_l and k_r large makes a very small contribution compared to the part with k_l and k_r small. Therefore, one looks for important contributions, assuming that k_l and k_r lie within several widths of the Coulomb wave functions. An *ad hoc* number used in Ref. [2] was $4\alpha m$.

Moreover, there is no bound-state wave function limiting the intermediate momenta p_l and p_r, and integrals over them extend up to the cutoff $\Delta\to\infty$. Eq. (4.26) shows that, for large momenta, the spin-flip part of the potential dominates other parts. This dominating part is selected here and denoted by $\Delta v_{\downarrow\uparrow}$; the fermions have opposite spin orientations and both have their spins flipped in the interaction. For the purpose of estimating the order of magnitude of the large-momentum spin-flip contribution, p_l and p_r are considered larger than m, and $(E_0-H_0-V_{Coulomb})^{-1}$ is replaced by $-1/H_0$, neglecting terms that would vanish when $\alpha\to0$. The resolvent then becomes diagonal in momentum space, and $p_l=p_r$ are commonly denoted by k_2. Details of how the cutoff Δ was initially introduced are not important for the estimate of the order of magnitude. Accordingly, the cutoff function was slightly changed to simplify the integration: the initial r_Δ limits changes of invariant masses in each of the vertices (Fig. 4.1), producing a complex shape of the k_2-integration boundary with details that depend on the small momenta k_l and k_r, irrelevant to the divergence issue at hand. The main role of the cutoff in $\Delta E^{(2)}$ is to provide an upper limit on the range of integration over k_2. One can estimate the size of the large-momentum range contribution by introducing a new cutoff k_{max}, equal to the maximum value that k_2 can take, and let $k_{max}\to\infty$ when $\Delta\to\infty$. Dependence on k_{max} will indicate dependence on Δ.

[5]The numbers given here are slightly larger then given in Ref. [2] because in the meantime we have improved accuracy of our numerics.

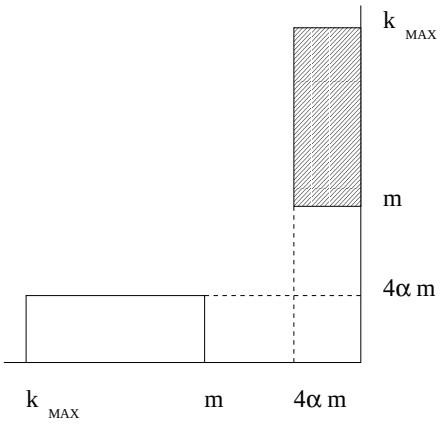


Figure 4.2: The shaded area represents the range of integration over $|\vec{k}_1|$ and $|\vec{k}_2|$ in Eq.(4.30) with the potential matrix $\Delta v_{\downarrow\uparrow}(\vec{k}_1,\vec{k}_2)$. In the lower right-hand corner, both momenta are equal to zero.

To summarize, the large relative-momentum part of the second-order energy correction can be estimated using the following expression (the tilde indicates the simplifications made in $\Delta E^{(2)}$):

$$-\Delta\tilde{E}^{(2)} = m^2 \int_0^{4\alpha m} \frac{d^3k_1}{\sqrt{E_1}} \int_m^{k_{max}} \frac{d^3k_2}{E_2} \int_0^{4\alpha m} \frac{d^3k_3}{\sqrt{E_3}} \phi_0(k_1) \Delta v_{\uparrow\downarrow}(\vec{k}_1,\vec{k}_2) \frac{1}{H_0} \Delta v_{\uparrow\downarrow}(\vec{k}_2,\vec{k}_3) \phi_0(k_3) \tag{4.30}$$

The range of integration over k_2 in this expression is shown in Fig. 4.2. Since the potential $\Delta v_{\uparrow\downarrow}$ approaches a constant for $k_2 \gg m$, one can expect a logarithmic dependence of $\Delta\tilde{E}^{(2)}$ on k_{max},

$$\int_m^{k_{max}} \frac{d^3k_2}{E_2} \Delta v_{\uparrow\downarrow}(\vec{k}_1,\vec{k}_2) \frac{1}{H_0} \Delta v_{\uparrow\downarrow}(\vec{k}_2,\vec{k}_3) \sim \log\frac{k_{max}}{m} \,. \tag{4.31}$$

A numerical evaluation of the 12-dimensional integral produces an estimate of the actual size of the logarithmically diverging correction. The results for different values of the coupling constant are given in Figure 4.3; the error bars indicate the standard deviation of a Monte Carlo routine used in the computation. No other parts of the second-order two-fermion bound-state mass correction (parts with external momenta bigger than $4\alpha m$, internal momenta smaller than m, or parts without change of the fermion spins) can compensate this divergence. Note that the corrections can quickly reach the order of 10% for coupling constants of the size expected in quark physics when the cutoffs are made larger than 100 quark masses and they continue to grow.

4.6 Effective particle approach – bound states of two effective fermions

This section briefly reviews the RGPEP for deriving Hamiltonians $\mathcal{H}_\lambda$ for effective particles of size λ^{-1}. It then applies $\mathcal{H}_\lambda$ in the Yukawa theory to a bound state of two effective fermions. In the presentation I refer to certain steps described in the previous section, and apply them now to the bound-state problem. I point out the key differences between the bare particles and the

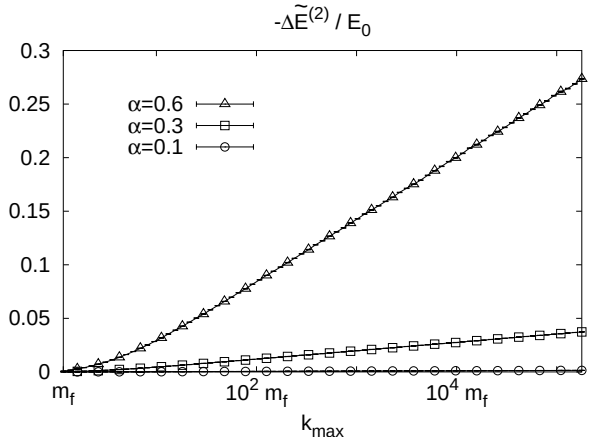


Figure 4.3: Dependence of the most singular part of the second-order correction (Eq. (4.30)) on the cutoff k_{max}. This correction diverges logarithmically for large k_{max} even for small coupling constants α, though the matrix elements of the two-body Hamiltonian do not depend on k_{max} for momenta smaller than k_{max}.
In the bare Tamm-Dancoff approach, physical results should be obtained using $k_{max} \sim \Delta$, in the limit $\Delta \to \infty$.

effective particles, that lead to the two-effective fermion dynamics that converges in the region of large relative momenta.

4.6.1 Renormalization group for effective particles

The RGPEP is defined by means of a unitary rotation for creation and annihilation operators [45, 46]

$$b^\dagger_\lambda = U_\lambda b^\dagger U^\dagger_\lambda \,. \tag{4.32}$$

The Hamiltonian can be expressed in terms of both sets of operators, b^+ or $b^\dagger_\lambda$, and each has different matrix elements in the Fock space basis built using the operators of each kind. There is no change in the physical content of the theory. The idea of the rotation (4.32) is that the Hamiltonian expressed in terms of b_λ (i.e., the effective Hamiltonian of width λ, $\mathcal{H}_\lambda$) contains vertex form factors f_λ of width λ in all interaction terms. The choice made here for estimating the large-momentum contributions to bound-state dynamics of effective fermions, is:

$$f_{ba} = \exp\left(-\frac{(M^2_{created} - M^2_{annihilated})^2}{\lambda^4}\right) , \tag{4.33}$$

where $M_{created}$ is the total free mass of all particles created by a given term in $\mathcal{H}_\lambda$ and $M_{annihilated}$ is the total free mass of particles annihilated by the term. This choice is based on the results obtained so far for asymptotic freedom in $\mathcal{H}_{\lambda\ QCD}$ [15] and Poincaré algebra in scalar theory [39].

If the unitary transformation U_λ were known exactly, there would be no λ dependence in the spectrum of $\mathcal{H}_\lambda$. But when U_λ (and $\mathcal{H}_\lambda$) are calculated in perturbation theory, the approximation leads to some residual λ dependence of theoretical predictions for observables. **The sensitivity of the results to variation of λ is a simple test of how large the errors are in the perturbative**

expansion for $\mathcal{H}_\lambda$, on top of the error margin resulting from the approximations used to solve the Schrödinger equation with H . On the one hand, one tries to **get down to as small** λ **as possible**, so that the nonperturbative diagonalization will require the smallest possible range of energy scales to handle explicitly, using a computer. On the other hand, one expects that the errors arising from the use of perturbation theory in evaluating $\mathcal{H}_\lambda$ will grow as λ is reduced, and so λ **should not be too small**. The reason for this is that, if $\lambda \to 0$, the Hamiltonian becomes almost diagonal, which is equivalent to solving the nonperturbative dynamics of bound states, and a perturbative calculus for $\mathcal{H}_\lambda$ must fail at some point before λ becomes equal to the scale of the nonperturbative phenomena.

4.6.2 Effective Hamiltonian for Yukawa theory

Effective Hamiltonian – Zeroth and First order

When one evaluates the Hamiltonian for effective fermions, the only change in the zeroth-order Hamiltonian (free part, order g^0) is that the bare operators such as $b^\dagger b$ are replaced by $b_\lambda^\dagger b_\lambda$. In order g^1, the effective Hamiltonian has the form:

$$\mathcal{H}_\lambda^{\Delta(1)} = g\sum_{i=1}^{2}\sum_{\sigma_1,\sigma_2}\int [p_1p_2q]\,\widetilde{\delta}^3(p_1-p_2-q)\exp\left(\frac{-\kappa_{p_2,q}^{\perp 2}}{\Delta^2}\right) r_\delta\left(\frac{x_q}{x_{p_1}}\right)\exp\left[-\frac{\left(M_{p_2q}^2-m^2\right)^2}{\lambda^4}\right]\times$$
$$\times\left[a_{q\lambda}^\dagger b_{p_2\sigma_2\lambda}^{(i)\dagger} b_{p_1\sigma_1\lambda}^{(i)}\bar{u}_{p_2\sigma_2}u_{p_1\sigma_1} - a_{q\lambda}^\dagger d_{p_2\sigma_2\lambda}^{(i)\dagger} d_{p_1\sigma_1\lambda}^{(i)}\bar{v}_{p_1\sigma_1}v_{p_2\sigma_2} + h.c.\right]. \quad (4.34)$$

Note that expressing b's by b_λ's has induced the form factor f_λ in $\mathcal{H}_\lambda$ (expression (4.34) comes both from first-order unitary rotation U_λ of H_0 and zeroth-order rotation of H_Y; this is not a rotated H_I only). This form factor means that the regularization factor r_Δ depending on Δ is equivalent to 1 when $\Delta/\lambda \to \infty$; it can therefore be omitted from this expression.

Effective Hamiltonian – Second order: Mass term

When one calculates the term in $\mathcal{H}_\lambda$ of order g^2 that contains $b_\lambda^\dagger b_\lambda$, one finds that it contains a mass-squared-like term with a divergent Δ-dependence. A counterterm must be added to the initial Hamiltonian that has exactly the same form (4.21) as in approach 1. The form of the effective mass term in $\mathcal{H}_\lambda$ is then (in the limit $\Delta \to \infty$):

$$\mathcal{H}_{\lambda\,\delta m} = \int [p] b_\lambda^\dagger b_\lambda \frac{\delta m_\lambda^2}{p^+}, \quad (4.35)$$

where

$$\delta m_\lambda^2 = \frac{g^2}{16\pi^2}\int_{m^2}^{\infty} dz \frac{1}{2}\left(1+\frac{6m^2}{z}+\frac{m^4}{z^2}\right)\exp\left[-\frac{2(z-m^2)^2}{\lambda^4}\right] + const. \quad (4.36)$$

Note that the renormalization is carried out now at the level of full theory in the whole Fock space, not after reduction to a specific Fock sector (accordingly, there are no sector-dependent mass counterterms). Since the regulators did not violate any kinematical light-front symmetries, the calculated mass term does not depend on particle momentum. In other words, the relativistic form of the dispersion relation does not change, and there is only a change in the value of the effective fermion mass.

Effective Hamiltonian – Second order: Potential term

Second-order terms in $\mathcal{H}_\lambda$ that contain two creation and two annihilation operators for effective fermions do not contain any dependence on Δ when $\Delta \to \infty$, and no counterterms are needed of such form. Therefore, the complete answer for these potential terms is:

$$\begin{aligned}\mathcal{H}^{\Delta(2)}_{\lambda\, b^\dagger b^\dagger bb} &= \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \int [p_1 p_2 p_3 p_4] b^{(1)\dagger}_{p_1\lambda} b^{(2)\dagger}_{p_3\lambda} b^{(2)}_{p_4\lambda} b^{(1)}_{p_2\lambda}\, \widetilde{\delta}^3(p_1+p_3-p_2-p_4) \\ &\quad \times v^{(2)}_\lambda(p_1,p_2,p_3,p_4,\sigma_1,\sigma_2,\sigma_3,\sigma_4)\,, \end{aligned} \tag{4.37}$$

where

$$v^{(2)}_\lambda(p_1\dots p_4,\sigma_1\dots\sigma_4) = \frac{-g^2\bar{u}_1 u_2 \bar{u}_3 u_4}{q^+} f_{ac}\mathcal{F}_2(a,b,c)|_{Fig.4.1a} + \text{the same}|_{Fig.4.1b}\,, \tag{4.38}$$

and $\mathcal{F}_2(a,b,c) = [(xba+(1-y)bc)/(ba^2+bc^2)](f_{ba}f_{bc}-1)$, with $x = p_1^+/(p_1^+ + p_3^+)$ and $y = p_2^+/(p_2^+ + p_4^+)$. The notation used is that of Fig. 4.1, as for approach 1. However, the potential[6] v_λ is quite different from the OBE potential of Eq. (4.25): for example, the denominators are different and there are key form factors f_{ac}. Other term in the effective Hamiltonian do not matter in further discussion.

4.6.3 Solving the eigenvalue problem with $\mathcal{H}_\lambda$

In the case of bound states of two effective fermions, the reduction procedure is based on the same rules as in the approach 1, except that the effective particles interact with vertex form factors of width λ and the large-relative-momentum convergence is improved. The change of particle number is also severely limited in strength, since massive particles cease to be produced when λ is lowered below their mass. Even the emission of massless scalar particles is severely limited. As in approach 1, the departure point in solving the bound state dynamics is the eigenvalue equation for a single fermion.

Reduction to one-effective-fermion subspace

This step produces an equation $H_R|k\rangle = P^-|k\rangle$, where

$$P^- = \frac{k^{\perp 2}+m_f^2}{k^+}\,, \tag{4.39}$$

and m_f^2 is the physical fermion mass of the same value as in the approach 1, by definition. It comes out independent of Δ by virtue of adjusting once and for all the mass-squared counterterm in H^Δ. The same adjustment involves fixing the free finite constant in Eq. (4.36) so that, for a certain value of $\lambda = \lambda_0$, the physical fermion mass eigenvalue m_f equals the value found experimentally. Interestingly, the same eigenvalue is subsequently obtained for all values of λ automatically and the physical dispersion relation satisfies all the requirements of special relativity. This is the simplest manifestation of the general rule that physical results should be independent of λ.

[6] Note that v denotes here a potential, not a single spinor of an antiparticle.

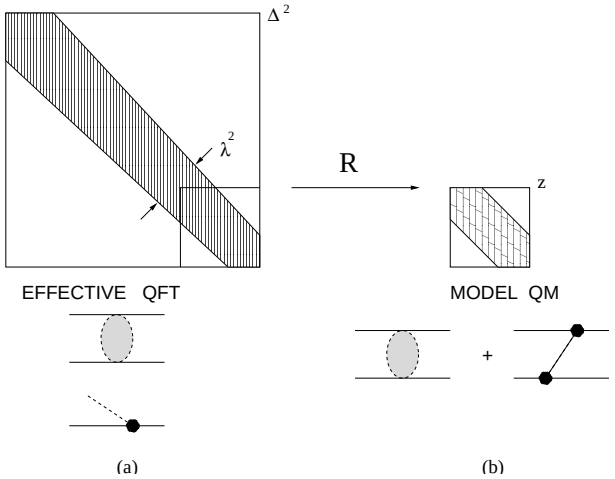


Figure 4.4: The reduction of effective QFT (a) to relativistic quantum mechanics in a model subspace (b) is possible in approach 2 due to the form factors in the effective interaction vertices.

Reduction to two-effective fermions

Using the transformation R to reduce $\mathcal{H}_\lambda$ to the two-effective-particle subspace without restrictions on the relative momenta, one obtains a quantum mechanical interaction that can change the invariant mass of the two particles by a certain Λ if, and only if, the interaction acts more than Λ/λ times. Thus, approach 2 produces an effective Hamiltonian that is free from the overlapping divergence problem discussed in [88], and in Section 4.5 in approach 1. However, in order to make a connection with the nonrelativistic two-particle Schrödinger quantum mechanics that was not available in approach 1, one now needs to limit the relative momenta in the effective two-particle Fock sector to $k < z$ (where z is a new parameter required for defining the new operation R that enables one to define the procedure of introducing the nonrelativistic limit).

Accordingly, we define a new transformation R that leads to a model Hamiltonian H_R. This H_R acts only in the subspace of the two-effective-particles Fock sector with limited invariant masses (Fig. 4.4). Not only the number of effective particles is limited, but also the range of their relative momenta. It is required that H_R has the same spectrum of low-lying energy levels as $\mathcal{H}_\lambda$ has in the whole space. This step is no longer related in any way with the infinite renormalization problem as in the approach 1. The existence of such reduction is plausible only because $\mathcal{H}_\lambda$ has a small width λ.

The projection operator used here in constructing R is

$$\hat{P} = \sum_{\sigma_1\sigma_2} \int [p_1 p_2] b_{1\lambda}^{(1)\dagger} b_{2\lambda}^{(2)\dagger} |0\rangle \langle 0| b_{2\lambda}^{(2)} b_{1\lambda}^{(1)} \;\; \theta\left(z - |\vec{k}|\right) \;, \tag{4.40}$$

where $\vec{k}$ is the relative momentum of effective particles of momenta p_1 and p_2. Although introducing z is useful from a conceptual point of view, the form factors f_λ imply that z is not important in practice, see Fig. 4.4.

The effective Schrödinger equation has the form of Eq. (4.24), with v_{OBE} replaced by a new potential, denoted $v_{R\lambda}$, the sum of two terms (Fig. 4.4b). The first term is the projection of $\mathcal{H}^{\Delta(2)}_{\lambda\, b^\dagger b^\dagger bb}$, cf. Eq. (4.37), on the two-body space restricted by z. The second term comes from

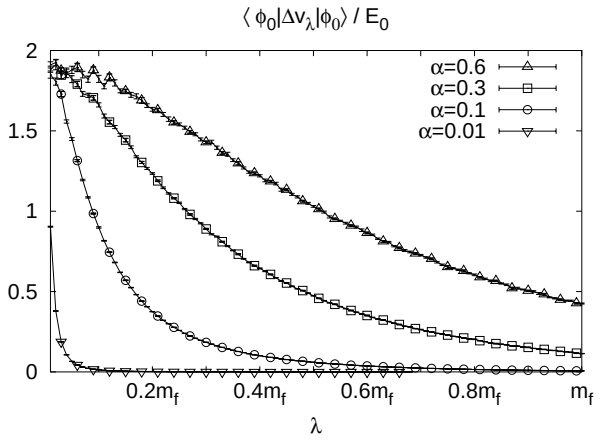


Figure 4.5: First-order correction to the ground-state binding energy as a function of λ for $z=\infty$, which shows the magnitude of corrections to the well-known Schrödinger equation with a Coulomb potential expected in the approach 2 in QFT. The λ-axis of the plot starts at $\lambda = 0.01 m_f$, and not zero, because the computational method used here is not applicable for $\lambda = 0$.

the one-effective boson exchange (OEBE), and has a form similar to (4.25),

$$v_{OEBE\lambda}(\sigma_1\sigma_2\sigma_3\sigma_4\vec{k}\vec{k}') = -\frac{\pi\alpha}{2m^2} f_\lambda \bar{u}_1 u_2 f_\lambda \bar{u}_3 u_4 \frac{1}{q^+}\left(\frac{p^+_{ba}}{ba}+\frac{p^+_{bc}}{bc}\right)_{Fig.4.1a} + \text{the same}|_{Fig.4.1b}\,, \tag{4.41}$$

except for the form factors f_λ in vertices and the overall limitation of the momenta by z [not indicated explicitly in Eq (4.41)]. Both of these terms (i.e., the projection of $\mathcal{H}^{\Delta(2)}_{\lambda\, b^\dagger b^\dagger bb}$ and v_{OEBE}) behave for $k,k' \ll m$ like the Coulomb potential (4.27), with form factors f_λ that limit the changes of the fermion kinetic energies.

One can approximate the Schrödinger equation with this QFT potential by the equation with a Coulomb potential plus a correction, and one may estimate the size of the correction using bound-state perturbation theory. For this purpose, the difference between potentials $v_{R\lambda}$ and $v_{Coulomb}$ is denoted by Δv_λ. The first-order correction, $\Delta E^{(1)}_\lambda = \langle\phi_0|\,\Delta v_\lambda\,|\phi_0\rangle$, is a function of the parameters λ and z. A numerical calculation confirms that for $z > \lambda$ there is no noticeable z-dependence of this matrix element. Figure 4.5 shows how the matrix element depends on λ for $z=\infty$. As expected in Section 4.6.1 for small λ, there is some λ-dependence in the result. It appears because, when lambdas are too small, the similarity factors f_λ start to limit the Hamiltonian in the momentum region that is important for the bound-state formation, and the derivation of $\mathcal{H}_\lambda$ cannot be carried out precisely using the perturbative renormalization group procedure down to so small lambdas. When λ and z are large enough, the correction $\Delta E^{(1)}_\lambda$ tends to a finite value that depends on α. This happens because the wave function ϕ_0 has a width $a = \alpha\mu$ (see Appendix J.1) and limits the integration over both momenta in the matrix element $\langle\phi_0|\,\Delta v_\lambda\,|\phi_0\rangle$. As shown already in Section 4.5.2, the first-order correction is small for small coupling constants due to the fast fall-off of the Coulomb wave function at large momenta, independently of the details of Δv that one obtains in the approaches 1 or 2. The correction is small even for a divergent potential such as a δ function.

We now look at the second order of the bound-state perturbation theory to check the self-consistency of the effective-particle picture and to compare it to approach 1. To demonstrate

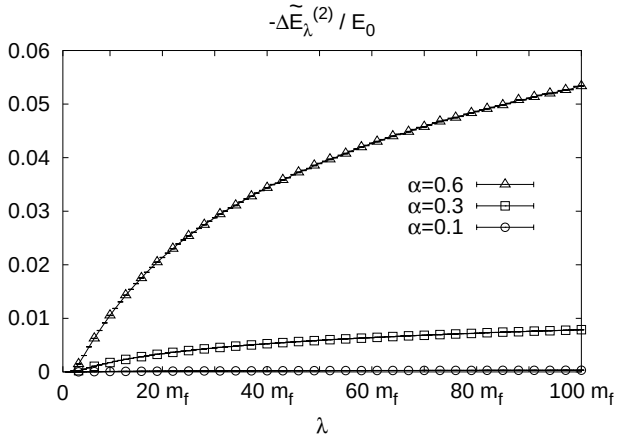


Figure 4.6: Dependence of the large-relative-momentum contribution to the second-order bound-state mass correction on λ, for the cutoff Δ (or k_{max}) sent to ∞. When one works with effective fermions, cutoffs can be sent to ∞ for any given value of λ. This feature is not available in the approach 1 for bare fermions, as shown in Fig.4.3.
In the effective-particle approach, physical results for a two-fermion bound state should be obtained using λ on the order of m_f, or other momentum scale physically relevant for the binding mechanism.

that the effective theory does not exhibit the consistency problems faced by approach 1 (see Fig. 4.3), one can follow closely the derivation of Eq. (4.30), but now with the OBE potential v_{OBE} replaced by $v_{R\lambda}$. Again, one may investigate whether there is a logarithmically divergent dependence on k_{max}.

It turns out that, for finite values of λ, there is no such divergent dependence. One can safely take the limit $k_{max} \to \infty$, since λ itself already cuts off sums over intermediate states in the correction,

$$-\Delta\tilde{E}_\lambda^{(2)} = m^2 \int_0^{4\alpha m} \frac{d^3k_1}{\sqrt{E_1}} \int_m^\infty \frac{d^3k_2}{E_2} \int_0^{4\alpha m} \frac{d^3k_3}{\sqrt{E_3}} \phi_0(k_1) \Delta v_{\uparrow\downarrow\lambda}(\vec{k}_1,\vec{k}_2) \frac{1}{H_0} \Delta v_{\uparrow\downarrow\lambda}(\vec{k}_2,\vec{k}_3) \phi_0(k_3) \,. \tag{4.42}$$

Here, $\Delta v_{\uparrow\downarrow\lambda}$ is defined similarly to $\Delta v_{\uparrow\downarrow}$, but with v_{OBE} replaced by $v_{R\lambda}$. Numerical results for this matrix element for different values of λ (and for the cutoffs $k_{max} \sim \Delta \to \infty$), are shown in Fig. 4.6.

Two important points should be noted. First, the results in Fig. 4.6 can be considered a good approximation to the whole second-order correction only for $\lambda \gg m$ (i.e. in the right part of the figure). If λ is comparable to m, the similarity factors f_λ limit the potential $v_{R\lambda}$ and the high-low and low-high corners of the potential matrix (Fig. 4.2) are practically eliminated. The correction coming from the large momentum region selected in the integration in Eq. (4.42) is therefore also reduced and the other of the 2^4 parts of the whole correction can contribute more significantly than they do for large lambdas. Hence, for small lambdas, the results given in Fig. 4.6 are not necessarily a good approximation of the whole second-order energy correction.

Second, in practical work, one needs to lower λ as far down as possible, possibly below m. Thus, Fig. 4.6 provides only evidence for the consistency of the effective fermion dynamics in which the convergence in the large-relative momentum region is secured by the presence of λ, and the original QFT cutoffs can be safely sent to infinity. Additional analysis can be found

in Ref. [2].

4.7 Summary of the mechanism of removal of divergences by RGPEP form factors

Let me summarize the two approaches to bound states of fermions. I used the example of the bound states of two fermions in the Yukawa theory.

Approach 1 starts from the sector of two bare fermions. This approach leads to overlapping divergences in the light-front Hamiltonian dynamics and lacks consistency in its handling of the large-relative-momentum region, when one attempts to send the bare cutoff to infinity without including infinitely many bare particles. It is possible to remove this defect through sector-dependent counterterms, but the required construction of the full renormalization group triangle with growing numbers of bare particles is both highly complex and not fully understood. The basic ultraviolet problem arises from short distances in the transverse directions, and no practical tool yet exists for handling huge numbers of bare particles with the precision required by the rotational, parity, and other symmetries of the initial Lagrangian.

Approach 2 is free of the difficulties connected with the large-relative-momentum convergence. In this approach description of the bound state starts from two effective fermions. The decisive convergence factor is introduced by solving renormalization-group equations for effective particles. This solution includes form factors of width λ in the interaction vertices, and these form factors suppress the large-momentum domain. This can be verified by numerical estimates. The well-known one boson exchange (OBE) potentials that are deduced from the on-shell S-matrix elements are replaced by new one-effective-boson exchange (OEBE) and additional interactions that are derived in the Hamiltonian. One can also take advantage of a subsequent S-matrix calculation in choosing free finite parts of the counterterms (this is discussed in Chapter 5).

The accuracy of this approximate treatment can be estimated by inspecting the variation in results if one changes the renormalization group parameter λ. An exact theory would exhibit no dependence on λ; in the approximate treatment, the variation depends on the size of the coupling constants.

For α smaller than about 0.3, there is a wide range of lambdas in the Yukawa theory with massless bosons, in which the results for the two-fermion bound-state mass are stable and do not differ significantly from the results of a nonrelativistic Schrödinger equation with Coulomb potential. These values of α can be called nonrelativistic. For α larger than about 0.3, one has to allow λ to grow to the size of the order of m_f to achieve λ-independence of the corrections to the bound-state mass. This means that relativistic momenta do, in fact, matter, and the nonrelativistic Schrödinger equation is not a good approximation of the effective dynamics. Still, the effective theory is well contained in the range determined by λ, and one may look for solutions of the eigenvalue problem without making a nonrelativistic approximation. For the first-order bound-state perturbation theory, corrections to the nonrelativistic approximation are quite considerable, while the large-relative-momentum region in second-order corrections contributes only about 10%. Being limited by λ, it does not introduce any diverging contributions and one does not – and should not – attempt sending λ to infinity, in contrast to the cutoff $\Delta \to \infty$ in approach 1.

The light-front form of Hamiltonian dynamics enables us to separate the relativistic motion of bound states from their internal constituent dynamics. Thanks to this separation, one can

reduce the description of the binding mechanism to the Schrödinger equation for internal motion of the constituents without choosing a specific frame of reference. All eigenvalue equations derived here, in both approaches, are independent of the total momentum of the states they describe. Thus, the boost symmetry allows us to understand moving bound states in arbitrary motion as soon as we understand them at rest. This is not possible in the standard form of Hamiltonian dynamics.

Chapter 5

Covariance of scattering amplitudes calculated with an effective Hamiltonian

5.1 Introduction

Scattering experiments play an important role in the development of the theory of particles. These experiments provide the only way to probe directly the inside of hadrons and perhaps even leptons and gauge bosons. The main tool for describing scattering processes in perturbation theory is Feynman diagrams, whose invention, renormalization, and application in the standard model is one of the greatest triumphs of the theory of particles – a quantum approach that satisfies the principles of special relativity as long as interactions are weak and bound-state dynamics is of secondary importance. Therefore, any attempt to construct a Hamiltonian approach that aims to naturally include the bound-state eigenvalue problem must also match the success of Feynman diagrams in the perturbative domain of scattering phenomena. The analysis of Hamiltonian approach presented here does not fully rise to the challenge of verifying if the Hamiltonian approach can produce covariant results to all orders, because this analysis is limited to calculations of order not higher than the third. It is analogous to one-loop level in Feynman's approach. I also limit my calculations to the simplest asymptotically free theory that could be considered. Nevertheless, even the challenge of the third-order calculations – to obtain the covariant results with non-covariant Hamiltonians for effective particles that have form factors of small widths in three-dimensional kinematical momentum space in interaction vertices – turns out to require extensive studies in an entirely new calculational scheme for Hamiltonians. I should stress that the loops one obtains in the calculation of a Hamiltonian operator itself are different from the loops that occur in Feynman diagrams for scattering amplitudes. The difference originates from the choice of light-front kinematical momentum variables and special non-covariant regularization factors specific to RGPEP. Moreover, the initial Hamiltonian is regularized once and for all of its vertices. The effective Hamiltonians $\mathcal{H}_\lambda$ have vertex form factors f_λ that cannot be freely changed or adjusted, and the only freedom left is hidden in the counterterms in H^Δ that have to repair the damage to explicit covariance introduced by regularization of the Hamiltonian operator. In fact, it was not clear when I began this research whether any Hamiltonian approach could actually produce covariant results in a similar procedure as the one outlined in the simple model of Chapter 2. But the principles of RGPEP appeared conservative enough so that one might ask the question: Can one regulate a canonical Hamiltonian of any theory (even a ridiculously simple one, treated only perturbatively, but preferably asymptotically free), apply the principles of RGPEP, and obtain a covariant answer for an amplitude that

even remotely resembles standard Feynman results for amplitudes such as $e^+e^- \to hadrons$?

The answer to this question turns out to be positive, as described in this chapter. But the description that follows cannot, unfortunately, be brief, and must start with rather basic details, since it was necessary to regulate the entire theory in its Hamiltonian form *ab initio*, and the loop integrals obtained could not be regulated in a way that was as convenient for making Wick rotation as dimensional regularization is. I never encounter Wick rotation, because the unregulated integrals in k^- are not the same as integrals in k^0.[1] It was necessary to step back to the Lehmann-Symanzik-Zimmermann reduction formalism and carry out all steps of the LSZ formalism with explicit control over all regularization effects, keeping track of Hamiltonian counterterms and tracing how many terms combine and how they combine in the RGPEP scheme for Hamiltonians to a covariant answer.

I will begin by listing the issues that the calculation presented here involves.

Scattering processes in QFT

The most important complexities relate to the fact that the scattering of strongly interacting particles in fact involves bound states rather than the free particles. This means that one cannot limit the calculation to perturbation theory alone. While in the first approximation most problems with high-energy scattering may be avoided by noting that QCD is an asymptotically free theory [89, 90] (and therefore quark interactions become weak in high-energy scattering), one expects complications wherever the nonperturbative structure of the bound states is involved.

In all circumstances where the bound-state formation matters, it is crucial to have a formulation that applies both to bound states and scattering in order to obtain a description based on first principles. It is natural to try to define a Hamiltonian description, since the basic connection between the Green functions and scattering is based on the LSZ formula that involves an assumption that a Hamiltonian of the theory exists, while the bound states must be described by an eigenvalue equation for the same Hamiltonian (see the discussion of divergences in the bound-state eigenvalue problem in Chapter 4).

However, the standard approach to scattering in QFT is based on Feynman diagrams and covariant regularization such as dimensional regularization. It is thought [9, 8], that the Hamiltonian approach cannot produce covariant answers because of its intrinsically non-covariant nature, due to the distinguishing of the time axis and non-covariant regularization.

In other words, the basic issue with Hamiltonians is: can one renormalize the Hamiltonian calculation and obtain covariant results? The prospects of applying all known methods of Hamiltonian quantum mechanics (linear algebra of eigenvalue equations, variational methods, and explicitly unitary evolution of states) are exciting, but divergent QFTs need regularization, and there is the question of whether one can define non-covariant regulated Hamiltonians in the Fock space, renormalize them and obtain covariant results for scattering. It turns out that when one switches from the standard-time evolution to Dirac's front form of Hamiltonian dynamics an important change takes place: kinematical rotational invariance is turned into kinematical boost invariance and it is much easier to regulate rotations that involve angles between 0 and π than boosts that involve imaginary angles between 0 and infinity. In other words, it is hard to regulate boost operators in the equal-time formulation and control states of moving particles. Light-front dynamics provides an opportunity to treat boosts like in a free theory, and the cal-

[1] Perhaps, higher-order analysis of the Hamiltonian approach may take advantage of the connection between k^-, k^+, k^0 and k^3, and employ analytic continuation in k^0 in the amplitudes after they are shown to be finite and independent of regularization.

culations outlined here suggest that rotational symmetry – even if dynamical – may be brought under control in perturbation theory.

The standard approach is to derive a formal expression for the S matrix using an unregularized Hamiltonian, and to learn how to regulate the expressions that result from the formal procedure later, i.e., when calculating physical observables using Feynman diagrams. This approach proved extremely fruitful. It leads to simple, yet powerful renormalization group equations which describe the behavior of the Green functions. The most powerful regularization is the dimensional regularization [91], which leads to covariant and gauge-invariant expressions at each order of perturbative S-matrix calculations (see also [92, 93]).

The main difficulty of a loop-by-loop (i.e., perturbative) regularization procedure for the Green functions is that it is not clear how to apply it to nonperturbative bound-state problems, where powers of the coupling constant are mixed up in a way non-expandable into a power series. In asymptotically free theories, one faces the additional problem that the coupling constant is large in the domain of momenta where the binding mechanism is most active, and remains unknown. This problem occurs, for example, in the context of the Bethe-Salpeter equation. To use such an equation in a controlled fashion, one would have to design a reliable regularization and renormalization procedures that would apply in the the region of the binding mechanism.

Given the above, asymptotically free QFT can be used to describe high-energy scattering, but bound states have to be described using more or less phenomenological constituent models. There remains the problem of how to construct a single formalism that would incorporate both aspects of the theory [53].

Hamiltonian description of scattering

There are several reasons why the Hamiltonian description may be hard to develop, even for perturbative scattering processes in the femtouniverse [94], where there are no bound states involved.

1. Interference with the ground state: The light-front Hamiltonian theory considered here is designed to avoid the following problems: (1) In equal-time theory old-fashioned perturbation theory is affected by a (complicated) vacuum structure through Z-diagrams (Fig. 3.2b). There are no such diagrams in the formalism considered here, which means that both the vacuum itself and the old-fashioned perturbation theory are simpler than in the equal-time approach. (2) In equal-time perturbation theory regularized corrections to energies of one-physical-particle states do not automatically have the form required by special relativity (cf. Section 3.2.6). In the light-front approach one does not face the difficulty with different denominators that must be somehow combined in equal-time approach to produce relativistic expressions, and how they combine should not be destroyed by regularization. I don't know how to solve this problem in equal-time approach.

2. Non-covariant regularization: The question here is how to describe scattering processes involving particles moving with arbitrary velocities, in particular, velocities close to the speed of light. It must be investigated whether some proper choice of finite parts of counterterms can lead to a covariant scattering matrix, even when the Hamiltonian is formulated using non-covariant cutoffs.

3. The need for going beyond cancellations such as those of the Ward identities: Feynman calculus uses Ward identities, and the scattering amplitude $e^+e^- \to hadrons$ is finite in

QCD coupled to QED in one loop: namely, the ultraviolet divergence of wave function renormalization factors Z corresponding to the external quark lines cancels the divergence in the $\gamma \to q\bar{q}$ vertex. In the asymptotically free scalar theory considered in this chapter, such Ward identities are absent. The Hamiltonian approach to scalar theories has to solve the ultraviolet problems independently of Ward identities.

It should be noted, however, that in Hamiltonian dynamics of gauge theories one has to choose a gauge (for example, the light-front gauge $A^+ = 0$ (E.19)). Therefore, there is no explicit local gauge symmetry in the Hamiltonian, and it is not immediately apparent how, in such a theory, analogs of Ward identities emerge: in particular, any extra terms added to the Hamiltonian (e.g., the required counterterms) may depend on the choice of gauge and regularization. In fact, the structure of the Hamiltonians of gauge theories is complicated and the ultraviolet renormalization problem is plagued with singularities related to the choice of gauge. On this point, it is instructive to study some divergent terms in Hamiltonians in gauge theories and observe the structure of their ultraviolet divergences. Examples of such divergent terms are given in Appendix F. All these terms, despite their additional complexities, share some common divergence structure with a plain ϕ^3 theory when treated perturbatively. The results for counterterms in QED and QCD shown in Appendix F explain why I had to restrict this initial study to scalar theories that are less singular than gauge theories (they do not contain small-x divergences). Since gauge theories are more complicated than scalar theories, one first has to show that a Hamiltonian formulation can lead to covariant answers in an asymptotically free scalar theory.

The aim of this chapter is thus to understand the ultraviolet (high-energy) Hamiltonian renormalization procedure in a theory that describes particles analogous to electrons, photons, quarks, and gluons, in the presence of interactions that resemble in their ultraviolet structure the structure of QED and QCD. The idea of simplifying a complex theory in order to address basic issues goes far back in time, and the key example regarding old-fashioned perturbation rules in the infinite momentum frame (isomorphic to the light-front scheme) was provided by Weinberg [95].

One of key features of any theory, determining how complicated the UV problem is, is the degree of divergence of the loop diagrams appearing in perturbative calculation of the S matrix. In the case of QED and QCD, these are quadratic and logarithmic divergences. The main part of this chapter deals with scalar theory in six space-time dimensions (five space dimensions and one time), where there is the same degree of divergence. This means, for example, that the wave-function renormalization factors Z are logarithmically divergent. I trace below which counterterms remove this and all other divergences from physical results, and how this happens.

The most important feature of the scalar ϕ^3 theory considered here is that it is asymptotically free. This means that the effective coupling constant gets small when the momentum transfers in a process grow, and this enables us to believe in a consistent description of the high energy structure of the theory based on the perturbative approach.[2] But every coin has two sides: in this case, the flip side is that the coupling constant becomes large for small energies, and this

[2]Note, however, that although the motivation of the renormalization program does not apply strictly to theories that are not asymptotically free, one usually expects that a similar program of ultraviolet renormalization would work in this case too, with few modifications. The prime example is QED, which is not asymptotically free, yet it is analyzed (e.g. when scattering amplitudes are derived by Feynman diagrams) by means of a standard UV renormalization using smallness of the coupling constant and the fact that it varies only logarithmically.

includes the critical domain of the formation of bound states. In this domain, it is essential that a perturbative calculation of a Hamiltonian may still be justified while the perturbative treatment of scattering is entirely inappropriate.

The plan of this chapter

The main results for scattering are presented in the following order in this chapter.

In Section 5.2, I review the derivation of Feynman diagrams in the case of a light-front Hamiltonian, which from the very beginning is regulated. Thus, all steps of the derivation are mathematically well-defined. This leads to an analogue of LSZ formula for the scattering matrix, and its perturbative expansion leads to an analogue of Feynman diagrams, but with regularization coming directly from the Hamiltonian that includes counterterms. Details are given in Appendix G.

In Section 5.3, I describe a change of basis in the Fock space and in the space of operators that act in the Fock space, and show that effective particles described by an effective Hamiltonian $\mathcal{H}_\lambda$ can be used for obtaining the same scattering amplitude as the bare canonical particles can. This has two important consequences. First, it means that one and the same Hamiltonian can be employed in a scattering formalism and for description of bound states: our regularization and renormalization for the effective Hamiltonian produces an operator that has all properties that we found necessary for description of bound states in Chapter 4 (the binding is described in terms of low-energy effective particles). The second consequence is that the physical scattering amplitude is totally independent of the ultraviolet cutoff Δ in the initial canonical Hamiltonian, and also independent of the effective Hamiltonian width λ (in perturbation theory).

All calculations are carried out in a scalar theory similar to ϕ^3 scalar theory in 5+1 dimensions. In Section 5.6, I present RGPEP for this theory and the resulting counterterms (see Section 3.4 for details of RGPEP). In fact, I consider two ways of constructing the same counterterms. One of them is much simpler than the other, and both are sufficient for the description of a scalar analogue of $e^+e^- \to$ *hadrons* scattering amplitude up to order e^2g^2. In Section 5.7, I show that the RGPEP counterterms remove divergences from the amplitude.

Another issue is the finite dependence of the Hamiltonian on regularization, and the constraints on that finite dependence that follow from the Poincaré symmetry of the physical scattering amplitude. In particular, the S matrix is not automatically covariant unless one makes sure that there exist finite counterterms in the Hamiltonian that remove finite violation of covariance due to regularization. I provide explicit expressions for the Hamiltonian terms that assure covariance of the scattering amplitude. As a result, the scattering amplitude depends on momenta of scattered particles as in the Feynman diagrams. This is how the rigorous Hamiltonian approach justifies the formal procedure that leads to the Feynman diagrams for my model of the amplitude $e^+e^- \to$ *hadrons*. Overall, therefore, this section implies that the heuristic model of Chapter 2 is useful in providing patterns that one can attempt to reproduce in QFT.

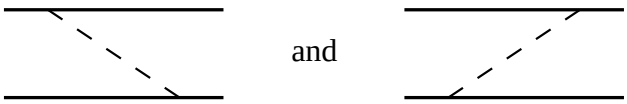
5.2 Calculation of S-matrix elements using regularized light-front Hamiltonians for bare particles

The perturbative formula for the S matrix in terms of covariant Feynman diagrams is derived in the standard QFT textbooks such as [96-98] and many others. However, the derivation presented in these works does not take into account the divergences of the QFT from the very beginning. Instead of introducing regulators explicitly, formal expressions for physical observables are derived first, and only then useful loop-by-loop regularizations of the final expressions are introduced (see also [99] in the context of infinite-momentum-frame calculations of an S matrix).

Such an approach is not directly useful if one wants to use the same Hamiltonian for describing both bound states and scattering processes. The ultimate goal is to determine the counterterms including their finite parts, using symmetries of the S matrix (this might concern only bound states in the case of confinement) and fixing values of all free parameters using data for selected scattering experiments or some selected bound states, and then use the Hamiltonian without free parameters to describe all other processes and states. The entire procedure of the S-matrix calculation using regularized Hamiltonians is described in Appendix G.

The key differences between the the standard Lagrangian procedure and the Hamiltonian procedure are as follows:

- The canonical Hamiltonian is regularized: All terms are regulated, but different regulating factors appear in differently ordered terms, for example:



(5.1)

  The regularization factors that we introduce do not have to solve problems with the creation of particles from the vacuum (see Section 3.3.3).

- The Hamiltonian is not a local product of pointlike fields: This happens due to the regularization. Instead of local fields we deal with the creation and annihilation parts of the fields in apparently separate ways. However, I make sure that all vertices are regulated without reference to variables that depend on spectators, and our regulators depend only on the kinematical momenta of the particles involved. This is why the demonstration of the covariance of the resulting amplitude is a challenge – it does not refer to mathematical operations that are only formally valid, and would be invalidated if a regularization was taken into account.

- An entire family of different Hamiltonians H_λ is calculated: The family of Hamiltonians is parametrized by a renormalization-group parameter λ and, in the RGPEP approach, the parameter λ is also a width of the momentum-space form factors that appear in interaction vertices. The Hamiltonian $\mathcal{H}_\lambda$ is expressed in terms of creation and annihilation operators, $a^\dagger_\lambda$ and a_λ, that create or annihilate effective particles whose size in configuration space can be thought equal to $1/\lambda$. When one switches to the effective-particle picture, one faces the challenge of showing that it is possible to develop scattering theory using the effective particles and corresponding Hamiltonians and obtain the same results as in the initial canonical theory, independently of the value of λ. The most critical test is whether one obtains Poincaré symmetry for scattering amplitudes.

- Light-front Hamiltonian dynamics: The evolution operator is chosen to be $H = P^-$, as was advocated by Dirac in 1940s [16]. Therefore, analysis is carried out in terms of the evolution parameter x^+ instead of the usual x^0, and I do not have energy denominators in my S-matrix calculus, but rather p^- denominators.

The standard derivation of Feynman diagrams consists of three main steps:

1. Matrix elements of interacting fields are assumed to evolve in the far future or far past in the same way as the matrix elements of properly chosen free fields.

2. S-matrix elements are expressed in terms of vacuum expectation values of the time-ordered products of fields (the LSZ formula).

3. These vacuum expectation values are expanded in a perturbative series in the interaction Hamiltonian.

I review these three steps below very briefly for the purpose of our light-front analysis; details are presented in Appendix G. In Section 5.3, I show how different auxiliary fields, all unitarily equivalent, can be used for the calculation (e.g. $\phi_0 = \phi_{0\infty}$ or $\phi_0 = \phi_{0\lambda}$). For a different choice of ϕ_0, one has a different perturbation theory for the same physical quantity: ϕ_0 may be chosen differently, but the interaction Hamiltonian H_I and the wavefunction renormalization factors Z will change accordingly, and the result for the S matrix will not change.

5.2.1 *In* and *out* fields and states

Section 3.3 introduced the construction of free and interacting fields in our approach (see also Appendix G.1). Let us consider the scattering of two particles in the initial state, well separated from each other; this is ensured by using proper wave packets. Final particles emerging from the interaction region are assumed not to interact with each other; this is also ensured by using proper wave packets.

The group velocity of a packet made of massive particles is smaller than the speed of light, c. Massive particles propagate within the light cone of past and future and, for them, the conditions $x^0 \to -\infty$ and $x^+ \to -\infty$, or $x^0 \to +\infty$ and $x^+ \to +\infty$ are equivalent (Fig. 5.1). For massless particles – not considered here – these two conditions are not strictly equivalent, as they differ in the case of propagation along the direction where the light cone touches the front[3], (see Fig. 5.2).

It is assumed that matrix elements of packets made of fields[4] $\phi(x)$ between any normalizable states $|\alpha\rangle$, $|\beta\rangle$ behave for the time $x^+ \to -\infty$ as matrix elements of similar packets of certain free fields ϕ_{in}

$$\lim_{x^+\to-\infty} \langle\beta|\,\phi(x^\mu)\,|\alpha\rangle = \sqrt{Z} \lim_{x^+\to-\infty} \langle\beta|\,\phi_{in}(x^\mu)\,|\alpha\rangle \ . \tag{5.2}$$

Thus, the creation operators $a^\dagger_{in}$ create states that have an interpretation of physical states for the time x^+ approaching $-\infty$. The statement that ϕ_{in} is a free field means that the creation operators $a^\dagger_{in}$ evolve only by a change of a phase [see (G.7)].

[3] One would expect that the quantum effects wash out the line singularity and make it less severe through the uncertainty principle than in the classical theory of partial differential equations.

[4] For a definition of the smeared fields see, e.g. [100]. The specific definition does not matter for the following considerations; to simplify the notation, the smearing is not indicated in the equations.

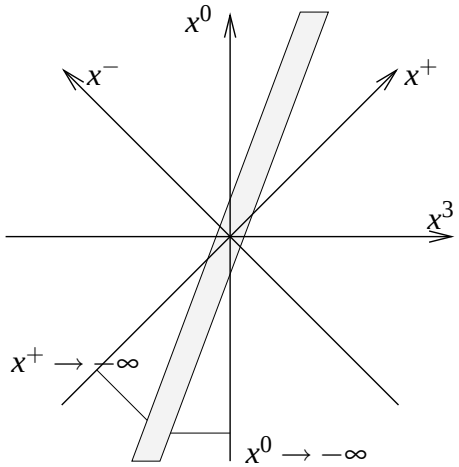


Figure 5.1: A qualitative picture of a wave packet (the stripe) moving with a group velocity smaller than the speed of light. For massive particles (which cannot move with the speed of light), the limits $x^0 \to -\infty$ and $x^+ \to -\infty$ are equivalent. Thus, when defining scattering theory, x^+ ordering may be used (instead of time ordering), and *in* and *out* states and fields may be defined by the limits $x^+ \to \mp\infty$. This figure does not take into account the expected spreading of the wave packet in time.

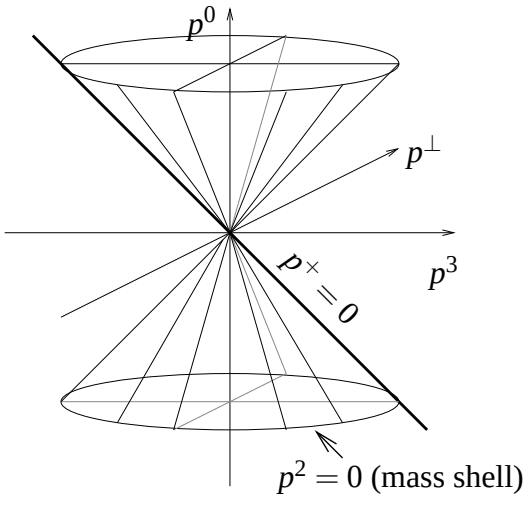


Figure 5.2: For particles of zero mass, only one direction in the space-time corresponds to the zero value of p^+ momentum.

The asymptotic condition (5.2) defines not only the fields ϕ_{in}, but also the physical mass: a wave-packet describing a single, well-separated particle, which evolves with x^+ as a packet of free states, each of its Fourier components evolving with a factor $\exp(-ik^-x^+/2)$. The physical mass m^2_{phys} is defined by the condition $k^- = \left(k^{\perp 2} + m^2_{phys}\right)/k^+$. Likewise, the physical outgoing states and field ϕ_{out} are introduced by considering the limit $x^+ \to +\infty$.

5.2.2 One-particle states

Let us consider the equation (5.2) in the case of $|\alpha\rangle = |0\rangle$ and $|\beta\rangle = |p\rangle_{in}$[5]:

$$\lim_{x^+\to-\infty} {}_{in}\langle p|\phi(x^\mu)\,|0\rangle = \sqrt{Z}\lim_{x^+\to-\infty} e^{ip_m x}\,, \tag{5.3}$$

where I used ${}_{in}\langle p|\phi_{in}(x^\mu)\,|0\rangle = e^{ip_m x}$. The x^+ dependence cancels on both sides because:

$$\begin{aligned} {}_{in}\langle p|\phi(x^\mu)\,|0\rangle &= {}_{in}\langle p|e^{iP^-_m x^+/2}\phi(\vec{x},x^+=0)e^{-iP^-_m x^+/2}\,|0\rangle = & (5.4)\\ &= {}_{in}\langle p|e^{ip^-_m x^+/2}\phi(\vec{x},x^+=0)\,|0\rangle\,, & (5.5)\end{aligned}$$

and this leads to:

$${}_{in}\langle p|\phi(\vec{x},x^+=0)\,|0\rangle = \sqrt{Z}e^{-i\vec{p}\vec{x}}\,. \tag{5.6}$$

I use the notation $\vec{p}\vec{x} := -\frac{1}{2}p^+x^- + p^\perp x^\perp$.

The matrix element on the left-hand side can be looked at in two ways. The equation (5.6) says that the probability of finding a one physical particle state $|p\rangle_{in}$ in a state created by ϕ (namely, $\phi(x^\mu)\,|0\rangle$) for a large time x^+ is Z. This way of looking at the equation will provide a straightforward connection to the spectral representation [96, 101, 102], and Z will be equal to a residue of a pole of a propagator.

However, one can also look at the equation (5.6) from a different angle. The creation operators in the Fourier transform of the field $\phi(\vec{x},x^+=0)$ are the bare creation operators in terms of which the Hamiltonian is initially expressed. Therefore, $\phi(\vec{x},x^+=0)\,|0\rangle$ is a superposition of one- and more-bare-particle states. One can look at this expression in terms of the Fock basis of the bare particles. If the physical state is expressed as a superposition

$$|p\rangle_{in} = N\left\{|p\rangle + \int [p_1,p_2]\widetilde{\delta}(p-p_1-p_2)\phi_p(p_1,p_2)\,|p_1p_2\rangle + \dots\right\}\,, \tag{5.7}$$

where the second term denotes two-bare particles component of $|p\rangle_{in}$, $\sqrt{Z}$ is equal to the amount of one-bare particle content of the physical state $|p\rangle_{in}$:

$$\sqrt{Z} = N\,. \tag{5.8}$$

5.2.3 The reduction formula for scalar fields

We can imagine a situation in which in the distant past there were particles forming a set α. Such a situation is described by a state $|\alpha\rangle_{in}$. If, for example, there were two particles of momenta p_1 and p_2, this state would be:

$$|p_1,p_2\rangle_{in} = a^\dagger_{p1,in}a^\dagger_{p2,in}\,|0\rangle\,. \tag{5.9}$$

[5] States $|p\rangle_{in}$ are considered limits of wave-packets whose momentum widths tend to zero.

Due to interactions, this state has a complicated structure in the future. The probability that the state contains asymptotic outgoing particles of a set β in the distant future is measured by a square of a matrix element $S_{\beta,\alpha}$:

$$S_{\beta,\alpha} = {}_{out}\langle \beta \mid \alpha \rangle_{in} \,. \tag{5.10}$$

This defines the scattering matrix S for this process. In Appendix G.2 it is shown that for a set of $q_1, q_2, \dots$ incoming particles and $p_1, p_2, \dots$ outgoing, and for all $p_i \neq q_j$, the S matrix element is:

$$\begin{aligned} {}_{out}\langle p_1 \dots p_m \mid q_1 \dots q_n \rangle_{in} \;=\; & \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \prod_{i=1}^{m} \int \mathrm{d}^4 x_i \prod_{j=1}^{n} \int \mathrm{d}^4 y_j e^{-iq_{im}x_i} \left(\overrightarrow{\Box}_{xi} + m^2\right) \times \\ & \times \langle 0 | \, T_{(+)} \left[\phi(y_1) \dots \phi(y_n) \phi(x_1) \dots \phi(x_m)\right] | 0 \rangle \times \\ & \times \left(\overleftarrow{\Box}_{yj} + m^2\right) e^{ip_{mj}y_j} \,, \end{aligned} \tag{5.11}$$

(cf. (G.38)). In this equation $\mathrm{d}^4 x = \mathrm{d}x^+ \mathrm{d}x_+ \mathrm{d}^2 x^\perp = \mathrm{d}x^+ \mathrm{d}x^- \mathrm{d}^2 x^\perp / 2$. This is a light-front version of the Lehmann-Symanzik-Zimmermann (LSZ) formula [103, 104]. This equation directly corresponds to equation (16.81) of the Bjorken and Drell textbook [96], with x^+ ordering instead of x^0 ordering and a different Fourier expansion of the fields.

Based on the spectral representation, one expects that the Green function $\langle 0 | \, T_{(+)} [\phi \dots \phi] \, | 0 \rangle$ has poles corresponding to each of the external particles. I will review in Section 25 (page 92) how this kind of pole structure emerges in perturbation theory, but the spectral representation argument (see [96, 98]) is more general, and shows this independently of perturbation theory.

For each external particle, there appears a factor

$$\frac{Z}{p^2 - m^2_{phys} + i\varepsilon} \tag{5.12}$$

– the denominator is exactly canceled with the $(\Box + m^2_{phys})$ of the LSZ formula, and the wave-function renormalization factor Z partially cancels with the $1/\sqrt{Z}$ of (5.11).

In fact, this structure is universal, and one can calculate the full propagator (5.12) (two-point Green function) first, and use it to determine the Z factor and the position of the pole.

Note, also, that the equation (5.11) substitutes for the momenta of particles their physical values (for example, all $p^+ > 0$).

5.2.4 Perturbative expansion of the tau functions and the S matrix

One can assume that the operators $a_{\vec{k}}(x^+)$ and some complete set of free operators $a_{0\vec{k}}(x^+)$ are (unitarily) equivalent, that is, that there exists an operator $U(x^+)$ such that

$$a_{\vec{k}}(x^+) = U^{-1}(x^+) a_{0\vec{k}}(x^+) U(x^+) \tag{5.13}$$

and the same for $a^\dagger$. Note that this is consistent with the fact that, for a given time x^+, operators $a_{\vec{k}}(x^+)$ and $a_{0\vec{k}}(x^+)$ fulfill the same commutation relations. Also, they carry the same quantum numbers (for the scalar theory the only quantum number of a particle represented by $a^\dagger_{\vec{k}}$ is its three-momentum).

In fact, rather than looking for the U transformation it is easier to look for a product:

$$U(x^+,x'^+) := U(x^+)U^{-1}(x'^+)\,. \tag{5.14}$$

(cf. G.50). This leads to the following expression for the Green function of the LSZ equation:

$$\widetilde{\tau} = \langle 0|\, T_{(+)}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \\ = \langle 0|\, T_{(+)}\left[\phi_0(x_1)\ldots\phi_0(x_n)\exp\left(-i\int_{x'^+}^{x^+} H_I(\xi^+)\cdot\frac{1}{2}\mathrm{d}\xi^+\right)\right]|0\rangle\,. \tag{5.15}$$

(cf. Eq.(G.69)), where

$$H_I(x^+) \;:=\; H_0(a_0) - H(a_0)\,, \tag{5.16}$$

and H_0 involves physical eigenvalue m:

$$H_0(a_0) \;:=\; \int [k]k_m^- a^\dagger_{0\vec{k}} a_{0\vec{k}}\,. \tag{5.17}$$

Equation (5.15) is a light-front analog of the equation (17.22) in the Bjorken and Drell textbook [96]. It is interesting to notice that its derivation (see Appendix G) remains valid for a regularized Hamiltonian which is not an integral of a local product of fields.

5.2.5 Feynman diagrams with Hamiltonian regularization

The standard derivation of the Feynman diagrams uses Wick theorem at this point to reduce the matrix elements (5.15) to products of the two-point Green functions (i.e., the propagators) and interaction vertices. The Feynman propagators emerging this way have two poles at $k^0 = \pm\left(\sqrt{\vec{k}^2+m^2} - i\varepsilon\right)$.

In our case, however, the situation is slightly different. In the light-front Hamiltonian all particles have positive p^+ momenta, and for each value of p^+ there is only one pole in p^-. Nevertheless, the structure of the expressions is very similar to the casual Feynman propagator obtained formally using the equal-time form.

For some quantities, limiting of p^+ to positive values does not matter at all. For example, when one calculates a propagators for the incoming or outgoing particles as a part of an S-matrix calculation, the physical values of three-momenta are substituted through the LSZ formula, and only the physical pole of the propagator contributes (this is true both for the equal-time and light-front calculations). Thus, the τ-function presented here can agree with one defined by the standard Feynman diagrams.

The situation is slightly different for internal lines. For example, diagrams (5.1) in the Hamiltonian formalism presented here are two separate expressions (especially if there is regularization, which does depend on an ordering of the interaction terms). However, the relation of such diagrams to the corresponding (not regulated) Feynman diagram is simpler in light-front coordinates, because the form of the Feynman propagator appears for each of these lines separately (rather than as a result of summing two diagrams that occur in the equal time case). One thus obtains Feynman propagators in a standard form, but with momenta limited by Hamiltonian regularization factors in vertices.

The old-fashioned rules in scalar ϕ^3 theory were considered by Weinberg in 1960s [95]. See also an article by Bardakci and Halpern [18]. An entire study for scalar, Yukawa and

vector-gluon theories in light-front formalism was considered by Yan and collaborators in 1970s [20-23]. Yan and his collaborators developed a theory in which the Hamiltonians were not regularized, but the formal expression for the S matrix could be manipulated and eventually shown to be the same as in the Feynman approach. The main difficulty Yan and collaborators were dealing with was that the standard time-ordering of interactions is different than light-front time x^+-ordering, and the Coulomb term in equal-time Hamiltonian is very much different from an analogous Coulomb-like term in light-front Hamiltonians. But Yan and collaborators have shown that the different orderings and different interactions combine formally to the same scattering amplitude. Examples of recent discussions that follow up on these works can be found in [105, 106]. As far as I know, nobody has ever considered a complete calculation with fully regulated Hamiltonians, and nowhere in the literature I have found evaluation of Hamiltonians for effective particles, like $\mathcal{H}_\lambda$, except from references [40, 74] which however did not consider scattering amplitude. Here, I discuss what happens when the Hamiltonian is regularized before developing perturbation theory for the S matrix. It is necessary to follow this procedure if one wants the Hamiltonians to produce well-defined eigenvalue equations for bound states. In the formation of bound states, the perturbative S-matrix calculus of asymptotically free theories is not directly applicable.

The procedure that I effectively employ in the one-loop calculation described here consists of the following steps:

1. Renormalized Hamiltonians are used in perturbative expression for the Fourier transform of the τ-function (5.15).
2. Theta functions of x^+-ordering are replaced by integrals (I.4) with $i\varepsilon$ in denominators.
3. Integrals over x^+ lead to $\delta(\Delta k^-)$ in vertices.
4. Every denominator introduced in (2) is multiplied by corresponding k^+, and the product is the same as in a standard Feynman propagator for each ordering.
5. Self-interaction loops on external lines lead to a geometric series, which may be summed up. This leads to propagators with poles at physical masses, and with modified residues Z.
6. In the case of external lines, the resulting physical propagator is exactly canceled by the factors $\left(\Box + m^2\right)$ in the LSZ equation [cf. (5.11)]. What is left is the factor $\sqrt{Z}$.

In the case of scalar theories in $n+2$ dimensions[6] (such as (5.77)), these rules obtained using (5.11) may be summarized as follows:

- For each internal line there is an integral

$$\int \frac{\mathrm{d}^n k^\perp \mathrm{d}k^+ \mathrm{d}k^- \theta(k^+)}{2(2\pi)^2} \frac{i}{k^2 - m^2 + i\varepsilon} , \tag{5.18}$$

  where m is the physical mass. The factor $\theta(k^+)$ means, in particular, that I am considering each of the orderings in (5.1) separately.

- For each external line ending at a point x^μ

$$\int \frac{\mathrm{d}^n k^\perp \mathrm{d}k^+ \mathrm{d}k^-}{2(2\pi)^2} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik_\mu x^\mu} ; \tag{5.19}$$

[6] n is the number of transverse dimensions.

- For each interaction vertex

$$(-ig)2(2\pi)^{(n+2)}\delta(k^-_{cre}-k^-_{ann})\delta(k^+_{cre}-k^+_{ann})\delta^{(n)}(k^\perp_{cre}-k^\perp_{ann})\boxed{r_\Delta}\,, \tag{5.20}$$

  where the regularization factors r_Δ (coming from the regularized Hamiltonian) depend only on perpendicular and plus momenta. These regularization factors are key to the following calculation and they distinguish the Hamiltonian diagrams from the Feynman diagrams. They are the same for all loops, because they originate from one interaction term in the Hamiltonian operator.

The necessary combinatorial factors can be figured out by inspecting the way creation and annihilation operators are commuted. Also, depending on the vertex considered, the coupling constant g in (5.20) is replaced by e, e_q or one half of these coupling constants (cf. (5.77)).

In the case of vertices other than the regularized bare vertex coming from H^Δ (such as in (5.77)), for example for the terms coming from the counterterms or terms in the effective Hamiltonian, g in (5.20) is replaced by a corresponding factor form the Hamiltonian term with appropriate $(2\pi)\delta(k^-_{cre}-k^-_{ani})$.

In the the regulated Feynman diagrams derived here, the regulators do not depend on k^- momentum. This in turn allows us to reduce them to an old-fashioned perturbation theory for the S matrix by replacing the integral over k^- by the sum over residues at poles in k^-.

Finally, one should note that a similar correspondence between the Hamiltonian expressions and the Feynman diagrams does not seem possible if one uses an equal-time Hamiltonian. When regularization is needed, one could introduce a cutoff either on each particle momentum independently (which leads to a theory that cannot be boosted to an arbitrary frame) or on momentum differences in the vertex (which means different regularization in each of the two diagrams in Figure 3.2b and thus with such regularization these diagrams do not add up to a Feynman-propagator form). Compared to this, the situation with a light-front Hamiltonian is much simpler.

5.3 Calculation of S-matrix elements using light-front Hamiltonians for effective particles

5.3.1 S matrix in terms of H_λ

RGPEP equations (3.76) define effective-particle creation operators as operators unitarily equivalent to the bare-particle creation operators:

$$a_\infty = U_\lambda^\dagger a_\lambda U_\lambda\,. \tag{5.21}$$

The effective Hamiltonians are the same Hamiltonian operator expressed in terms of effective-particle operators, instead of the bare-particle operators:

$$\mathcal{H}_\lambda(a_\lambda) = H_\infty(a_\infty) = U_\lambda^\dagger H_\infty(a_\lambda)U_\lambda. \tag{5.22}$$

The basis of the scattering theory is another unitary equivalence: namely, the equivalence of bare creation operators a_∞ and the physical a_{in} operators (5.13):

$$a_{\vec{k},\infty}(x^+) = U^{-1}(x^+)a_{0\vec{k}}(x^+)U(x^+)\,, \tag{5.23}$$

where the index $\lambda = \infty$ was added to indicate the bare-particle annihilation operator. Substituting into this equation expression (5.21), one has:

$$a_{\vec{k},\lambda}(x^+) = W_\lambda^{-1}(x^+) a_{0\vec{k}}(x^+) W_\lambda(x^+) \,, \tag{5.24}$$

with[7]

$$W_\lambda(x^+) = U(x^+) U_\lambda^\dagger \,. \tag{5.25}$$

Equation (5.24) for operators a_λ means, that the physical external one-particle state (or a matrix element of a corresponding field) can be represented not only using a one-bare-particle state (as in Appendix G.3), but also as a one-effective-particle state. However, the proportionality coefficient of the spectral representation changes if one changes the set of representing operators from the bare ones ($a_\infty^\dagger$) to the effective ones ($a_\lambda^\dagger$). Thus, not only is the perturbative expansion changed (see below), but also the LSZ formula must be modified accordingly.

It is now possible to repeat the steps shown in Appendix G.3, using $W_\lambda(x^+)$ instead of $U(x^+)$: differentiate Eq. (5.24) over x^+; introduce $W(x^+, x'^+) := W(x^+) W^{-1}(x'^+)$; and so on. This leads to exactly the same expression for the S matrix, but with an interaction Hamiltonian:

$$\mathcal{H}_{\lambda\,,I} = \mathcal{H}_\lambda(a_0) - H_0(a_0) \,. \tag{5.26}$$

The details of this calculation are presented in Appendix H. The results can be summarized as the following theorem:

Theorem: The same S matrix describing scattering of physical particles can be obtained using:

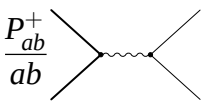
1. A bare Hamiltonian H^Δ, and representing the incoming/outgoing particles by bare-particle creation and annihilation operators a_∞
2. Or using an effective Hamiltonian H_λ and effective particles a_λ.

In each order of perturbation theory, the result for the S matrix is the same, provided the connection between a_∞ and a_λ (H^Δ and $\mathcal{H}_\lambda$) is fulfilled up to this order.

In both cases, the perturbation is done in powers of a difference between the full Hamiltonian $H^\Delta = \mathcal{H}_\lambda$ (with creation operators, $a_\infty^\dagger$ or $a_\lambda^\dagger$ respectively, replaced by free operators a_0) and a free Hamiltonian H_0 with physical spectrum.

5.3.2 Example: tree amplitude for $e^+e^- \to$ *hadrons*

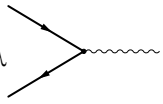
The bare Hamiltonian presented below in Section 5.4.1 leads to $e^+e^- \to$ *hadrons* S-matrix of order ee_q:

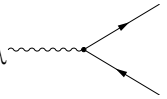


(5.27)

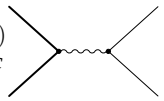
[7] Note that expression (5.21) can be written at any x^+, which means that $U_\lambda^\dagger$ also depends on x^+ (it is a fixed function of creation and annihilation operators, but these depend on the time x^+). The dependence of $U_\lambda^\dagger$ on x^+ is not marked explicitly here to make it easier for the reader to distinguish the $U(x^+)$ (scattering theory) from the $U_\lambda(x^+)$ (RGPEP).

I review below briefly how the same result emerges when one uses the effective Hamiltonian $\mathcal{H}_\lambda$. $\mathcal{H}_\lambda$ has three terms that may contribute: the order-e vertices with form factors

$\mathcal{H}_{\lambda\,\succ\!-} \;=\; f_\lambda$  , (5.28)

$\mathcal{H}_{\lambda\,-\!\prec} \;=\; f_\lambda$  , (5.29)

and an e^2 term

$\mathcal{H}_{\lambda\,\succ\!\prec} = f_{ac}\mathcal{F}^{(2)}_{abc}$  . (5.30)

When $\mathcal{H}_{\lambda\,\succ\!\prec}$ contributes to the scattering matrix, due to energy conservation $f_{ac}\equiv 1$ and $\mathcal{F}^{(2)}_{abc}$ simplifies to:

$$\mathcal{F}^{(2)}_{aba} = \frac{P^+_{ba}}{ba}(f^2_{ba}-1)\,, \tag{5.31}$$

(cf. (3.118)). There are thus two contributions: one from $\mathcal{H}_{\lambda\,Y}$ acting twice:

$$\mathcal{H}_{\lambda,\succ}\frac{1}{E_0-H_0+i\varepsilon}\mathcal{H}_{\lambda,\prec} = f_\lambda H_\succ \frac{1}{E_0-H_0+i\varepsilon} f_\lambda H_\prec\,, \tag{5.32}$$

and the other from $\mathcal{H}_{\lambda\,\succ\!\prec}$:

$$\mathcal{H}_{\lambda\,\succ\!\prec}\big|_{ac=0} = (1-f^2_{ab})\frac{1}{E_0-H_0}H_\succ H_\prec \tag{5.33}$$

The f^2_λ term cancels the $\mathcal{H}_{\lambda\,Y}\mathcal{H}_{\lambda\,Y}$, and the rest reproduces the S matrix obtained from the bare Hamiltonian, (5.27).

The above example also shows that the analytic structure of the amplitude is not changed: if one goes close to the $i\varepsilon$-regularized pole, ab goes to zero (and $f_{ab}\approx 1$). Thus the whole contribution to the pole comes from (5.32), and the result for the residue in the pole is the same as that calculated using the bare Hamiltonian (5.27).

5.3.3 Consequences of the theorem

I now turn to the two questions posed in the introduction to this chapter: (1) do counterterms found through the RGPEP procedure lead to a divergence-free S matrix; and (2) is the S matrix calculated using the effective Hamiltonian $\mathcal{H}_\lambda$ independent of λ?

RGPEP fixes the counterterms in the initial bare Hamiltonian H^Δ by requiring that coefficients of $\mathcal{H}_\lambda$ be independent of Δ. This is not the same as requiring that the S matrix be independent of Δ, hence the question (1).

The theorem states, that both $\mathcal{H}_\lambda$ and H^Δ produce the same S-matrix elements. Because of form factors in interaction terms of $\mathcal{H}_\lambda$, when one calculates scattering amplitude using this Hamiltonian the results do not depend on Δ in corresponding order of perturbation theory when $\Delta\to\infty$. These two statements, taken together, prove that the scattering amplitude can be obtained using the bare renormalized Hamiltonian H^Δ, and that the RGPEP counterterms in H^Δ lead to result which is not divergent. This is not a trivial result, as the counterterms

were found based on different conditions, and because the ultraviolet cutoff dependence of the S matrix comes from different expressions. For example, the factor Z_Δ, appearing both in the LSZ formula and in the full propagator can be a divergent function of Δ. Based on the theorem, however, one would expect that Δ-dependence will vanish. It is interesting to see how this happens in practice, at least at low orders – which parts of the counterterms remove which parts of the divergences. I investigate this in Sections 5.6 and 5.7 for a scalar theory in 5+1 dimensions in one loop.

The theorem also predicts that, when using $\mathcal{H}_\lambda$ to calculate the S matrix, the result will be independent of λ. The argument here is as follows: if one calculates a scattering amplitude using H^Δ, the result does not depend on λ since there is no such parameter in H^Δ. However, the same result can be obtained using $\mathcal{H}_\lambda$. This means that the effective Hamiltonian $\mathcal{H}_\lambda$ leads to λ-independent results for the scattering matrix in a given order of perturbation theory in appropriately defined coupling constant g_λ, whose dependence on λ is calculated to the same order. This, again, is not obvious without the theorem. For example, wave-function renormalization factors for effective particles Z_λ depend on λ, and there are many terms in the effective Hamiltonian that do not appear in the bare Hamiltonian.

The example in Section 5.3.2 demonstrates in the lowest order that the scattering matrix calculated using $\mathcal{H}_\lambda$ is indeed independent of λ. This particular calculation was simple, because energy conservation meant that, in this order, the factor f_{ac} was equal to 1. In higher orders, however, corresponding form factor f_λ is no longer equal to 1. The structure of f_λ is related to the renormalization group flow. In particular, the particle mass used to defined *ac* invariant mass difference has to be the same as that used in the denominators of RGPEP (Eq. (3.81)), i.e., the free mass from H_0. If it is not the same, f_λ would not fulfill one of its main tasks: to prevent the appearance of small energy denominators in the perturbative calculation of the renormalization group flow of $\mathcal{H}_\lambda$ (cf. Eq. (3.116)). At the same time, in the denominators of the S matrix, there are differences of energies that are defined using physical particle mass. The real, physical energy of *in* and *out* states (defined using physical masses of particles) is conserved, and so the energy defined using free masses is not conserved: *ac* of f_{ac} is no longer zero, and f_{ac} is not equal to 1. Calculating the S matrix using effective Hamiltonians will thus not be as simple in higher orders as it is in the low-order case presented in Section 5.3.2.

5.4 Simplest canonical example: ϕ^3 in 1+1 dimensions

Perturbative ϕ^3 theory is the simplest example of QFT that one can usefully analyze. No ϕ^3 theory exists beyond perturbation theory due to the fact that the ground state collapses. However, before confronting more difficult theories, such as QCD, it is helpful to examine some questions using ϕ^3 theory. The ϕ^3 theories in 1+1 and 5+1 dimensions presented below can thus help us understand the perturbative structure of the S matrix, while ignoring the nonperturbative problems of these theories will not lead to inconsistencies within the purely perturbative analysis.

ϕ^3 theory in 1+1 dimensions does not lead to any ultraviolet divergences in perturbation theory, so no ultraviolet cutoff is required. The scattering amplitude can be calculated using the canonical Hamiltonian without any modifications. Thus, in 1+1 dimension the Hamiltonian calculation presented here is exactly the same as standard, finite and covariant Feynman diagrams. Massive theory considered here is also infrared finite.

1+1 dimensional theory allows me to outline the calculation for the scattering amplitude,

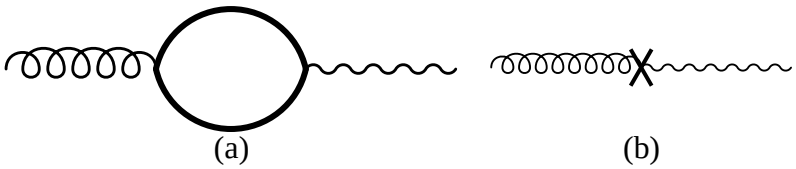


Figure 5.3: In the scalar theory (5.34), scalar gluons and scalar photons mix in perturbation theory through terms such as (a). Moreover, in 3+1 or more dimensions, such terms are divergent and require counterterms such as (b). Such terms are not found in QCD coupled to QED because of color conservation, and do not need to be discussed in this thesis.

even before the complexities of regularization and renormalization are introduced. This provides explicit examples of the general properties mentioned in previous sections. An example is the connection between, on the one hand, the physical mass and normalization of the Hamiltonian eigenstate and, on the other hand, the position of the pole and value of the residue of a full propagator. The 1+1 dimensional model also allows me to show all the steps that will be later subject to alteration when the regularized and renormalized Hamiltonian vertices enter in 5+1 dimensions. In this way the description of the divergent 5+1 case (Sec.5.6 and 5.7) can focus on how counterterms fit in this general picture.

5.4.1 Hamiltonian

As a starting point I take the Lagrangian density:

$$\mathcal{L} = \sum_{j\in\{e,q,g\}} \frac{1}{2}\left(\partial_\mu\phi_j\partial^\mu\phi_j - m^2\phi_j^2\right) + \frac{1}{2}\left(\partial_\mu\phi_\gamma\partial^\mu\phi_\gamma - m_\gamma^2\phi_\gamma^2\right) - \frac{e}{2}\phi_e^2\phi_\gamma - \frac{e_q}{2}\phi_q^2\phi_\gamma - \frac{g}{2}\phi_q^2\phi_g \quad (5.34)$$

Theory defined this way describes four types of real scalar fields, mimicking an interaction of electrons, quarks, photons and gluons. I will refer to these scalar fields by the names of the particles they mimic (i.e."scalar electrons", "scalar quarks" etc.); the subscripts in Eq. (5.34) correspond to these names. In 1+1 dimensional theory, g has dimension of $mass^2$ (this changes in higher dimensions – see Appendix A.5).

In the Lagrangian above, the mass m (common to scalar electrons, quarks and gluons), and the scalar photon mass m_γ are different from zero. Having three particles of the same mass m simplifies the calculation considerably, without affecting issues of finiteness and covariance of the S matrix. Note, also, that even though the same mass is chosen for scalar electrons, quarks and gluons, their physical masses will differ because of the different interactions they are subject to. The photon mass is chosen so that it is different from the gluon mass, in order to avoid irrelevant complication of degenerate perturbation theory for physical masses (cf. Fig.5.3).

The Lagrangian contains three interaction terms:

- Electrons ϕ_e interact with photons ϕ_γ; the coupling constant of this interaction is e,
- Quarks ϕ_q interact with photons ϕ_γ with coupling constant e_q,
- Quarks ϕ_q interact with gluons ϕ_g with coupling constant g.

In each interaction there is one boson-like field (photon or gluon) and two fermion-like fields (electrons or quarks). In the real world, for low-energy processes, the QED coupling constant

analogous to e is much smaller than the QCD coupling constant analogous to g, while electrons and quarks have electric charges of similar value. Therefore I choose for this model

$$e \simeq e_q \ll g \,. \tag{5.35}$$

A canonical Hamiltonian derived from the Lagrangian (5.34), according to the rules described in Section 3.3.3, is:

$$H = H_0 + H_e + H_{e_q} + + H_g \,. \tag{5.36}$$

Here, H_0 denotes the free Hamiltonian (kinetic energy):

$$H_0 \;=\; \int [k] \frac{m^2}{k^+} b^\dagger_{\vec{k},e} b_{\vec{k},e} + \int [k] \frac{m^2}{k^+} q^\dagger_{\vec{k},q} q_{\vec{k},q} + \int [k] \frac{m_\gamma^2}{k^+} c^\dagger_{\vec{k},e} c_{\vec{k},e} + \int [k] \frac{m^2}{k^+} a^\dagger_{\vec{k},e} a_{\vec{k},e} \,, \tag{5.37}$$

where $[k] = dk^+ \theta(k^+)/(4\pi k^+)$. Despite the different notation, all the creation operators ($a^\dagger$ for gluons, $b^\dagger$ for electrons, $c^\dagger$ for photons, and $q^\dagger$ for photons) are bosonic operators in this model. In 1+1 dimensions there are no perpendicular directions and the free energies [e.g. Eq. (3.48)] simplify to mass squared over k^+. Note that k^+ ranges from 0 to $+\infty$ only.

All the interaction parts have two creation operators and one annihilation operator, or their hermitian conjugation. They can thus be split into two types: those mimicking interactions that involve creation or annihilation of a fermion-anti-fermion pair:

$$H_{Ys} = \int [123] \widetilde{\delta}(1+2-3) \left[\frac{e}{2} \left(b_1^\dagger b_2^\dagger c_3 + c_3^\dagger b_1 b_2 \right) + \frac{e_q}{2} \left(q_1^\dagger q_2^\dagger c_3 + c_3^\dagger q_1 q_2 \right) + \right.$$
$$\left. + \frac{g}{2} \left(q_1^\dagger q_2^\dagger a_3 + a_3^\dagger q_1 q_2 \right) \right] , \tag{5.38}$$

and those mimicking an emission or absorption of bosons by a fermion:

$$H_{Yt} = \int [123] \widetilde{\delta}(1+2-3) \left[e \left(b_1^\dagger c_2^\dagger b_3 + b_3^\dagger b_1 c_2 \right) + e_q \left(q_1^\dagger c_2^\dagger q_3 + q_3^\dagger q_1 c_2 \right) + \right.$$
$$\left. + g \left(q_1^\dagger a_2^\dagger q_3 + q_3^\dagger q_1 a_2 \right) \right] . \tag{5.39}$$

The factor of one-half by which the terms H_{Ys} and H_{Yt} differ is a standard difference.

5.4.2 Properties of the state of one physical particle

Using the above Hamiltonian in perturbation theory, one can calculate the energy and structure (i.e., Fock-space components) of a state of one physical particle. Although the eigenvalues of H_0 for scalar quarks, gluons and electrons are the same, one can use non-degenerate perturbation theory, as the interactions do not mix corresponding one-particle degenerate states. For example, a matrix of any power n of the interaction Hamiltonian vanishes between one-quark and one-electron states:

$$\langle e | \left(H_I \right)^n | q \rangle \;=\; 0 \,. \tag{5.40}$$

Since we are dealing with perturbation theory only, we can consider quarks as physical particles in the femto-universe [94], and look for the energy of one physical quark of momentum k^+ and the Fock-space structure of this state. A similar calculation can also be done for other

particles. The general structure of the physical state of one quark expanded in perturbation theory, beginning the perturbative expansion with the state $q^\dagger|0\rangle$, is as follows:

$$\begin{aligned}\left|q,k^+\right\rangle_{phys} &= N_k\Big(\left|q,k^+\right\rangle^{(0)}_{phys}+\left|q,k^+\right\rangle^{(g)}_{phys}+\left|q,k^+\right\rangle^{(e)}_{phys}+\\ &\quad+\left|q,k^+\right\rangle^{(g^2)}_{phys}+\left|q,k^+\right\rangle^{(ge)}_{phys}+\left|q,k^+\right\rangle^{(e^2)}_{phys}+\ldots\Big) \qquad (5.41)\end{aligned}$$

where superscripts indicate the order of the calculation and N_k is the normalization factor. The results for successive orders are:

$$\left|q,k^+\right\rangle^{(0)}_{phys} = q^\dagger_{k^+}|0\rangle \qquad (5.42)$$

$$\left|q,k^+\right\rangle^{(1)}_{phys} = \frac{1}{E^{(0)}-H_0}H_{>-}q^\dagger_{k^+}|0\rangle = \{H_{>-}\}_0\,q^\dagger_{k^+}|0\rangle \qquad (5.43)$$

$$\left|q,k^+\right\rangle^{(2)}_{phys} = N_k\left(\left\{H_{>-}\{H_{>-}\}_0\right\}_0 q^\dagger_{k^+}|0\rangle\right)\,, \qquad (5.44)$$

where superscript (1) refers to e or g, and (2) to e^2, eg or g^2. The normalization condition:

$${}_{phys}\!\left\langle q,k^+\mid q,p^+\right\rangle_{phys} = {}^{(0)}\!\left\langle q,k^+\mid q,p^+\right\rangle^{(0)} \qquad (5.45)$$

reduces to:

$$|N_k|^2\left(1+\delta_N(k^+)\right) = 1 \qquad (5.46)$$

$$\delta_N(k^+) := \frac{1}{\langle k\mid p\rangle}\langle k|H^{(g)}_{-<}\left(\frac{1}{E^{(0)}-H_0}\right)^2 H^{(g)}_{>-}|p\rangle\,. \qquad (5.47)$$

Here, on the assumption that $e_q \ll g$, I have kept only the most important terms – those of order g^2.

$$\delta_N(k^+) = \frac{g^2}{4\pi m^4}\int_0^1\frac{\mathrm{d}x}{x(1-x)}\frac{1}{\left[1-\frac{1}{x(1-x)}\right]^2} = \frac{g^2}{18\sqrt{3}\pi m^4}\left(3\sqrt{3}-\pi\right) \qquad (5.48)$$

$$N = \frac{1}{\sqrt{1+\delta_N}} \approx 1-\frac{1}{2}\delta_N\,. \qquad (5.49)$$

The physical energy (i.e., the eigenvalue of the full Hamiltonian) is also expanded in a power series:

$$E_{q,k^+} = E^{(0)}_{q,k^+}+E^{(g)}_{q,k^+}+E^{(e)}_{q,k^+}+E^{(g^2)}_{q,k^+}+E^{(ge)}_{q,k^+}+{}^{(e^2)}_{q,k^+}+\ldots \qquad (5.50)$$

Up to order g^2 this reads:

$$\frac{m^2_{phys}}{k^+} := E^{(0)}_{q,k^+}+E^{(1)}_{q,k^+}+E^{(2)}_{q,k^+} = \frac{m^2}{k^+}+0+\frac{1}{\langle k\mid k\rangle}\langle k|H^g_{-<}\frac{1}{E^{(0)}_k-H_0}H^g_{>-}|k\rangle \qquad (5.51)$$

$$E^{(2)}_{q,k^+} = \frac{1}{k^+}\left[\frac{-g^2}{4\pi m^2}\int_0^1\mathrm{d}x\frac{1}{x^2-x+1}\right] = \frac{1}{k^+}\left[\frac{-g^2}{6\sqrt{3}m^2}\right] \qquad (5.52)$$

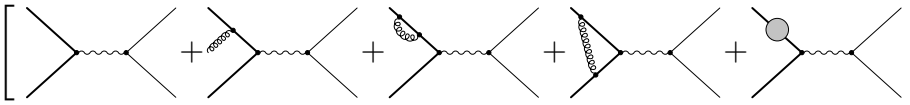
$$m^2_{phys} = m^2-\frac{g^2}{6\sqrt{3}m^2}\,. \qquad (5.53)$$

Note that this equation applies only for small g, and one thus avoids the problem that m^2_{phys} is less than zero for a sufficiently large g.

5.5 Scalar model of the process $e^+e^- \to hadrons$

I now proceed to the calculation of a scattering amplitude that is analogous to $e^+e^- \to hadrons$. I use the equation (5.11) and the perturbative expansion of the τ function appearing in this expression.

Up to the order e^2g^2, the scattering amplitude has the following terms:

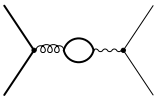
$S_{\beta\alpha} = \frac{1}{(\sqrt{Z_q})^2}$  . (5.54)

The last of these terms comes from:

$$H_{\delta m} = \int [k] q_k^\dagger q_k \frac{m_0^2 - m_{phys}^2}{k^+} \tag{5.55}$$

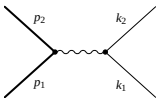
which is an interaction term due to the fact that H_I in (5.16) is a difference of H^Δ and an H_0 with the physical mass (rather than the initial mass parameter m_0). To simplify notation, henceforth in this chapter I denote the physical mass as m, and use a symbol m_0 to refer to the initial mass parameter in (5.37).

In this model amplitude for the process $e^+e^- \to hadrons$ I do not include the diagram:

 (5.56)

because there are no similar terms in the case of QCD coupled to QED. This term is connected to mixing of the scalar gluons and scalar photons in this model. Because of the structure of the color SU(3) factors there are no terms of this type in QCD coupled to QED.

Tree diagram of order e^2

The tree diagram  in momentum space is:

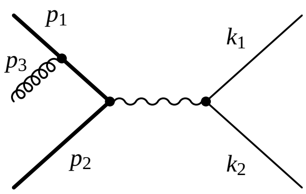
$$\widetilde{\tau}^{(e_q e)}_{ee\to qq}(p_1,p_2,k_1,k_2) = 4\cdot\left(\frac{-i}{2}\right)^2 e_q e 2(2\pi)^2\delta^2(p_1^\mu + p_2^\mu - k_1^\mu - k_2^\mu)\frac{i}{p_1^2 - m^2 + i\varepsilon}\frac{i}{p_2^2 - m^2 + i\varepsilon}\times$$
$$\times \frac{i}{k_1^2 - m^2 + i\varepsilon}\frac{i}{k_2^2 - m^2 + i\varepsilon}\frac{i}{(k_1+k_2)^2 - m_\gamma^2 + i\varepsilon}\theta(p_1^+)\theta(p_2^+)\theta(k_1^+)\theta(k_2^+)\,. \tag{5.57}$$

Using equation (5.11), one obtains the matrix element:

$${}_{out}\langle p_1,p_2 \mid k_1,k_2\rangle_{in}^{(e_q e)} = -ie_q e\cdot\ 2(2\pi)^2\delta^2(p_1^\mu + p_2^\mu - k_1^\mu - k_2^\mu)\frac{1}{(k_1+k_2)^2 - m_\gamma^2 + i\varepsilon} \tag{5.58}$$

The θ factors automatically equal 1 when the physical momenta of the initial and final particles are substituted for p_1, p_2, k_1 and k_2, (5.11). The μ superscripts are $+$ or $-$ in this expression.

Tree diagram of oder e^2g



The S-matrix element obtained in an analogous way in order e^2g is:

$$\begin{aligned} {}_{out}\langle p_1, p_2, p_3 \mid k_1, k_2\rangle_{in}^{(e_q eg)} &= 4\cdot\ (-i)^3 \frac{e_q}{2}\frac{e}{2} g\, 2(2\pi)^2 \delta^2(p_1^\mu + p_2^\mu + p_3^\mu - k_1^\mu - k_2^\mu) \times \\ &\times \left(\frac{i}{(P-p_1)^2 - m^2 + i\varepsilon} + \frac{i}{(P-p_2)^2 - m^2 + i\varepsilon} \right) \times \\ &\times \frac{i}{P^2 - m_\gamma^2 + i\varepsilon} \end{aligned} \tag{5.59}$$

For the calculation of the total cross-section, it is convenient to introduce Feynman x parameters, defined as

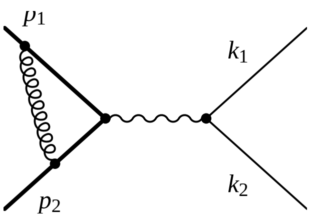
$$x_i^F \ := \ \frac{2P^\mu p_{i\mu}}{s}\ . \tag{5.60}$$

In this way:

$$\begin{aligned} {}_{out}\langle p_1, p_2, p_3 \mid k_1, k_2\rangle_{in}^{(e_q eg)} &= \frac{e_q eg}{s}\, 2(2\pi)^2 \delta^2(p_1^\mu + p_2^\mu + p_3^\mu - k_1^\mu - k_2^\mu) \times \\ &\times \left(\frac{1}{(1-x_1) + i\varepsilon} + \frac{1}{(1-x_2) + i\varepsilon} \right) \frac{1}{s - m_\gamma^2 + i\varepsilon} \end{aligned} \tag{5.61}$$

The same expression emerges from the old-fashioned light-front Hamiltonian perturbation theory (i.e., with the k^- integrated over residues).

Triangle e^2g^2 diagrams



The part of the scattering amplitude corresponding to this diagram is[8]:

$$\begin{aligned} &{}_{out}\langle p_1, p_2, p_3 \mid k_1, k_2\rangle_{in, \triangleright\!-}^{(e_q eg^2)} = 4\cdot(-i)^4 \frac{e}{2}\frac{e_q}{2} g^2\, 2(2\pi)^2 \delta^2(k_1^\mu + k_2^\mu - p_1^\mu - p_2^\mu) \frac{i}{(k_1+k_2)^2 - m_\gamma^2 + i\varepsilon} \times \\ &\times \int \frac{\mathrm{d}k^+ \mathrm{d}k^-}{2(2\pi)^2} \frac{i}{k^2 - m^2 + i\varepsilon} \left[\theta(k^+)_{\triangleright\!-} + \theta(k^+)_{\triangleright\!-}\right] \frac{i}{(k+p_2)^2 - m^2 + i\varepsilon} \frac{i}{(p_1-k)^2 - m^2 + i\varepsilon} = \\ &= e e_q g^2\, 2(2\pi)^2 \delta^2(k_1 + k_2 - p_1 - p_2) \frac{1}{s - m_\gamma^2 + i\varepsilon} (-i) A(s). \end{aligned} \tag{5.62}$$

$$A(s) = i \cdot \int \frac{\mathrm{d}k^+ \mathrm{d}k^-}{2(2\pi)^2} \frac{1}{k^2 - m^2 + i\varepsilon} \frac{1}{(p_1-k)^2 - m^2 + i\varepsilon} \frac{1}{(k-p_2)^2 - m^2 + i\varepsilon} \tag{5.63}$$

[8] In this expression, $\theta(k^+)_{\triangleright\!-}$ and $\theta(k^+)_{\triangleright\!-}$ denote all θ-factors corresponding to a given ordering.

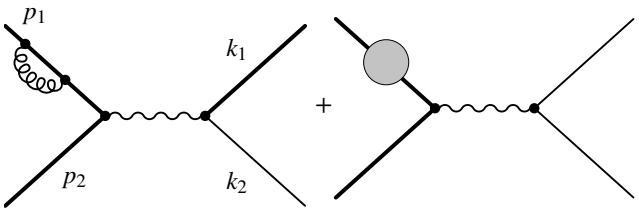
The function $A(s)$ can be calculated by introducing Feynman parameters or by first replacing integral over k^- by a sum of residues. The result is:

$$Im\,(A(s)) = -\frac{1}{4m^4}\,\frac{\delta}{\sqrt{1-4\delta}}\,\frac{1-2\delta}{1-3\delta} \tag{5.64}$$

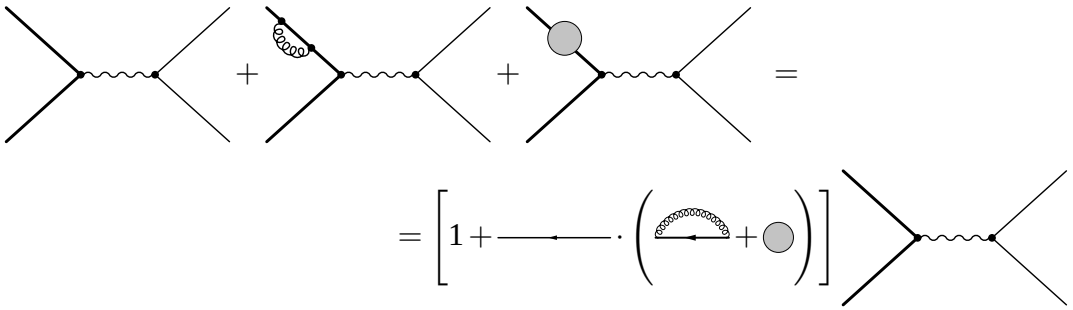
$$Re(A(s)) = \frac{1}{4\pi m^4}\,\frac{\delta}{1-3\delta}\left[\frac{2\pi}{3\sqrt{3}}+\frac{1-2\delta}{\sqrt{1-4\delta}}\log\left(\frac{\left(1+\sqrt{1-4\delta}\right)^2}{4\delta}\right)\right], \tag{5.65}$$

where $\delta := m^2/s < 1/4$.

Diagrams with loops on external lines

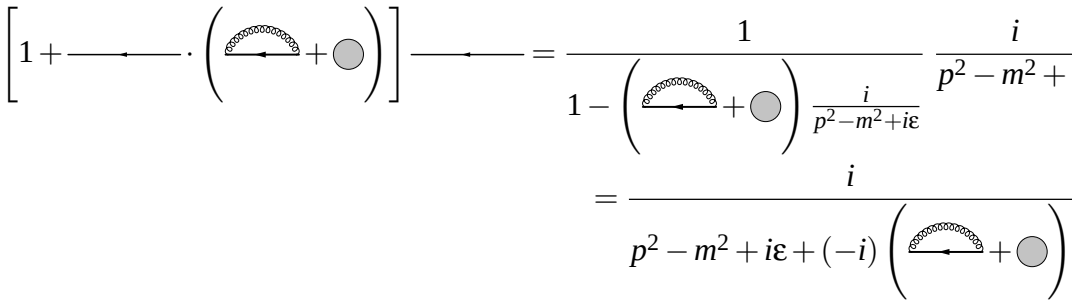


When one adds these diagrams to the tree diagram, the latter can be factorized:



$$= \left[1+\quad\cdot\left(\quad+\right)\right]\quad . \tag{5.66}$$

The sum in the square brackets can be treated as the two first terms of a geometric series. Summing such a series and multiplying the result by the propagator of the external line gives:



$$= \frac{1}{1-\left(\quad+\right)\frac{i}{p^2-m^2+i\varepsilon}}\,\frac{i}{p^2-m^2+i\varepsilon} = \frac{i}{p^2-m^2+i\varepsilon+(-i)\left(\quad+\right)} \tag{5.67}$$

$$f := (-i)\left(\quad+\right)\ . \tag{5.68}$$

Note that, again, the functions $\theta(k^+)$ for each of the momenta do not prevent factorization. This is because only one physical value of $k^+ > 0$ appears in this expression when it is inserted into the LSZ formula (5.11).

The result is a scattering amplitude with a modified ("physical") propagator (5.67) on the external lines. Loops on other external lines can be summed out in the same way. The propagator (5.67) is analyzed in Section 5.5.1.

5.5.1 Relationship between quark eigenstates and properties of the propagator

The loops on external lines (5.67) sum up to exactly what one would get if one simply considered a two-point Green's function $\langle 0|\, T_{(+)}\phi(x)\phi(x')\,|0\rangle$, independently of the full S-matrix calculation. The detailed calculation of this two-point Green's function for a light-front ϕ^3 Hamiltonian is presented in Appendix I.1. Below, I analyze the result of this calculation.

Position of the pole

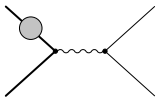
The full propagator is:

$$\tau^{(0+g^2)}(x_1,x_2) = \int \frac{\mathrm{d}^2p}{(2\pi)^2} e^{-ip(x_1-x_2)} \frac{i}{p^2-m^2+i\varepsilon+f(p^2)} \tag{5.69}$$

$$f(p^2) := \frac{g^2}{4\pi m^2}\frac{\delta}{\sqrt{\delta-1}}\arctan\frac{1}{\sqrt{\delta-1}} + \left(m^2-m_0^2\right) \tag{5.70}$$

where $\delta := 4m^2/p^2$ is assumed to be in the region $\delta \in (1,\infty)$. The value of function f at m^2 is zero. Therefore the pole of this propagator remains unchanged, namely the pole is in p^2 equal to the physical scalar quark mass, m^2.

There is another way of writing the perturbation series. Namely, one could use H_0 with the initial mass m_0^2 for denoting the leading term in the evolution of asymptotic fields, and the physical mass emerges when corrections are included. The only difference will be, that the mass in denominator of Eq. (5.69) will be the initial mass m_0, and there would be no terms such as:



(5.71)

in Eq. (5.54). Accordingly, $f(p^2)$ would not have the term $\left(m^2-m_0^2\right)$. Still the pole of the propagator will be at:

$$p^2_{pole} = m_0^2 - f(m_0^2) = m_0^2 - \frac{g^2}{m_0^2}\frac{1}{6\sqrt{3}} = m^2\,, \tag{5.72}$$

which is exactly equal to the physical mass calculated from the Hamiltonian eigenvalue equation (5.53). Since this second way of writing the perturbation series is shorter (due to absence of terms like (5.71)), this is the way how the expressions will be written in the 5+1 dimensional case in Section 5.7.

Residue

In the vicinity of the pole, function f can be expanded into the Taylor series:

$$f(s) = f(m^2) + \left(\frac{\mathrm{d}f}{\mathrm{d}s}(m^2)\right)(s-m^2) + \ldots \tag{5.73}$$

From this, the denominator of the full propagator (5.69) can be approximated near its zero by

$$p^2-m^2+i\varepsilon+f(p^2) \overset{p^2\to m^2}{\approx} \left(p^2-m^2\right)\left[1+\left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{m^2}\right]. \tag{5.74}$$

The derivative can be calculated using the explicit form of the function $f(p^2)$, Eq. (5.70). Thus, in the vicinity of the pole $p^2 \to m^2$,

$$\tilde{\tau}^{(0+g^2)}(p) = \frac{Z}{p^2-m^2}\,, \tag{5.75}$$

with

$$\sqrt{Z} = 1 - \frac{g^2}{36\sqrt{3}\pi m^4}\left(3\sqrt{3}-\pi\right)\,, \tag{5.76}$$

which agrees with the normalization factor of the physical state (5.49), as expected in (5.8).

5.6 One-loop calculation of a renormalized Hamiltonian for effective particles in asymptotically free theory: ϕ^3 in 5+1 dimensions

5.6.1 Canonical Hamiltonian

I consider here a ϕ^3 QFT in 5+1 space-time dimensions defined by a Hamiltonian:

$$H = H_0 + H_e^\Delta + H_{e_q}^\Delta + H_g^\Delta + X^\Delta \tag{5.77}$$

$$H_0 = \int [k] \frac{k^{\perp 2}+m^2}{k^+} b^\dagger_{\vec{k},e} b_{\vec{k},e} + \int [k] \frac{k^{\perp 2}+m^2}{k^+} q^\dagger_{\vec{k},q} q_{\vec{k},q} + \tag{5.78}$$

$$+ \int [k] \frac{k^{\perp 2}+m_\gamma^2}{k^+} c^\dagger_{\vec{k},e} c_{\vec{k},e} + \int [k] \frac{k^{\perp 2}+m^2}{k^+} a^\dagger_{\vec{k},e} a_{\vec{k},e}\,, \tag{5.79}$$

$$H_e^\Delta = \frac{e}{2}\int [123]\widetilde{\delta}(1+2-3)\boxed{r_\Delta}\left(b_1^\dagger b_2^\dagger c_3 + c_3^\dagger b_1 b_2 + 2b_1^\dagger c_2^\dagger b_3 + 2b_3^\dagger b_1 c_2\right) \tag{5.80}$$

$$H_{e_q}^\Delta = \frac{e_q}{2}\int [123]\widetilde{\delta}(1+2-3)\boxed{r_\Delta}\left(q_1^\dagger q_2^\dagger c_3 + c_3^\dagger q_1 q_2 + 2q_1^\dagger c_2^\dagger q_3 + 2q_3^\dagger q_1 c_2\right) \tag{5.81}$$

$$H_g^\Delta = \frac{g}{2}\int [123]\widetilde{\delta}(1+2-3)\boxed{r_\Delta}\left(q_1^\dagger q_2^\dagger a_3 + a_3^\dagger q_1 q_2 + 2q_1^\dagger a_2^\dagger q_3 + 2q_3^\dagger q_1 a_2\right)\,, \tag{5.82}$$

where $[k] = \mathrm{d}^4k^\perp \mathrm{d}k^+/2(2\pi)^5 k^+$. As in the 1+1 dimensional model, H_e couples scalar analogs of electrons to scalar photons, H_{e_q} couples scalar quarks to scalar photons, and H_g couples scalar quarks to scalar gluons. This is the simplest theory with asymptotic freedom and the structure of diagrams that resembles realistic cases of QCD coupled to QED.

In comparison to the 1+1-dimensional case [cf. (5.36)], we have here integrations over four additional perpendicular directions $k^\perp$. This leads to ultraviolet-divergent integrals and requires regularization. The form of the regularization factors r_Δ is quite arbitrary. I choose a simple regularization, depending on perpendicular momenta only, namely a factor:

$$\boxed{r_\Delta(x,\kappa)} = \exp(-2\kappa^2/\Delta^2) \tag{5.83}$$

in each vertex (5.80)-(5.82), creating or annihilating a pair of particles of relative perpendicular momentum $\kappa^\perp$. The same regularization was used earlier for the bound-state problem discussed in Chapter 4.

I present here two different definitions of the effective Hamiltonians, and thus two ways to perform the renormalization. **The first approach**, described in Section 5.6.2, is based on

a **simplified RGPEP** transformation. I construct the effective scalar-quark basis based on the strong-interaction Hamiltonian H_g only, and express the full Hamiltonian H^Δ in terms of this basis. This approach defines the effective scalar quarks on the basis of their strong interactions alone, neglecting dressing due to electromagnetic interactions; it does not introduce effective scalar photons or scalar electrons. This is a very natural approach, especially for the description of the scattering of bound states: to describe the bound states of quarks itself one naturally looks first at their strong interactions. These interactions define the degrees of freedom natural for a strong bound-states description; the electromagnetic interactions are less important.

The $e^+e^- \to hadrons$ scattering amplitude up to order e^2g^2 is unusual, in that all the divergences that appear in it are due to strong corrections to electromagnetic interactions. The counterterms from the simplified RGPEP, based on the strong interactions only, are thus sufficient to make the amplitude finite up to the order e^2g^2. This would not be true if the electromagnetic interactions of quarks (e.g., in the order e^4) were included. Furthermore, the calculation is considerably simpler, for example, it is enough to calculate the U_λ transformation of RGPEP in the second order (i.e., g^2). The drawback of this simplified approach is that the resulting Hamiltonian is applicable to only a limited set of processes. Details of the formulae for this approach are given in Appendix C.3.

In **the second approach**, described in section 5.6.3, the **RGPEP is based on the entire Hamiltonian**. It defines counterterms and the effective Hamiltonian applicable to all processes at a given order of perturbation theory. To calculate all the counterterms needed for a description of the $e^+e^- \to hadrons$ scattering amplitude to order e^2g^2, it is now necessary to apply RGPEP to the third order, rather than just the second. Although this approach is more complicated, it turns out that it leads to exactly the same counterterms contributing to the $e^+e^- \to hadrons$ amplitude in the order e^2g^2. Other counterterms would only be needed if one wanted to perform higher-order calculations (e.g., the e^4 correction to this amplitude) or to describe processes other than $e^+e^- \to hadrons$; such counterterms could only be found using the full RGPEP.

Both approaches are presented below up to orders that allow one to calculate counterterms of the order g^2 and eg^2, since this is sufficient for calculating the $e^+e^- \to hadrons$ amplitude up to the order e^2g^2. It should be stressed that I calculate here counterterm operators in a Hamiltonian, which in general are not the same as counterterms for a specific term in an S matrix (cf. Sec. 5.3.2 showing a simple example of the difference between an S-matrix element and a full off-shell operator).

5.6.2 RGPEP based on strong interactions only, up to order g^2

Interactions of scalar quarks and scalar gluons

In the simplified approach, RGPEP is based on the strong-interaction Hamiltonian. Thus, the resulting strong part of the effective Hamiltonian (i.e., order g^n) is similar in its structure to the effective Hamiltonian for Yukawa theory, as described in Chapter 4. Below, I list the corresponding terms (see Sections 4.6 and 3.4 for details).

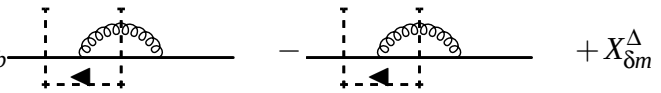
The zeroth order Hamiltonian for quarks and gluons does not change:

$$\mathcal{H}_{\lambda\,0,q,g} = H_{0,q,g}\,. \tag{5.84}$$

In the first order, (g), the effective Hamiltonian $\mathcal{H}_\lambda^{(g)}$ is simply the sum of all strong vertices (Eq. (5.82)) with similarity form factors f_λ. 

In the second order, (g^2), there are two kinds of terms that contribute to the scattering amplitude in question: The first kind is an effective "potential" term, that is, the term with two creation operators and two annihilation operators,  (not divergent and not requiring any counterterms). The second kind is an effective scalar-quark mass term . This term is quadratically divergent in Δ:

$$\mathcal{H}_{\lambda,\delta m} = \int [1] b_1^\dagger b_1 \frac{\delta m_\lambda^2}{p_1^+} =$$

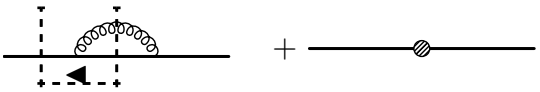


$$= f_{ab}^2 \quad - \quad + X_{\delta m}^\Delta \tag{5.85}$$

$$\delta m_\lambda^2 = \frac{g^2}{2(2\pi)^5 P^+} \int \frac{\mathrm{d}x \mathrm{d}^4\kappa}{x(1-x)} \frac{P_{ab}^+}{ba} (f_{ba}^2 - 1) \boxed{r_\Delta^2} + \delta m_\Delta^2 + \delta m_{phys}^2 \,, \tag{5.86}$$

where the last two terms come from the counterterm. The rule for constructing counterterms is that they should remove regularization dependence from the effective Hamiltonian. The part of the integral (5.86) with the form factor f_λ is not sensitive to the cutoff r_Δ, but the part without f_λ is quadratically divergent. The counterterm required in the canonical Hamiltonian H^Δ is therefore:

$$X_{\delta m}^\Delta = \int [1] b_1^\dagger b_1 \frac{\delta m_\Delta^2 + \delta m_{phys}^2}{p_1^+} =$$



$$= \quad + \tag{5.87}$$

$$\delta m_\Delta^2 = \frac{g^2}{2(2\pi)^5 P^+} \int_0^1 \frac{\mathrm{d}x}{x(1-x)} \int_a^\infty \mathrm{d}^4\kappa \frac{P_{ab}^+}{ba} (+1) \boxed{r_\Delta^2} = \tag{5.88}$$

$$= \frac{g^2}{(4\pi)^3} \left[\frac{\Delta^2}{4} - \frac{5}{6} m^2 \ln \frac{\Delta^2}{m^2} + \text{finite terms} \right] . \tag{5.89}$$

The finite part of the counterterm, δm_{phys}^2, has the following physical interpretation: the energy of one physical scalar quark in perturbation theory using H^Δ is:

$$E^{(2)} = \frac{p^{\perp 2} + m^2}{p^+} + \frac{1}{\langle p \mid p \rangle} \langle p | \quad + \quad | p \rangle = \tag{5.90}$$

$$= \frac{p^{\perp 2} + m^2 + \delta m_{phys}^2}{p^+} = \frac{p^{\perp 2} + m_{phys}^2}{p^+} \,. \tag{5.91}$$

Hence, the finite part of the counterterm δm_{phys}^2 shifts the scalar-quark mass from the initial value m^2 to the physical value:

$$\delta m_{phys}^2 = m_{phys}^2 - m^2 \,. \tag{5.92}$$

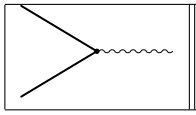
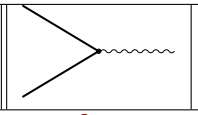
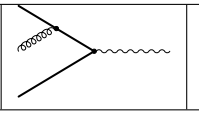
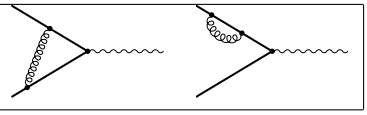
The same value of the physical mass is obtained if one calculates it using the effective Hamiltonian $\mathcal{H}_\lambda$: the lambda dependent part of the effective mass term (the part of Eq.(5.85) with f_{ab}^2) cancels with the effective vertices $f_\lambda H_Y$, and the result for the physical one-particle Hamiltonian eigenvalue is again (5.91).

Interactions of scalar electrons and scalar photons

The effective Hamiltonian $\mathcal{H}_\lambda$ is calculated by expressing bare-particle creation and annihilation operators in terms of effective-particle creation and annihilation operators. However, in the simplified RGPEP considered here, only the scalar-quark and scalar-gluon degrees of freedom change. Thus, the interaction of the scalar photons that do not involve scalar quarks, and the interactions of the scalar electrons in the canonical Hamiltonian, are the same in the simplified effective Hamiltonian.

As a consequence, the H_e^Δ interactions of scalar electrons are not renormalized. The theory defined in this simplified way is thus sufficient to describe quantities such as the scattering amplitude similar to $e^+e^- \to hadrons$ up to the order e^2g^2, but not sufficient for other purposes.

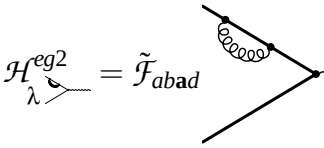
Interactions of scalar quarks with scalar photons

	effective $\mathcal{H}_\lambda$		
H_{can}	no change	$u_1 H_{can}$	$u_{HH} H_{can}$
			

The above terms[9] emerge in the effective Hamiltonian in the following way: the effective Hamiltonian expressed in terms of the effective particles is simply equal to the canonical Hamiltonian expressed in terms of the bare particles. The forms of both Hamiltonians differ, because the same operator expressed in terms of two different operator bases has two different forms.

Having derived the bare creation operators expressed in terms of the effective ones (see Appendix C.3), one can simply insert this expression into all terms of the canonical Hamiltonian to get the effective Hamiltonian. The table above shows all the terms that appear in the effective Hamiltonian when the bare scalar quarks are re-expressed in terms of the effective scalar quarks and gluons.

One might also expect terms corresponding to the following diagram:  (coming from the mass counterterm contribution to $u_{H2}^{(g^2)} H_e$), because they would be of the order $g^2 e$, which is included in $\mathcal{H}_\lambda$. However, because of the structure of the RGPEP transformation U_λ, there are no such terms in $\mathcal{H}_\lambda$. Both terms marked $u_{HH} H_{can}$ in the above table are divergent, and they require counterterms.

$$\mathcal{H}_\lambda^{eg2} \; \raisebox{-0.5em}{  } = \tilde{\mathcal{F}}_{ab\mathbf{a}d} \; \raisebox{-0.5em}{} = g^2 \frac{e_q}{2} \int [123] a_1^\dagger a_2^\dagger a_3 \widetilde{\delta}(1+2-3) \boxed{r_\Delta} \times$$

$$\times \frac{1}{p_1^+} \int [45] \widetilde{\delta}(4+5-1) \left(-\frac{1}{2}\right) \frac{(p_1^+)^2}{\left(M_{45}^2 - m^2\right)^2} (1-f_{ba})^2 \boxed{r_\Delta^2} \,. \quad (5.93)$$

There is also another term, with $(1 \leftrightarrow 2)$, leading to exactly the same contribution. The tilde over $\tilde{\mathcal{F}}$ indicates that it refers to the simplified RGPEP transformation ($\mathcal{F}$ without a tilde will be used to refer to the full RGPEP in Sec. 5.6.3). In this term, $\boxed{r_\Delta^2} = e^{-d_\eta \kappa^2/\Delta^2}$ ($d_\eta = 4$ for regulators (5.83), but this is arbitrary; finite d_η-dependence should also in principle be removed

[9] A similar table of terms for QCD coupled to QED is given in the Appendix F.1.2.

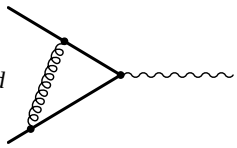
from the result). The regularization-dependent part of this expression has the following form:

$$\mathcal{H}^{eg2}_{\lambda\ +\ ,\Delta} = \int [123] a_1^\dagger a_2^\dagger a_3 \boxed{r_\Delta} \widetilde{\delta}(1+2-3) \left(\gamma_{\ ,\Delta} + \gamma_{\ ,\Delta} \right) \tag{5.94}$$

$$\gamma_{\ ,\Delta} + \gamma_{\ ,\Delta} = -\frac{e_q g^2}{2(4\pi)^3} \frac{1}{6} \left[\ln \frac{\Delta^2}{m^2} - 6 \int_0^1 \mathrm{d}x\, x(1-x) \ln d_\eta + const. \right] \tag{5.95}$$

This requires a counterterm X, of exactly the same form and opposite sign. This is an example of a term in $\mathcal{H}_\lambda$, which in the simplified RGPEP does not feature the overall formfactor f_λ in electromagnetic vertices. This term contains a regularization factor $\boxed{r_\Delta}$ that cannot be replaced by 1 (i.e., by its limit for $\Delta \to \infty$ when λ is finite).

The second divergent part is the triangle term:

$$\mathcal{H}^{eg2}_{\lambda\ } = \tilde{\mathcal{F}}_{abcd}$$  (5.96)

The $\tilde{\mathcal{F}}$ function is given explicitly in Eq. (C.72). Its divergent part is:

$$\mathcal{H}^{eg2}_{\lambda\ ,\Delta} = \int [123] q_1^\dagger q_2^+ q_3 \widetilde{\delta}(1+2-3) \gamma_{\ ,\Delta} \tag{5.97}$$

$$\gamma_{\ ,\Delta} = \frac{e_q g^2}{(4\pi)^3 x_2} \int_{x_1}^1 \mathrm{d}x (1-x) \int_0^\infty \mathrm{d}z \frac{z}{(z+m^2)^2} \boxed{r_\Delta} \tag{5.98}$$

$\boxed{r_\Delta}$ denotes here the product of three regulators corresponding to the three Hamiltonian vertices of the triangle term (5.96), in the effective Hamiltonian. For the choice of regulators (5.83) $\boxed{r_\Delta}$ in this expression is:

$$\boxed{r_\Delta} = \exp\left(-c_\eta \frac{z}{\Delta^2} \right) , \tag{5.99}$$

with

$$c_\eta = \left(2 + \frac{x_1^2}{x^2} \right) . \tag{5.100}$$

Although the specific value of c_η changes the result only in a finite way, c_η is arbitrary and all c_η-dependent parts of $\mathcal{H}_\lambda$ should be removed by counterterms. After adding contributions of both orderings of the scalar-gluon emission and absorption vertices, $\ $ and $\ $, one obtains:

$$\gamma_{\left(\ + \ \right),\Delta} = \frac{g^2 e_q}{2(4\pi)^3} \left\{ \ln \frac{\Delta^2}{m^2} - 2 \left[\frac{1}{x_2} \int_{x_1}^1 \mathrm{d}x (1-x) \ln c_\eta(x_1) + \right. \right.$$
$$\left. \left. + \frac{1}{x_1} \int_{x_2}^1 \mathrm{d}x (1-x) \ln c_\eta(x_2) \right] + const. \right\} . \tag{5.101}$$

A counterterm $X_{\ }$ of opposite sign is required in H^Δ.

5.6.3 RGPEP based on entire Hamiltonian, up to orders $g^2 e$ and $g e^2$

Section 5.6.2 described the construction of the effective Hamiltonian based on the introduction of effective scalar quarks as required by the strong coupling Hamiltonian.

In a fully renormalized theory, however, it is necessary to define the effective interactions in such a way that they all contain form factors (cf. Sec.3.4). This requires the introduction of effective particles of each type – not only quarks and gluons, but also effective photons and electrons. Moreover, when constructing effective quarks one has to take into account both their strong and their electromagnetic interactions. Below, I show only the terms needed for the comparison it with the simplified approach above, and in order to calculate the $e^+e^- \to hadrons$ scattering amplitude up to the order e^2g^2.

In this full approach, both H^Δ and U_λ now involve all the interactions, and thus the effective Hamiltonian has more terms coming from more sources. For example, a term[10]  comes from: (1) H^Δ of order g, and $u^{(eg)}$; (2) H^Δ of order g, $u^{(g)}$ and $u^{\dagger(e)}$; (3) H^Δ of order 0 (the free part) and $u^{(g^2e)}$; and so on. This can be compared to the previous case, where  came from only one source: $H^{(e)}$ and $u^{(g^2)}$.

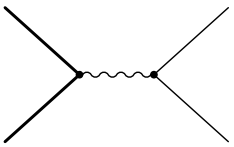
All the terms of the effective Hamiltonian given below are derived from the general formulae of the Appendix C.2.

In the zeroth order one gets the sum of free Hamiltonians.

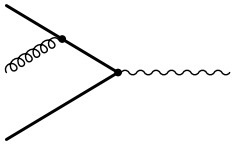
In the first order one gets all the H_Y vertices with form factors f_λ.

In the second order there are three terms which are not divergent:

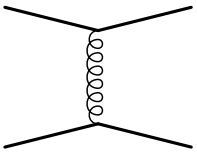
$$\mathcal{H}^{(e^2)}_\lambda = f_{ac}\mathcal{F}^{(2)}_{abc}\cdot H^{(e)}H^{(e)} = f_{ac}\mathcal{F}^{(2)}_{abc}\cdot \tag{5.102}$$



$$\mathcal{H}^{(eg)}_\lambda = f_{ac}\mathcal{F}^{(2)}_{abc}\cdot H^{(g)}H^{(e)} = f_{ac}\mathcal{F}^{(2)}_{abc}\cdot \tag{5.103}$$



$$\mathcal{H}^{(g2)}_{\lambda,V} = f_{ac}\mathcal{F}^{(2)}_{abc}\cdot H^{(g)}H^{(g)}\big|_V = f_{ac}\mathcal{F}^{(2)}_{abc}\cdot \tag{5.104}$$

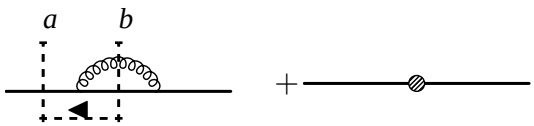


and one term that is divergent, namely the term which shifts the effective mass of the scalar-quark:

$$\mathcal{H}^{g2}_{\lambda,\delta m} = \mathcal{F}^{(2)}_{aba}\cdot \tag{5.105}$$

(I do not list here divergent terms which do not contribute to the g^2e^2 S matrix, for example terms changing effective mass of scalar electrons). The scalar-quark mass shift $\mathcal{H}^{g2}_{\lambda,\delta m}$ is exactly the same as the term described in the previous section (cf. Eq. (5.85)). This means that a counterterm is needed, which has the form:

$$X^\Delta_{\delta m} =$$

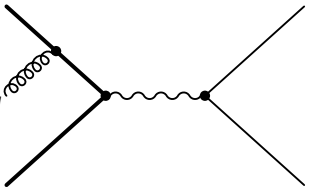


$$. \tag{5.106}$$

[10] "A term  " means a term in $\mathcal{H}_\lambda \sim q^\dagger q^\dagger a$, coming from two interactions of order g and one of order e, and with momentum configuration corresponding to this form of the diagram, enforced by corresponding δ-functions.

In the third order (i.e., terms with $\mathcal{F}^{(3)}$), there is one term that does not diverge. It is of order e^2g:

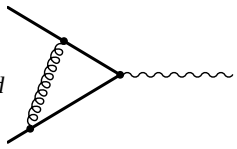
$$\mathcal{H}^{e2g}_{\lambda\,\nearrow\!\!\searrow} \;=\; f_{ad}\,\mathcal{F}^{(3)}_{abcd}$$



. (5.107)

There are also two divergent terms of order g^2e – the triangle term:

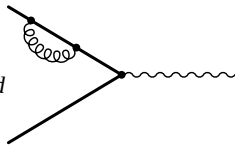
$$\mathcal{H}^{eg2}_{\lambda\,\nearrow} \;=\; f_{ad}\,\mathcal{F}^{(3)}_{abcd}$$



, (5.108)

and the scalar-quark self-interaction term:

$$\mathcal{H}^{eg2}_{\lambda\,\nearrow} \;=\; f_{ad}\,\mathcal{F}^{(3)}_{ab\mathbf{a}d}$$

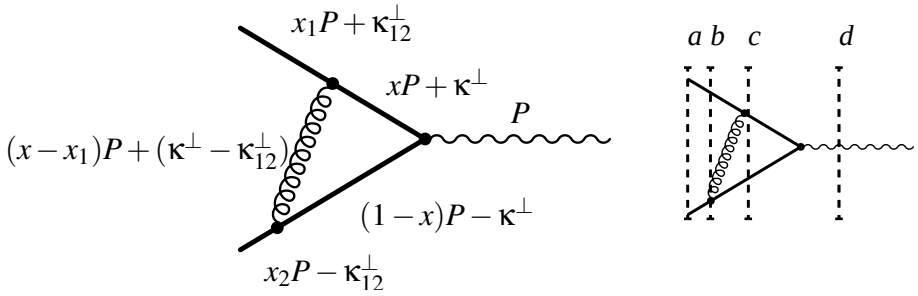


. (5.109)

In the first of these, $\mathcal{H}^{eg2}_{\lambda\,\nearrow}$, the divergence comes from the parts of $\mathcal{F}^{(3)}_{abcd}$ (C.59)-(C.60) that do not have any f_λ factors. These are:

$$\begin{aligned}
\mathcal{F}^{(3)}_{\nearrow+\nearrow,\Delta} \;=\;& \Pi_{bcd}\Big\{\pi_{abd}\left[\frac{-1}{ab^2+bc^2+cd^2+bd^2}+\frac{1}{ab^2+bd^2}\right]+\\
&+\zeta_{db}\left(bc^2+cd^2\right)\left[\frac{-1}{ab^2+bc^2+cd^2}+\frac{1}{ab^2+bc^2+cd^2+bd^2}\right]\Big\}+\\
&+\Pi_{abc}\Big\{\pi_{acd}\left[\frac{-1}{ab^2+bc^2+cd^2+ac^2}+\frac{1}{ac^2+cd^2}\right]+\\
&+\zeta_{ac}\left(ba^2+bc^2\right)\left[\frac{-1}{ab^2+bc^2+cd^2}+\frac{1}{ab^2+bc^2+cd^2+ca^2}\right]\Big\}\,,
\end{aligned} \tag{5.110}$$

where the subscript Δ in $\mathcal{F}^{(3)}_{\nearrow+\nearrow,\Delta}$ indicates that only the divergent part is listed. Definitions of the momenta combinations: Π, π and ζ, are given in Eqs. (C.49)-(C.51). I denote the momenta in the triangle term as shown in the figures:



(5.111)

In Eq. (5.110), the following combinations of momenta appear:

$$\mathcal{M}^2 := \frac{\kappa^2+m^2}{x(1-x)} = \mathcal{M}_c^2 \tag{5.112}$$

$$ba = x_2^2\frac{\left(\kappa-\frac{1-x}{x_2}\kappa_{12}\right)^2+m^2}{(x-x_1)(1-x)} - m^2 \to \frac{x_2^2 x}{x-x_1}\mathcal{M}^2 \tag{5.113}$$

$$cd = \mathcal{M}^2 - m_\gamma^2 \to \mathcal{M}^2 \tag{5.114}$$

$$ca = \mathcal{M}^2 - \mathcal{M}_{12}^2 \to \mathcal{M}^2 \tag{5.115}$$

$$bc \to \frac{x(1-x)x_1}{(x-x_1)}\mathcal{M}^2 \tag{5.116}$$

$$bd \to \frac{xx_2}{x-x_1}\mathcal{M}^2 \tag{5.117}$$

$$ad = \mathcal{M}_{12}^2 - m_\gamma^2 \text{ (fixed)}, \tag{5.118}$$

where the limits in each case mean the ultraviolet limit of the intermediate momenta, $\mathcal{M}^2 \gg m^2, s, \mathcal{M}_{12}^2$. Substituting these limiting values to $\mathcal{F}^{(3)}_{\triangleright\!\!-+\triangleright\!\!-}$, Eq. (5.110), leads to:

$$\mathcal{F}^{(3)}_{\triangleright\!\!-,\Delta,\mathcal{M}^2\gg m^2,s} = \frac{x-x_1}{xx_2}\frac{(P^+)^2}{\mathcal{M}^4}\,. \tag{5.119}$$

Using this limit, the divergent part of $\mathcal{G}^{(3)}$ corresponding to the triangle diagrams can be written as:

$$\mathcal{G}^{(3)}_{\triangleright\!\!-,\Delta} = \int [123]\widetilde{\delta}(1+2-x)q_1^\dagger q_2^\dagger a_3 \gamma_{\triangleright\!\!-,\Delta} \tag{5.120}$$

$$\gamma_{\triangleright\!\!-,\Delta} = g^2 e_q \frac{1}{2(2\pi)^5 P^+}\int \frac{\mathrm{d}x\mathrm{d}^4\kappa}{x(1-x)(x-x_1)P^+}\boxed{r_\Delta}\mathcal{F}^{(3)}_{\triangleright\!\!-,\Delta} =$$

$$= \frac{g^2 e_q}{2(2\pi)^5}\frac{\Omega_4}{2}\frac{1}{x_2}\int_{x_1}^1 \mathrm{d}x(1-x)\int_a^\infty \frac{\mathrm{d}z}{z}\boxed{\exp\left(-c_\eta\frac{z}{\Delta^2}\right)}\,. \tag{5.121}$$

Above expressions, and in particular definitions of momenta (5.111), referred to only one term,  . In fact, there is another term with different ordering of the interaction vertices,  . Adding the effective Hamiltonian terms coming from both orderings leads to:

$$\gamma_{\left(\triangleright\!\!-+\triangleright\!\!-\right),\Delta} = \frac{g^2 e_q}{2(4\pi)^3}\left\{\ln\frac{\Delta^2}{m^2} - 2\left[\frac{1}{x_2}\int_{x_1}^1 \mathrm{d}x(1-x)\ln c_\eta(x_1) + \right.\right.$$

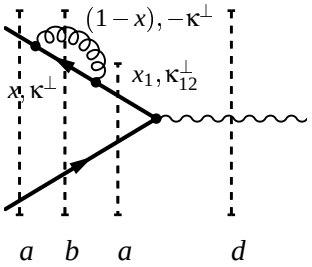
$$\left.\left. + \frac{1}{x_1}\int_{x_2}^1 \mathrm{d}x(1-x)\ln c_\eta(x_2)\right] + const.\right\} \tag{5.122}$$

where *const*. denotes a constant independent of regularization. The choice of regularization (5.83) means that

$$c_\eta(x_1) = \left(2+\frac{x_1^2}{x^2}\right) \tag{5.123}$$

but this choice is arbitrary, and properly chosen counterterms should also remove the finite dependence on regularization (see also [40]).

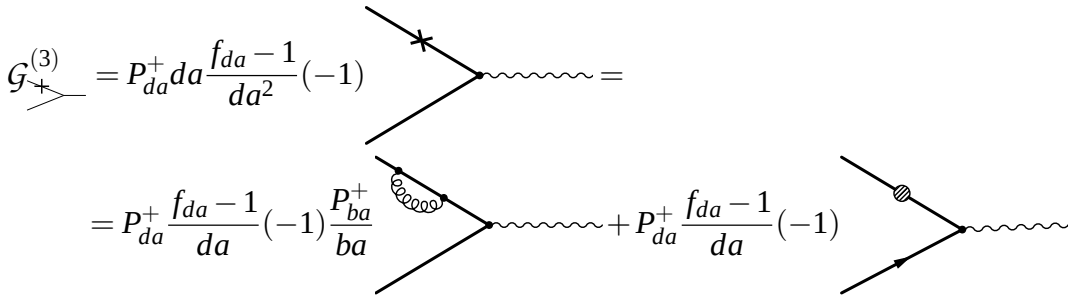
The second divergent third-order contribution corresponds to the following diagram:



(note the repeated index a). The divergent part of the corresponding $\mathcal{F}^{(3)}_{abad}$ is:

$$\begin{aligned}\mathcal{F}^{(3)}_{abad,\Delta} &= \Pi_{bad}\left\{\pi_{abd}\left[\frac{-1}{2ab^2+ad^2+bd^2}+\frac{1}{ab^2+bd^2}\right]+\right.\\ &\left.\zeta_{db}\left(ba^2+ad^2\right)\left[\frac{-1}{2ab^2+ad^2}+\frac{1}{2ab^2+ad^2+bd^2}\right]\right\}+\\ &+\underline{\zeta_{ba}P^+_{da}da\frac{-1}{da^2}}+\zeta_{ba}P^+_{da}da\frac{1}{2ab^2+ad^2}\end{aligned} \tag{5.124}$$

The underlined term could lead to quadratic divergence, but there is another contribution to $\mathcal{G}_\lambda$ coming from the mass counterterm, 



(5.125)

The first part of this expression cancels exactly the underlined part in Eq. (5.124). The second part does not depend on Δ. Substituting the ultraviolet limits:

$$\mathcal{M}^2 := \frac{\kappa^2+m^2}{x(1-x)} = \mathcal{M}^2_{b,loop} \tag{5.126}$$

$$ba = \frac{\kappa^2+m^2}{x(1-x)} - m^2 \to \mathcal{M}^2 \tag{5.127}$$

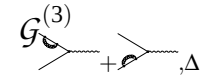
$$ad = \frac{\kappa^2_{12}+m^2}{x_1x_2} - m^2_\gamma \quad \text{(fixed)} \tag{5.128}$$

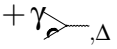
$$bd = \frac{ba}{x_1} + ad \to \frac{\mathcal{M}^2}{x_1} \tag{5.129}$$

to all the remaining parts of Eq. (5.124) leads to the following form of the part of $\mathcal{F}^{(3)}$  which leads to the divergence:

$$\mathcal{F}^{(3)}_{\,,\Delta} = -\frac{x_1^2(P^+)^2}{2\mathcal{M}^4}\,, \tag{5.130}$$

which in turn leads to a logarithmic divergence in the effective Hamiltonian term. As a result, the divergent part of this term in $\mathcal{H}_\lambda$ takes the following form:

$\mathcal{G}^{(3)}$  $= \int [123]\widetilde{\delta}(1+2-x)q_1^\dagger q_2^\dagger a_3 \boxed{r_\Delta}\gamma$ $+\gamma$ (5.131)

γ  $+\gamma$  $= -\frac{g^2 e_q}{2(2\pi)^5}\frac{\Omega_4}{2}\cdot\frac{1}{2}\int_0^1 \mathrm{d}x\, x(1-x)\int_a^\infty \frac{\mathrm{d}z}{z}\exp\left(-d_\eta \frac{z}{\Delta^2}\right) =$ (5.132)

$= -\frac{g^2 e_q}{2(4\pi)^3}\left[\frac{1}{6}\ln\frac{\Delta^2}{m^2} - \int_0^1 \mathrm{d}x\, x(1-x)\ln d_\eta + const.\right]$ (5.133)

Again, *const*. denotes the parts that do not depend on the regularization. In the case of the regulators considered here, $d_\eta \equiv 4$, but since regularization is arbitrary, d_η could be different – it introduces a finite dependence on regularization and must thus be removed by counterterms.

5.6.4 Summary of counterterms

The complete renormalized canonical Hamiltonian is given in Eq. (5.77) with a mass counterterm of order g^2 and vertex counterterms of order $g^2 e_q$. The mass counterterm is:

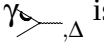
$$X_{\delta m} = \int [p] b_p^\dagger b_p \frac{\delta m_\Delta^2}{p^+}\,, \tag{5.134}$$

where δm_Δ^2 is given in Eq. (5.89). The vertex counterterms are:

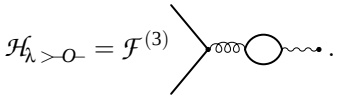
$X^{(g^2 e)}$  $= X^{(g^2 e)}$ $+ X^{(g^2 e)}$  (5.135)

$X^{(g^2 e)}$  $= \int [123]\widetilde{\delta}(1+2-3)q_1^\dagger q_2^\dagger a_3 \boxed{r_\Delta}\left(-\gamma\right.$  $\left.+(1\leftrightarrow 2)+\text{finite parts}\right)$ (5.136)

$X^{(g^2 e)}$  $= \int [123]\widetilde{\delta}(1+2-3)q_1^\dagger q_2^\dagger a_3 \boxed{r_\Delta}\left(-\gamma\right.$  $\left.+(1\leftrightarrow 2)+\text{finite parts}\right)$, (5.137)

where γ  is given in Eq. (5.133) and γ is given in Eq. (5.122).[11]

[11] In order $g^2 e$ there are also logarithmic counterterms corresponding to a term:

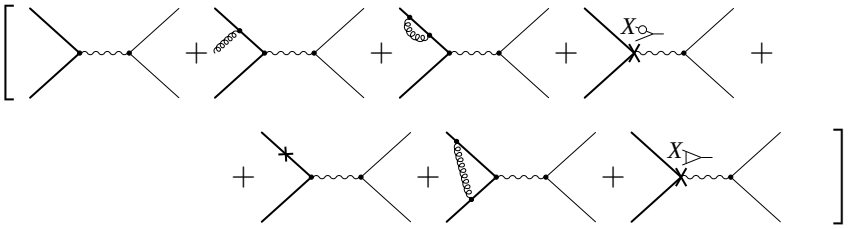
$\mathcal{H}_\lambda$  .

This is not given here, as the corresponding terms in the S matrix are not analyzed in Section 5.7.

5.7 Scalar model of the process $e^+e^- \to hadrons$

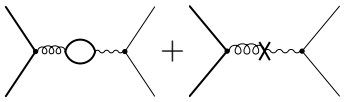
I calculate below a scattering matrix for a scalar analogue of a process $e^+e^- \to hadrons$ using Hamiltonian (5.77) in 5+1 dimensions. This model amplitude looks similar to the 1+1 dimensional case, i.e. Eq. (5.54). However, there are some important differences related to the fact that in 5+1 dimensions regulators (5.83) were introduced into the Hamiltonian together with counterterm operators (5.134)-(5.137) constructed in the canonical Hamiltonian based on RGPEP. The questions I address in this section are: (1) do the counterterms found using RGPEP without referring to the S matrix lead to a finite S matrix? And (2) can finite parts of these counterterms be chosen in such a way that the resulting S matrix is covariant? The answer to both questions is found to be positive.

Up to order e^2g^2, the scattering amplitude has the following terms:

$S_{\beta\alpha} = \frac{1}{(\sqrt{Z_q})^2}$  , (5.138)

where terms like those in Eq. (5.71) were excluded, and external-line propagators have poles in initial masses m_0.[12] All terms with loops are divergent. They are defined using a regularized Hamiltonian, and depend on the regularization in both a divergent and a finite way.

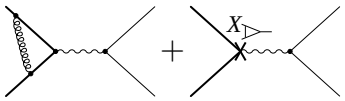
In the above model amplitude for the process $e^+e^- \to hadrons$ I do not include the following diagrams:

 (5.139)

because there are no similar terms in the case of QCD coupled to QED (cf. (5.56)).

5.7.1 Triangle S-matrix terms and triangle Hamiltonian counterterms

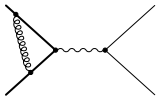
I start with analyzing the triangle term and triangle Hamiltonian counterterm contribution to the S matrix:

 . (5.140)

In this section I show, that this sum is finite, and that a proper choice of finite parts of the triangle counterterm $X^{(g^2e)}$ (Eq. (5.137)) in the Hamiltonian H^Δ makes the part of S matrix shown in (5.140) covariant.

[12] Alternatively, one may use the physical mass in all propagators, and include terms like those in Eq. (5.71). This leads to exactly the same result as presented in this section, when the result is expressed in the same set of parameters in order e^2g^2.

The first part of (5.140) is:



$$= {}_{out}\langle p_1, p_2, p_3 \mid k_1, k_2\rangle^{(e_q e g^2)}_{in,} =$$

$$= e e_q g^2\, 2(2\pi)^2 \delta^2(k_1+k_2-p_1-p_2)\frac{1}{s-m_\gamma^2+i\varepsilon}(-i)A(p_2^{+,\perp}, p_1^{+,\perp}, \boxed{r_\Delta}) \,, \quad (5.141)$$

where the function $A(p_2^{+,\perp}, p_1^{+,\perp}, \boxed{r_\Delta})$ is:

$$A(p_2^{+,\perp}, p_1^{+,\perp}, \boxed{r_\Delta}) = i\int \frac{\mathrm{d}^6 k}{(2\pi)^6}\left\{ \left[\prod_{\text{vertex } a} \boxed{r_{\Delta,a}(k^{+,\perp}, p_2^{+,\perp}, p_1^{+,\perp})} \prod_{\text{internal line } b} \theta(k_b^+)\right]_{\diagdown\!\!\!\!\diagup} \times \right.$$
$$\times \frac{1}{k^2-m^2+i\varepsilon}\frac{1}{(p_1+k)^2-m^2+i\varepsilon}\frac{1}{(p_2-k)^2-m^2+i\varepsilon} +$$
$$+ \left[\prod_{\text{vertex } a} \boxed{r_{\Delta,a}(k^{+,\perp}, p_2^{+,\perp}, p_1^{+,\perp})} \prod_{\text{internal line } b} \theta(k_b^+)\right]_{\diagdown\!\!\!\!\diagup} \times$$
$$\left. \times \frac{1}{k^2-m^2+i\varepsilon}\frac{1}{(p_1-k)^2-m^2+i\varepsilon}\frac{1}{(p_2+k)^2-m^2+i\varepsilon}\right\} , \quad (5.142)$$

$\mathrm{d}^6 k = \mathrm{d}k^+ \mathrm{d}k_+ \mathrm{d}^4 k^\perp = \mathrm{d}k^+ \mathrm{d}k^- \mathrm{d}^4 k^\perp/2$, and sums go over three vertices a connected by three lines b in the loop.

I shall review briefly how this expression is derived using regularized Hamiltonians. The S-matrix triangle diagrams come from a term with the fourth power of H_I in the expansion of the exponent in Eq. (5.15). The two $H_I^{(g)}$ which lead to the two leftmost vertices can be ordered in two ways, corresponding to the two terms (5.144). For each of these terms separately, light-front denominators combine with p^+ of a relevant line to a factor like in a Feynman propagator (see also Fig.3.4). The regulators act differently for the two orderings. The orderings are distinguished even in the 6-dimensional form (5.142) by the value of the three-momenta, namely by whether the p^+ momentum of the quark in the loop is smaller or larger than the p^+ momentum of the outgoing quark ($p_1^+ - k^+$ or $p_1^+ + k^+$, respectively).

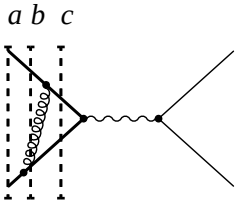
In the first part of (5.142), one can change the signs of all components of k, and the combined expression reads:

$$A(p_2^{+,\perp}, p_1^{+,\perp}, \boxed{r_\Delta}) = i\int \frac{\mathrm{d}^6 k}{(2\pi)^6}\left\{ \left[\prod_{\text{vertex } a} \boxed{r_{\Delta,a}(-k^{+,\perp}, p_2^{+,\perp}, p_1^{+,\perp})} \prod_{\text{internal line } b} \theta(-k_b^+)\right]_{\diagdown\!\!\!\!\diagup} + \right.$$
$$\left. + \left[\prod_{\text{vertex } a} \boxed{r_{\Delta,a}(k^{+,\perp}, p_2^{+,\perp}, p_1^{+,\perp})} \prod_{\text{internal line } b} \theta(k_b^+)\right]_{\diagdown\!\!\!\!\diagup} \right\} \times$$
$$\times \frac{1}{k^2-m^2+i\varepsilon}\frac{1}{(p_1-k)^2-m^2+i\varepsilon}\frac{1}{(p_2+k)^2-m^2+i\varepsilon} \,. \quad (5.143)$$

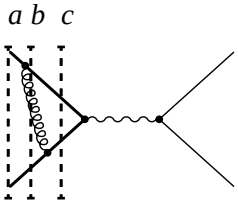
The expression (5.143), derivable from a regularized Hamiltonian, can be analyzed in the following way. There are three poles when integrating over k^-. Depending on the value of k^+ compared to P^+ and p_1^+, these poles can all be on the same side of the $Re\, k^-$ axis (and the result of the integration will be zero) or one of the poles can be on a different side than the other

two. The contour of integration can be closed in the half-plane with one pole. The non-zero contributions correspond to the θ-functions in (5.143). Replacing each integral over k^- with $(2\pi i)$ times the residue with appropriate signs leads to a sum of two diagrams:

$$\left[\frac{1}{\frac{ab}{x_2}+i\varepsilon}\,\frac{1}{ac+i\varepsilon}(P^+)^2\right]$$



$$+\left[\frac{1}{\frac{ab}{x_1}+i\varepsilon}\,\frac{1}{ac+i\varepsilon}(P^+)^2\right]$$



, (5.144)

where in both terms there are only five-dimensional (one $+$ and four $\perp$ dimensions) integrals, and the diagrams include regularization corresponding to the θ-functions. The value of k^+ determines which of the S-matrix diagrams of (5.144) is obtained.

Note that integration over k^- from $-\infty$ to $+\infty$ is carried out in the presence of regulators $\boxed{r_\Delta}$, and the result is well defined only because the regulators are there. The six-dimensional Feynman structure is in one-to-one correspondence with the Hamiltonian perturbation theory because the regulators do not depend on k^- (cf. [107]).

Attempting a similar construction of regularized Hamiltonians and a corresponding S matrix using equal-time Hamiltonians would lead to the following problem. The contributing Hamiltonian regularization depends on whether there are terms corresponding to the first or the second diagram (5.144). But these diagrams in the equal-time formalism are not distinguished by the three-momentum only. When using equal-time momenta, both orderings have to be summed up to arrive at the standard form of the Feynman propagators $i/(k^2-m^2+i\varepsilon)$, and this would not be possible for ordering-dependent regularization (such as introduced by a regulated Hamiltonian). In fact, in an equal-time calculation there would be many more orderings, arising from terms that create or annihilate three particles. However, it is not clear how to keep track of different orderings using three-momenta, although one can easily keep track of different orderings using k^+ in the light-front approach.

One can use the kinematics, Eqs. (5.112)-(5.118), to obtain the leading ultraviolet (divergent) part of (5.144). For example, the first term in (5.144) simplifies to:

$$\left[\frac{1}{\frac{ab}{x_2}+i\varepsilon}\,\frac{1}{ac+i\varepsilon}(P^+)^2\right] \approx \frac{x-x_1}{x_2 x}\,\frac{(P^+)^2}{\mathcal{M}^4}\,. \tag{5.145}$$

This is exactly equal to the divergent part of $\mathcal{F}^{(3)}$ (cf. Eq. (5.119)). Thus, when calculating the S matrix, the RGPEP counterterm $X^{(g^2e)}$  gives a diverging contribution with exactly the same value and the opposite sign, and it removes the divergence from the triangle diagram of the S matrix.

To summarize: When calculating the S matrix using the regularized Hamiltonian, one derives the expression for the function $A(p_2^{+,\perp}, p_1^{+,\perp}, \boxed{r_\Delta})$. This function has imaginary and real parts. The imaginary part comes from the pole $ac=0$ in (5.144). This means that the invariant mass of the intermediate state, c, is not large, and this part is not sensitive to the value and form of the regulator r_Δ. However, the real part is divergent for $\Delta\to\infty$. The logarithmic divergence (5.145) comes from the same expression as in the corresponding term in the effective Hamiltonian (5.119).

Note that the scattering amplitude (5.144) and the effective Hamiltonian term (5.108) contain different functions of external momenta. For example, the scattering amplitude is a complex function, with imaginary part coming from the pole in $ac = 0$, avoided by an introduction of $i\varepsilon$. In the effective Hamiltonian, the corresponding term is real: the region $ac \approx 0$ does not contribute because each such small denominator is accompanied by a factor of the type $1 - f_{ac}$, which vanishes in the limit $ac \to 0$ faster than the denominator. Also, for momenta other than near the pole the Hamiltonian and S-matrix expressions are different. Nevertheless (5.145) and (5.119) show that the ultraviolet parts of both the S matrix and in a coefficient in $\mathcal{H}_\lambda$ are the same. Because of this feature, the Hamiltonian counterterm which removes Δ-dependence from $\mathcal{H}_\lambda$, also removes such dependence from the S matrix.

I now turn to the question of choosing the finite part of the counterterm $X^{(g^2 e)}$.

Using Feynman parameters y_1 and y_2, the triangle contribution to the S matrix can be written as:

$$A(p_2^{+,\perp}, p_1^{+,\perp}, \boxed{r_\Delta}) = i \int_0^1 dy_1 \int_0^{1-y_1} dy_2 \int \frac{d^6 l}{(2\pi)^6} \times$$
$$\times \left\{ \left[\prod_{\text{vertex } a} \boxed{r_{\Delta,a}(l^{+,\perp}, p_2^{+,\perp}, p_1^{+,\perp})} \prod_{\text{internal line } b} \theta[-(l_b^+ - y_1 p_1^+ + y_2 p_2^+)] \right]_{\diagdown} + \right.$$
$$\left. + \left[\prod_{\text{vertex } a} \boxed{r_{\Delta,a}(l^{+,\perp}, p_2^{+,\perp}, p_1^{+,\perp})} \prod_{\text{internal line } b} \theta(l_b^+ - y_1 p_1^+ + y_2 p_2^+) \right]_{\diagup} \right\} \times$$
$$\times \frac{2}{\left(l^2 + s y_1 y_2 - m^2 \left[(y_1+y_2)^2 - (y_1+y_2) + 1\right] + i\varepsilon\right)^3} \qquad (5.146)$$

where

$$l^\mu = k^\mu + y_1 p_1^\mu - y_2 p_2^\mu \,, \qquad (5.147)$$

and $l_b^+ = k_b^+ + y_1 p_1^+ - y_2 p_2^+$ for internal line b. Note that this is not a covariant 6-dimensional expression: it depends on the external momenta not only through the invariant mass $s = (p_1 + p_2)^2$ in the denominator and θ-functions, but also through the regulating factors $\boxed{r_\Delta}$, which depend on the perpendicular and plus momenta of the outgoing scalar quarks.

The regulators in (5.143) depend on $k^{+,\perp}$, $p_2^{+,\perp}$, $p_1^{+,\perp}$ only, and not on the minus components. Since the variable change (5.147) does not mix $+$ and $\perp$ components of momenta with k^-, the regulators in the form (5.146) depend on l^+ and $l^\perp$ only, and not on l^-. Note, however, that one cannot integrate this expression over l^- over residues: this would suggest, that the result is zero, because there is only one triple pole in l^- with residue equal to zero. But one cannot integrate over l^- when $l^+ = 0$. This feature of expressions of the type (5.146) was analyzed by Yan [23]. Yan and his collaborators obtained similar integrals, in the sense that they combined the denominators in a similar way. However, their integrals did not involve regulator factors such as these used here. Yan considered a formal expression for the S matrix, and discussed loop-by-loop regularization and renormalization of the S-matrix diagrams only.

The triangle S-matrix diagram (5.141) is combined with the contribution to the S matrix from the triangle counterterm in the Hamiltonian. The question is whether this counterterm can produce a covariant answer with some choice of its finite part. The answer, as shown below, is that indeed it can. The requirement is that $X_{\diagdown}$ contribution to the S matrix be equal to the negative of (5.146) for some fixed value of s. Accordingly, I choose the counterterm operator

in H^Δ in the following form:

$$X_{\triangleright\!-} \;=\; \int [123]\widetilde{\delta}(1+2-3)q_1^\dagger q_2^\dagger a_3(-1)\gamma_\Delta(p_1^{+,\perp},p_2^{+,\perp},\Delta)\,. \tag{5.148}$$

$$\begin{aligned}\gamma_\Delta(p_1^{+,\perp},p_2^{+,\perp},\boxed{r_\Delta}) = g^2 e_q\, i\int_0^1 \mathrm{d}y_1\int_0^{1-y_1}\mathrm{d}y_2\int\frac{\mathrm{d}^6 l}{(2\pi)^6}\times\\ \times\Bigg\{\Bigg[\prod_{\text{vertex }a}\boxed{r_{\Delta,a}(l^{+,\perp},p_2^{+,\perp},p_1^{+,\perp})}\prod_{\text{internal line }b}\theta[-(l_b^+ - x_1p_1^+ + x_2p_2^+)]\Bigg]_{\triangleright\!-} +\\ +\Bigg[\prod_{\text{vertex }a}\boxed{r_{\Delta,a}(l^{+,\perp},p_2^{+,\perp},p_1^{+,\perp})}\prod_{\text{internal line }b}\theta(l_b^+ - x_1p_1^+ + x_2p_2^+)\Bigg]_{\triangleright\!-}\Bigg\}\times\\ \times\frac{2}{\left(l^2+s_0^2x_1x_2-m^2\left[(x_1+x_2)^2-(x_1+x_2)+1\right]+i\varepsilon\right)^3}\,,\end{aligned} \tag{5.149}$$

where $s_0 < 4m^2$. For these values of s_0, $\gamma_\Delta(p_1^{+,\perp},p_2^{+,\perp},\boxed{r_\Delta})$ is real. s_0 is a fixed parameter, but $\gamma_\Delta(p_1^{+,\perp},p_2^{+,\perp},\boxed{r_\Delta})$ continues to depend on momenta because of the regulating factors $\boxed{r_\Delta}$. I will show that the the above choice of γ is in agreement with RGPEP.

When one calculates scattering amplitude using the counterterm, one gets an expression similar to (5.141), but with g^2e_qA replaced by the sum:

$$\begin{aligned}g^2e_qA(p_2^{+,\perp},p_1^{+,\perp},\boxed{r_\Delta})+\gamma_\Delta(p_1^{+,\perp},p_2^{+,\perp},\boxed{r_\Delta}) = g^2e_q\,i\int_0^1\mathrm{d}y_1\int_0^{1-y_1}\mathrm{d}y_2\int\frac{\mathrm{d}^6l}{(2\pi)^6}\times\\ \times\Bigg\{\Bigg[\prod_{\text{vertex }a}\boxed{r_{\Delta,a}(l^{+,\perp},p_2^{+,\perp},p_1^{+,\perp})}\prod_{\text{internal line }b}\theta(-(l_b^+ - y_1p_1^+ + y_2p_2^+))\Bigg]_{\triangleright\!-} +\\ +\Bigg[\prod_{\text{vertex }a}\boxed{r_{\Delta,a}(l^{+,\perp},p_2^{+,\perp},p_1^{+,\perp})}\prod_{\text{internal line }b}\theta(l_b^+ - y_1p_1^+ + y_2p_2^+)\Bigg]_{\triangleright\!-}\Bigg\}\times\\ \times\Big[\frac{2}{\left(l^2+sy_1y_2-m^2\left[(y_1+y_2)^2-(y_1+y_2)+1\right]+i\varepsilon\right)^3}-\\ -\frac{2}{\left(l^2+s_0y_1y_2-m^2\left[(y_1+y_2)^2-(y_1+y_2)+1\right]+i\varepsilon\right)^3}\Big]\,.\end{aligned} \tag{5.150}$$

The two terms in the square brackets correspond to the two terms in (5.140). Both terms lead to exactly the same logarithmic divergence and their difference is finite. One can take the limit $\Delta\to\infty$, that is, $r_\Delta\equiv 1$. Then the result is finite and becomes covariant: the dependence on the external momenta comes only through $s = P^\mu P_\mu$ in the denominator of the first term in Eq. (5.150):

$$(g^2e_q)A(p_2^{+,\perp},p_1^{+,\perp},\boxed{r_\Delta})+\gamma_\Delta(p_1^{+,\perp},p_2^{+,\perp},\boxed{r_\Delta}) = (g^2e_q)A(s)\,, \tag{5.151}$$

$$\begin{aligned}A(s) \;=\; & i\int_0^1\mathrm{d}y_1\int_0^{1-y_1}\mathrm{d}y_2\int\frac{\mathrm{d}^6l}{(2\pi)^6}\times\\ & \times\Big[\frac{2}{\left(l^2+sy_1y_2-m^2\left[(y_1+y_2)^2-(y_1+y_2)+1\right]+i\varepsilon\right)^3}-\\ & -\frac{2}{\left(l^2+s_0y_1y_2-m^2\left[(y_1+y_2)^2-(y_1+y_2)+1\right]+i\varepsilon\right)^3}\Big]\,.\end{aligned} \tag{5.152}$$

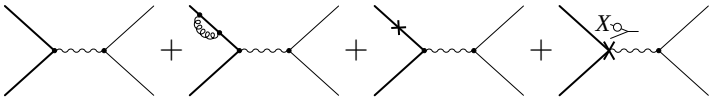
As I have shown before, the first term of (5.150) has both a real and an imaginary part. The imaginary part is finite and independent of regularization. The real part of $A(p_2^{+,\perp}, p_1^{+,\perp}, \boxed{r_\Delta})$ is divergent. However, the fact $A(s)$ is finite, means that the counterterm contribution in (5.150) removed this divergence. The diverging part of the counterterm (5.148) is exactly as predicted by RGPEP.

To summarize: the choice of the Hamiltonian counterterm (5.148) is in full agreement with the RGPEP requirements, and its finite part is such that the contribution of Eq. (5.140) to the scattering amplitude is fully covariant.

It is also interesting that Bardakci and Thorn [108] have provided a new formula for planar diagrams in scalar theories, and recently Thorn argued [109] that a small set of counterterms (including ghosts) is sufficient to obtain covariant results. Here I show that light-front Hamiltonians provide covariance at one-loop level with explicitly constructed non-trivial finite parts of the counterterms.

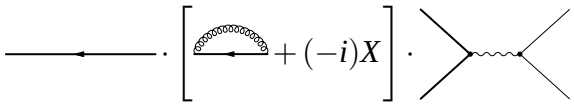
5.7.2 Contributions of loops on external lines

The remaining four contributions to the scattering matrix are:



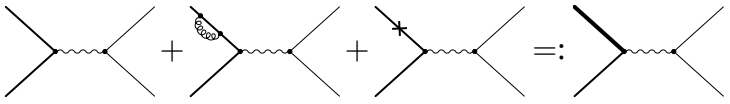
(5.153)

Below, I give the expressions corresponding to loops and counterterms on one line only (the second scalar-quark line contributes the same amount). I examine first three terms, as the fourth has a different structure. These three terms can be summed up in a way analogous to the sum in 1+1 dimensional case. The second and the third term can be written as the tree diagram times a propagator and a factor in square parenthesis:



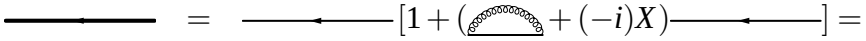
(5.154)

(in this expression, X denotes a $\left(\delta m_\Delta^2 + \delta m_{phys}^2\right)$ factor in the mass-counterterm operator (5.89)). Together with the first term, this can be written as the tree amplitude with a full propagator on one of the scalar-quark lines (marked with a bold line):



(5.155)

The full propagator is a result of summing a geometric series:



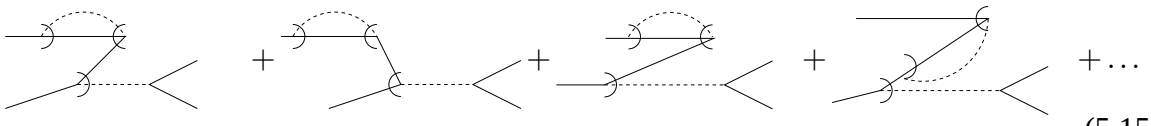
(5.156)

$$= \frac{i}{p^2 - m^2 + i\varepsilon + f(p^2)} \tag{5.157}$$

$$f(p^2) := (-i)[\ + (-i)X], \ . \tag{5.158}$$

This result has a similar form to (5.69) in the non-divergent 1+1 dimensional case.

A similar summation would not be possible in the case of a regularized equal-time Hamiltonian. Here, the regulating factors which depend on time orderings would lead to terms such as:



(5.159)

(where regularization is marked by half-circles). These terms do not reduce to a simple factor in front of the tree amplitude – one needs all the orderings summed up to reproduce Feynman propagators on each line – and, because of the regularization, summation is not possible.

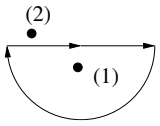
To return to the light-front approach, function $f(p^2)$, Eq.(5.158), needs to be evaluated because it contributes to the scalar quark mass and normalization of external quark states. The second term in (5.158) does not depend on external momenta, but does depend on Δ, cf. (5.89), and the regularization is present. I illustrate below the type of integrals that appear in $f(p^2)$. This shows, that $f(p^2)$ is indeed a function of scalar-quark $p^2 = p^\mu p_\mu$ only, even where regularization is present. The first term in Eq. (5.158) for $f(p^2)$ – the scalar quark-gluon loop without external propagators – is:

$$(-i)\,\text{  } = (-i)(-i)^2 g^2 (i)^2 \int \frac{\mathrm{d}k^+ \mathrm{d}k^- \mathrm{d}^4 k^\perp}{2(2\pi)^6} \times$$

$$\times \frac{\boxed{r_\Delta(k^{+,\perp}, p^{+,\perp}, \Delta)}}{(k^+k^- - k^{\perp 2} - m^2 + i\varepsilon)\left[(p^+ - k^+)(p^- - k^-) - (p-k)^{\perp 2} - m^2 + i\varepsilon\right]} = \tag{5.160}$$

$$= (-i)g^2 \int \frac{\mathrm{d}k^+ \mathrm{d}^4 k^\perp}{2(2\pi)^5} \frac{\theta(k^+)\theta(p^+ - k^+)}{k^+(p^+ - k^+)} \boxed{r_\Delta(k^{+,\perp}, p^{+,\perp}, \Delta)} \times$$

$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}k^-}{2\pi} \frac{1}{\left(k^- - \frac{k^{\perp 2}+m^2}{k^+} + i\varepsilon\right)\left(p^- - k^- - \frac{(p-k)^{\perp 2}+m^2}{p^+ - k^+} + i\varepsilon\right)} = \tag{5.161}$$



$$= \frac{-i}{2\pi}(-2\pi i)g^2 \int [k] \frac{\theta(p^+ - k^+)}{p^+ - k^+} \boxed{r_\Delta} \frac{1}{p^- - k_m^- - (p-k)_m^- + i\varepsilon} = \tag{5.162}$$

$$= (-1)g^2 \int [k_1 k_2] \widetilde{\delta}(p - k_1 - k_2) \boxed{r_\Delta} \frac{1}{p^- - k_{1m}^- - k_{2m}^- + i\varepsilon} = \left| z := \kappa^{\perp 2} \right|$$

$$= \frac{g^2}{2(2\pi)^5} \int_0^1 \mathrm{d}x \frac{\Omega_4}{2} \int_0^\infty z\mathrm{d}z \frac{1}{z + m^2 - p^2 x(1-x) - i\varepsilon} \boxed{\exp\left[-d_\eta \frac{z}{\Delta^2}\right]} \tag{5.163}$$

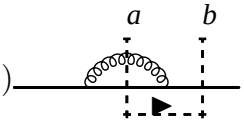
For regularization (5.83), $d_\eta = 4$ and this function depends on the external momentum p^μ only through p^2 (note that, here, p^2 is the square root of one-particle momentum, and not the square of the sum of the momenta of both scattered particles). The factor corresponding to the external line can thus be written as:

$$\frac{1}{p^2 - m^2 + i\varepsilon + f(p^2, \Delta)} \,. \tag{5.164}$$

This is similar to (5.69), but this time f depends on the regularization.

I now proceed as for the 1+1 dimensional case:

1) The physical mass is obtained by investigating position of the pole of (5.164):

$$m^2_{phys} = m^2 - f(m^2) = m^2 + (p^+)$$ 

$$+ \left(\delta m^2_\Delta + \delta m^2_{phys}\right) , \quad (5.165)$$

where, in the loop term, there is only a 5-dimensional integration and the terms in parentheses are defined in Eq. (5.89). This is exactly the same result for the physical mass as was obtained from the Hamiltonian perturbation theory (5.91) in agreement with the spectral-representation argument. This expression is finite (Δ-independent) and constant (momentum-independent).

2) The residue at the pole of (5.164):

$$Z = \frac{1}{1+\frac{df}{d(p^2)}(m^2)} = 1 - \frac{df}{d(p^2)}(m^2) \quad (5.166)$$

The mass counterterm does not contribute to this expression.

$$\frac{df}{d(p^2)}(m^2) = \frac{g^2}{2(2\pi)^5}\int_0^1 dx \left(\frac{\Omega_4}{2}\right)\int_0^\infty z dz \frac{x(1-x)}{\left[z+m^2(1-x+x^2)\right]^2}\boxed{\exp\left(-d_\eta \frac{z}{\Delta^2}\right)} \quad (5.167)$$

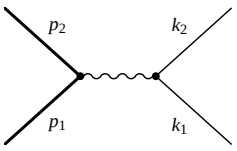
This expression is a momentum-independent constant, but it does have logarithmic dependence on Δ.

Putting them together, the first three terms of Eq. (5.153) can be written as the tree term (the first term of Eq. (5.153)) whose one external propagator is replaced with a propagator whose pole lies in the physical scalar-quark mass squared, and the residue at the pole is Z:

$$\frac{Z}{p^2 - m^2_{phys} + i\varepsilon} . \quad (5.168)$$

5.7.3 Interplay between Z and part of the vertex counterterm

Taken together with the factor $1/\sqrt{Z}$ of the LSZ formula (5.11), the terms analyzed in Section 5.7.2 in the vicinity of the pole $p^2 = m^2_{phys}$ lead to the following term in the S matrix:

$\sqrt{Z}\cdot$ 

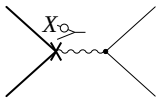
, (5.169)

where,[13] in the tree diagram, there are physical masses in the external propagators, and the wave-function renormalization factor is:

$$\sqrt{Z} = 1 - \frac{1}{2}\frac{g^2}{2(2\pi)^5}\left(\frac{\Omega_4}{2}\right)\int_0^1 dx\; x(1-x)\int_0^\infty z dz \frac{1}{\left[z+m^2(1-x+x^2)\right]^2}\boxed{\exp\left(-d_\eta \frac{z}{\Delta^2}\right)} . \quad (5.170)$$

[13] As explained above, I analyze here separately loops coming from corrections to one external scalar-quark line. The full amplitude has a factor $\sqrt{Z_q}^2\sqrt{Z_e}^2$.

The logarithmic divergence of this expression is canceled by the Δ-dependence of the only term that was yet not analyzed in the previous section, namely the last term in Eq. (5.153):



, (5.171)

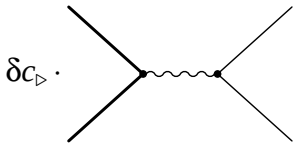
coming from the counterterm X  given in Eq. (5.132).

Thus, all the terms (5.153) give a finite contribution to the S matrix. The simplest choice of the finite part of the counterterm (marked *const*. in (5.132)), which makes this contribution covariant is a constant independent of momenta.

5.7.4 Summary of the S matrix calculations

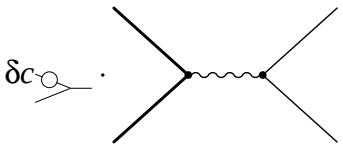
The result of the scalar model for the scattering amplitude of $e^+e^- \to hadrons$ has the following structure, up to order e^2g^2:

- A triangle part (analyzed in Section 5.7.1), is divergent and depends on the invariant mass s of the scattered particles. This part also depends on the momenta of the produced particles in a non-covariant way, through the regulator factors r_Δ. However, there is a contribution to the S matrix of the Hamiltonian counterterm operator X  calculated using RGPEP. This counterterm removes the infinite Δ-dependence of the triangle term in the S matrix. Moreover, the choice of the finite parts of the triangle counterterm, Eq. (5.148), leads to a fully covariant result. Within this form of the counterterm, one is free to choose one parameter, the renormalization point $s = s_0$. A different choice of $s_0 + \delta s_0$ will lead to a change in the S matrix, which has the form of the tree amplitude multiplied by a constant, finite factor $\delta c_\triangleright$:



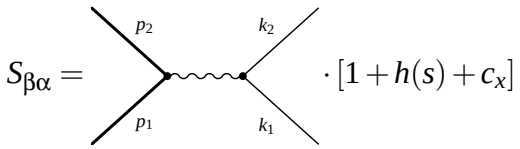
(5.172)

- Terms with self-interaction loops and mass counterterms on external lines (Sec. 5.7.2) sum up to propagators with poles in physical particle masses and with residue Z, which is a logarithmically divergent constant. The divergence is canceled by a contribution from the X  counterterm given in Eq. (5.132). A different choice of the finite constant part of the counterterm will shift the scattering amplitude by a finite constant factor δc  :



(5.173)

Taken together, the result for the S matrix is:



(5.174)

where $h(s)$ represents the s-dependence from the triangle term (5.140) and c_x is a constant depending on the finite parts of both logarithmic counterterms.

I now turn to the question of whether the covariant and finite result for the scattering amplitude calculated in the model using the regularized and renormalized Hamiltonian is the same as that predicted using Feynman diagrams. The tree amplitudes of order e^2 and e^2g are the same in both approaches. To discuss higher orders, I briefly review how the theoretical predictions compare with the experiment. When one compares the result for the S matrix with experimental findings one cannot fit all the parameters of the theory independently. This is because they appear in fixed combinations. For example, performing a scattering experiment for a given s_0 gives the value of

$$e_q\,[1+h(s_0)+c_x] \tag{5.175}$$

rather than the value of the electric charge parameter e_q for quarks in the canonical Hamiltonian. Theoretical predictions for a different s are changed by: (1) a change of s in the tree amplitude (which is the same in both approaches); (2) and by a change of the function $h(s)$ in the factor multiplying the tree amplitude.

Thus, to compare the Hamiltonian approach and Feynman diagrams it is enough to calculate the derivative of the function $h(s)$ over s (which means $\mathrm{d}f(s)/\mathrm{d}s$ in (5.150)). The result, when expressed in terms of results of the experiment for $s = s_0$ (Eq.(5.175)), is the same in both approaches. Note, however, that this can be done in the Hamiltonian approach discussed here only because I have previously shown that the result is independent of regularization and covariant. If f depended in a non-covariant way on $+$ and $\perp$ momenta components, the derivative $\mathrm{d}f(s)/\mathrm{d}s$ would not make sense.

To summarize: the result for the S matrix obtained using a canonical Hamiltonian with non-covariant cutoffs and counterterms found using RGPEP is finite. For a proper choice of the finite parts of the counterterms, it is also covariant, and the same as that obtained using the Feynman diagrams. Thus, as I have shown above, the program outlined in Chapter 2 in a very simple Hamiltonian model can be extended to ϕ^3 quantum field theory in 5+1 dimensions as far as the scattering amplitude analogous to $e^+e^- \to hadrons$ is concerned, in order e^2g^2.

Chapter 6

Summary of key findings

One of the most important questions in the theory of hadrons is how to employ QCD to describe the strong binding of quarks and gluons together with the weak-coupling phenomena of high-energy scattering processes in one formalism. The key feature of QCD is asymptotic freedom. At high energies, the effective charge of quarks and gluons is small, and decreases towards higher energies. The difficulty with bound states is that, when the energy scale is lowered towards the region where binding mechanism works, the coupling constant grows and becomes so large that experience from QED does not directly apply. The weak-coupling part is analogous to QED, but the strong binding with coupling constant on the order of 1 appears to be quite different from the binding in QED that is characterized by $\alpha \sim 1/137$. Lattice gauge theory is the most advanced candidate for dealing with the issues of binding. On the other hand, the phenomenology of hadrons, which speaks of quarks and gluons as almost pointlike partons in high-energy processes and as complex constituent particles in low-energy processes, suggests that there is a hope for a systematic approach based on the concept of effective particles. A natural candidate for connecting binding and scattering in one systematic description in terms of particles is the Hamiltonian formulation of QFT. This expectation stems from the fact that the theory of scattering is based on the time-evolution described by the operator e^{-iHt}, where H is the Hamiltonian, and the eigenvalue problem for the same Hamiltonian H defines bound states as its eigenstates. However, in the Hamiltonian formulation of QFT, renormalization is more complicated than in calculations based on Feynman diagrams, there is a question of covariance when one uses non-covariantly regulated Hamiltonians, and there is a need for a renormalization prescription which applies also to the bound-state problem, where the perturbative renormalizability of scattering matrix is not enough because one needs to control the off-shell behavior of the interaction.

In this thesis, I have attempted to find out how it is possible to describe bound states and scattering of particles in QFT using a Hamiltonian approach. I have given examples of a perturbative construction of the effective Hamiltonians $\mathcal{H}_\lambda$ in QFT which (1) can describe bound states of relativistic effective fermions in a meaningful way, and (2), in a theory with asymptotic freedom, can be used to describe scattering and lead to the same results as the Feynman diagrams. In order to obtain this result, it has been necessary to resolve a number of issues mentioned in the Introduction (see page 1). The results are summarized below.

In all the models considered here, I have introduced a universal regularization of all terms in the entire Hamiltonian operator (issue (1) in the Introduction). This means that the same regularized operator produces all the perturbative scattering diagrams, and it is also used for the nonperturbative description of bound states, although using different basis states than those

of bare particles. This also means that one set of counterterms removes ultraviolet divergences from all physical quantities derived using the Hamiltonian. The counterterms are constructed using a renormalization group procedure for effective particles (issue (2) from the Introduction). This procedure determines in a systematic way the ultraviolet structure of required counterterm operators in the Hamiltonian (which are not the same as counterterms in a Lagrangian approach, determined by divergences in the scattering amplitude calculations).

A heuristic outline of the problem considered in my thesis, of finding out if a Hamiltonian formulation of QFT can be used for relativistic description of bound states and scattering processes, is presented in Chapter 2 using an example of a Yukawa theory truncated to two sectors of the Fock space [1]. The general form of counterterms in the cutoff Hamiltonian can be found in a systematic way using a similarity transformation for Hamiltonian matrices. It follows that the proper choice of finite constant parts of the required counterterms leads to a relativistic S matrix. Further, for the same choice of counterterms, the bound-state equation reduces to a relativistic Dirac equation for a physical fermion. With the proper choice of counterterm parameters, the analytic structure of the scattering amplitude, including threshold behavior, is in agreement with the Dirac eigenvalue equation for a physical fermion. Thus, this heuristic model example suggests that it is possible to describe bound states and scattering processes with a single relativistic Hamiltonian within a well-defined renormalization group procedure.

One of the key problems with defining a bound-state equation for fermions in QFT on the light front – and we are bound to use light-front dynamics for a number of reasons, explained earlier in Section 3.3 – is that spin factors generate overlapping divergences. In particular, spinors introduce powers of perpendicular momenta that lead to potentials similar to a two-dimensional δ-function. This problem is common to all theories involving fermions, and the simplest example of such a theory is the Yukawa theory of fermions interacting with scalar particles. This theory is considered in Chapter 4. It is shown that, when one assumes that there is a bound state dominated by a two-bare-fermions Fock sector, the eigenvalue is unacceptably sensitive to the ultraviolet cutoff Δ due to the overlapping divergences. In the Tamm-Dancoff approach there is no systematic procedure leading to counterterms in the Hamiltonian which might remove this dependence, except perhaps for sector-dependent mass counterterms in a low-energy theory based on coupling coherence [4]. However, one cannot easily extend the coupling-coherence approach to high orders, and it is thus useful to study straightforwardly perturbative renormalization-group approach for effective particles.

Perturbative renormalization group procedure for effective particles (RGPEP) is useful not only because it can, in principle, be extended to high orders, but also because it naturally defines an alternative picture of bound states as being made of effective particles, and these are unitarily equivalent to the bare particles. As a result, the bound state of two effective relativistic fermions is well-defined and free from the overlapping divergences, since effective fermions have form factors in interaction vertices. These form factors provide convergence in the eigenvalue equation for the effective Hamiltonian $\mathcal{H}_\lambda$; they are also responsible for the dominance of Fock sectors with a low number of effective constituents in the lowest-mass bound states.

The RGPEP procedure allows one to accomplish a number of things. One can use it to define counterterms in the Hamiltonian H^Δ from canonical theory with an ultraviolet cutoff Δ. One can calculate the effective Hamiltonian $\mathcal{H}_\lambda$, which is equal to H^Δ, but expressed in terms of creation and annihilation operators for effective particles instead of the bare ones. Building on this, I have shown that a bound state of two fermions, defined by the eigenvalue equation for the Hamiltonian calculated to the order g^2, is well approximated by a Schrödinger equation for a two-body wave function, and it is not sensitive to the value of the artificial ultraviolet

cutoff Δ. Moreover, for reasonable values of the parameter λ of the renormalization group, the energies of the bound states in question depend on λ weakly, even if the interaction is only of the second order in g. Of course, the second order is a very low one as far as scattering matrices are concerned. However, when working with Hamiltonians, it must be remembered that the Coulomb potential that describes the entire chemistry of atomic bonds is only of the second order in electric charge e.

Thus, RGPEP appears to provide a method for evaluating interaction terms in Hamiltonians for bound states (issue (3) in the Introduction). The eigenvalue equations for the resulting effective Hamiltonians in QFT determine the wave functions of constituents. Because of the RGPEP form factors f_λ, there are no overlapping divergences in the derived equation (issue (5) in the Introduction). Also, the Fock sectors of different numbers of effective particles are coupled only weakly by the effective Hamiltonian. This happens even for sizable coupling constants, because the form factors limit the range of exchanged momenta to less than about λ, and λ can be small. This leads, in turn, to the approximation of the eigenstates by sectors with only a small number of constituents (issue (4)). These constituents have a complicated, but calculable structure when expressed in terms of bare particles. In this way, RGPEP allows us to derive well-defined equations for bound states of fermions from a local QFT.

For a detailed investigation of scattering processes using a Hamiltonian approach it was necessary to review the derivation of LSZ formula for S-matrix elements. An initial analysis in scalar theory shows that light-front Hamiltonians with non-covariant cutoffs can produce covariant answers. It is shown that the same amplitude for scattering of physical particles can be calculated either in terms of bare particles using the Hamiltonian H^Δ, or in terms of effective particles using the effective Hamiltonian $\mathcal{H}_\lambda$. Since the bound states are also described naturally in terms of the effective particles using $\mathcal{H}_\lambda$, this last result for scattering amplitude provides a stepping-stone towards a systematic relativistic Hamiltonian description of scattering processes in which there are bound states in the initial or final states.

The scalar theory under consideration here is an asymptotically free massive ϕ^3 theory in 5+1 dimensions, whose perturbative structure resembles in the lowest order QCD coupled to QED. Because two couplings of different strengths are introduced – e analogous to electric charge, and g, analogous to the color coupling constant – one could define two procedures based on the principles of RGPEP for evaluating the effective theory. These are (1) a simplified procedure, in which the definition of effective particles is based on their strong interactions only, and (2) a full procedure, in which RGPEP takes all the interactions into account on an equal footing.

In the case of Hamiltonian terms that contribute to the scattering amplitude analogous to $e^+e^- \to hadrons$ up to the order e^2g^2, both procedures lead to the same form of the counterterms in H^Δ. However, I do not employ the S-matrix calculation in order to determine the divergent parts of the counterterms. Only the finite parts are determined on the basis of the S-matrix calculation. In the case of confinement, it is expected that similar conditions can be found by considering scattering of bound states. Thus, this Hamiltonian approach seems not to be limited to perturbation theory in the description of physical scattering processes in theories with asymptotic freedom.

Nevertheless, in purely perturbative calculations up to one loop, one can also find finite parts of counterterms that render fully covariant scattering amplitude, and both triangle terms and wave-function renormalization terms have to be considered. The covariance is obtained despite the non-covariant regularization of the Hamiltonian (issue (6) in the Introduction) and intrinsically non-covariant nature of the Hamiltonian formulation. This non-covariant nature

is exhibited in the regularization factors r_Δ which depend only on kinematical momenta and do not involve the dynamical variables such as energy (I use p^+ and $p^\perp$ for momenta, and p^- for energy). It seems that the success of our procedure relies so heavily on the expression $p^- = (p^{\perp 2} + m^2)/p^+$ and the simplicity of the perturbative vacuum ($|\Omega\rangle = |0\rangle$), combined with a straightforward connection between energy denominators and Feynman denominators, that it is highly questionable if any similar result can be achieved using dynamical schemes other than the light-front scheme. It is also unclear how the unitary RGPEP procedure, which in the light-front formulation is independent of the vacuum structure, can be applied in the standard time-evolution theory, where the vacuum structure heavily mixes with the dynamics of particles.

Several problems that must be solved in gauge theories (especially QCD as a part of the standard model, or other theories that go beyond the standard model) could not be even formulated with the examples considered in this work. Two problems are particularly significant. The first concerns the divergences at small x-variable (analogous to Feynman and Bjorken x in the parton model). The second problem is that of spontaneous symmetry breaking. Both problems go far beyond the perturbative analysis of RGPEP discussed here, and may be related to basic problems with the concept of empty space. Spontaneous symmetry breaking can in principle be described using light-front Hamiltonians with additional terms corresponding to $x = 0$ [15]. For example, such terms in Hamiltonian formalism can reproduce standard results in sigma models. But the issue appears to be much more complicated in gauge theories in the presence of small-x singularities.

One immediate limitation of this work is that is does not describe fourth- and higher-order Hamiltonians. These may contain surprises, as the operators one calculates are explicitly known in their off-shell behavior and this implies that they can describe binding effects that are not accessible in perturbation theory. Until fourth and higher order calculations are completed, it will remain unclear how far the Hamiltonian approach can be relied on in physical studies. However, it is also possible that Hamiltonian calculations will offer new insights into the description of bound states and scattering processes in particle physics.

Bibliography

[1] T. MASŁOWSKI AND M. WIĘCKOWSKI. 4th order similarity renormalization of a model Hamiltonian. *Phys. Rev.*, **D57**:4976–4980, 1998. 2, 6, 13, 116, 145

[2] S. D. GŁAZEK AND M. WIĘCKOWSKI. Large-momentum convergence of Hamiltonian bound-state dynamics of effective fermions in quantum field theory. *Phys. Rev.*, **D66**:016001, 2002. 3, 34, 48, 51, 52, 53, 58, 61, 69, 149

[3] S. D. GŁAZEK AND K. G. WILSON. Renormalization of Hamiltonians. *Phys. Rev.*, **D48**:5863–5872, 1993. 6, 8, 45, 51, 145, 146

[4] R. J. PERRY, A. HARINDRANATH, AND K. G. WILSON. Light front Tamm-Dancoff field theory. *Phys. Rev. Lett.*, **65**:2959–2962, 1990. 6, 12, 54, 116

[5] S. D. GŁAZEK AND R. J. PERRY. Special example of relativistic Hamiltonian field theory. *Phys. Rev.*, **D45**:3740–3754, 1992. , and references therein; see also [110]. 6, 11, 13

[6] A. B. BYLEV, S. D. GŁAZEK, AND J. PRZESZOWSKI. Relativistic Hamiltonian dynamics of pions in nucleons. *Phys. Rev.*, **C53**:3097–3110, 1996. 7, 14

[7] M. GELL-MANN AND M. L. GOLDBERGER. The formal theory of scattering. *Phys. Rev.*, **91**:398–408, 1953. 10

[8] F. J. DYSON. The radiation theories of Tomonaga, Schwinger, and Feynman. *Phys. Rev.*, **75**:486–502, 1949. 72

[9] R. P. FEYNMAN. Space-time approach to quantum electrodynamics. *Phys. Rev.*, **76**:769–789, 1949. see also [111]. 72

[10] S-I. TOMONAGA. On infinite field reactions in quantum field theory. *Phys. Rev.*, **74**:224–225, 1948. and references therein.

[11] J. S. SCHWINGER. Quantum electrodynamics. I: A covariant formulation. *Phys. Rev.*, **74**:1439, 1948.

[12] J. S. SCHWINGER. Quantum electrodynamics. II: Vacuum polarization and self- energy. *Phys. Rev.*, **75**:651, 1948.

[13] J. S. SCHWINGER. Quantum electrodynamics. III: The electromagnetic properties of the electron: Radiative corrections to scattering. *Phys. Rev.*, **76**:790–817, 1949.

[14] I. Białynicki-Birula and Z. Białynicka-Birula. *Elektrodynamika kwantowa*. Państwowe Wydawnictwo Naukowe, 1974.

[15] K. G. Wilson, T. S. Walhout, A. Haridranath, W-M. Zhang, R. J. Perry, and S. D. Głazek. Nonperturbative QCD: A weak coupling treatment on the light front. *Phys. Rev.*, **D49**:6720–6766, 1994. 21, 30, 33, 36, 51, 63, 118

[16] P. A. M. Dirac. Forms of relativistic dynamics. *Rev. Mod. Phys.*, **21**:392–399, 1949. 24, 25, 77

[17] T. Heinzl. Light cone dynamics of particles and fields. 1998. habilitation thesis, Faculty of Physics, University of Regensburg; hep-th/9812190. 27

[18] K. Bardakci and M. B. Halpern. Theories at infinite momentum. *Phys. Rev.*, **176**:1686–1699, 1968. 81

[19] L. Susskind. Model of self-induced strong interactions. *Phys. Rev.*, **165**:1535, 1968.

[20] S-J. Chang, R. G. Root, and T-M. Yan. Quantum field theories in the infinite momentum frame. 1. Quantization of scalar and Dirac fields. *Phys. Rev.*, **D7**:1133–1148, 1973.

[21] S-J. Chang and T-M. Yan. Quantum field theories in the infinite momentum frame. 2. Scattering matrices of scalar and Dirac fields. *Phys. Rev.*, **D7**:1147–1161, 1973.

[22] T-M. Yan. Quantum field theories in the infinite momentum frame 3. Quantization of coupled spin one fields. *Phys. Rev.*, **D7**:1760–1779, 1973.

[23] T-M. Yan. Quantum field theories in the infinite momentum frame. 4. Scattering matrix of vector and Dirac fields and perturbation theory. *Phys. Rev.*, **D7**:1780–1800, 1973. 107

[24] M. Burkardt. Light front quantization. *Adv. Nucl. Phys.*, **23**:1–74, 1996. 28

[25] S. J. Brodsky, H.-C. Pauli, and S. S. Pinsky. Quantum chromodynamics and other field theories on the light cone. *Phys. Rept.*, **301**:299–486, 1998. 28, 131

[26] A. Casher. Gauge fields on the null plane. *Phys. Rev.*, **D14**:452, 1976. 30

[27] C. B. Thorn. Asymptotic freedom in the infinite momentum frame. *Phys. Rev.*, **D20**:1934, 1979. 30

[28] H. C. Pauli and S. J. Brodsky. Discretized light cone quantization: Solution to a field theory in one space one time dimensions. *Phys. Rev.*, **D32**:2001, 1985. 30

[29] H. C. Pauli and S. J. Brodsky. Solving field theory in one space one time dimension. *Phys. Rev.*, **D32**:1993, 1985. 30

[30] E. P. Wigner. On unitary representations of the inhomogeneous Lorentz group. *Annals Math.*, **40**:149–204, 1939. 30

[31] L. Susskind and M. Burkardt. A model of mesons based on chiSB in the light front frame. 1994. published in *Theory of hadrons and light front QCD*, edited by S. D. Głazek (Word Scientific, Singapore 1995), p.5. 121

[32] K. G. WILSON AND M. M. BRISUDOVA. Chiral symmetry breaking and light front QCD. 1994. published in *Theory of hadrons and light front QCD*, ([31]), p.166.

[33] K. G. WILSON AND D. G. ROBERTSON. Light front QCD and the constituent quark model. 1994. published in *Theory of hadrons and light front QCD*, ([31]), p.15.

[34] J. S. ROZOWSKY AND C. B. THORN. Spontaneous symmetry breaking at infinite momentum without p+ zero modes. *Phys. Rev. Lett.*, **85**:1614–1617, 2000.

[35] G. 'T HOOFT. Computation of the quantum effects due to a four-dimensional pseudoparticle. *Phys. Rev.*, **D14**:3432–3450, 1976. Erratum: *ibid.* D**18**,2199(1978). 30

[36] C. G. CALLAN JR., R. F. DASHEN, AND D. J. GROSS. Toward a theory of the strong interactions. *Phys. Rev.*, **D17**:2717, 1978. 30

[37] G. P. LEPAGE AND S. J. BRODSKY. Exclusive processes in perturbative quantum chromodynamics. *Phys. Rev.*, **D22**:2157, 1980. 30

[38] S. D. GŁAZEK AND A. P. SZCZEPANIAK. Special relativity constraints on the effective constituent theory of hybrids. *Phys. Rev.*, **D67**:034019, 2003. 30

[39] S. D. GŁAZEK AND T. MASŁOWSKI. Renormalized Poincaré algebra for effective particles in quantum field theory. *Phys. Rev.*, **D65**:065011, 2002. 31, 40, 52, 63

[40] S. D. GŁAZEK. Boost-invariant running couplings in effective Hamiltonians. *Phys. Rev.*, **D60**:105030, 1999. 33, 35, 82, 101, 143

[41] S. J. BRODSKY, J. R. HILLER, AND G. MCCARTOR. Pauli-Villars as a nonperturbative ultraviolet regulator in discretized light-cone quantization. *Phys. Rev.*, **D58**:025005, 1998. 33

[42] S. J. BRODSKY, J. R. HILLER, AND G. MCCARTOR. Application of Pauli-Villars regularization and discretized light-cone quantization to a (3+1)-dimensional model. *Phys. Rev.*, **D60**:054506, 1999. 33

[43] T. KINOSHITA (ED.). *Quantum electrodynamics*. 1990. Singapore, Singapore: World Scientific (1990) (Advanced series on directions in high energy physics, 7). 33

[44] K. G. WILSON. A model of coupling constant renormalization. *Phys. Rev.*, **D2**:1438, 1970. 34, 44, 55, 193

[45] S. D. GŁAZEK. Similarity renormalization group approach to boost invariant Hamiltonian dynamics. *Acta Phys. Polon.*, **B29**:1979–2064, 1998. 34, 35, 51, 54, 63

[46] S. D. GŁAZEK. Running couplings in Hamiltonians. *Acta Phys. Polon.*, **B31**:909–930, 2000. 34, 51, 54, 63

[47] S. D. GŁAZEK AND J. MŁYNIK. Optimization of perturbative similarity renormalization group for Hamiltonians with asymptotic freedom and bound states. *Phys. Rev.*, **D67**:045001, 2003. see also [112]. 35, 52, 54

[48] A. HARINDRANATH AND R. J. PERRY. Lowest order mass corrections for Yukawa (1+1) in light front perturbation theory. *Phys. Rev.*, **D43**:492–498, 1991. 43

[49] C. BLOCH. *Nucl. Phys.*, **6**:329, 1958. 44, 55, 193

[50] S. D. GŁAZEK AND K. G. WILSON. Perturbative renormalization group for Hamiltonians. *Phys. Rev.*, **D49**:4214–4218, 1994. 45

[51] I. TAMM. Relativistic interaction of elementary particles. *J. Phys. (USSR)*, **9**:449, 1945. 48

[52] S. M. DANCOFF. Nonadiabatic meson theory of nuclear forces. *Phys. Rev.*, **78**:382–385, 1950. 48

[53] R. P. FEYNMAN, M. KISLINGER, AND F. RAVNDAL. Current matrix elements from a relativistic quark model. *Phys. Rev.*, **D3**:2706–2732, 1971. 73

[54] D. GROMES AND I. O. STAMATESCU. Linearly rising quark potential and baryon spectrum. *Nucl. Phys.*, **B112**:213, 1976.

[55] W. FISCHLER. Quark - anti-quark potential in QCD. *Nucl. Phys.*, **B129**:157–174, 1977.

[56] J. L. RICHARDSON. The heavy quark potential and the upsilon, J / psi systems. *Phys. Lett.*, **B82**:272, 1979.

[57] D. P. STANLEY AND D. ROBSON. Nonperturbative potential model for light and heavy quark anti-quark systems. *Phys. Rev.*, **D21**:3180–3196, 1980.

[58] E. EICHTEN, K. GOTTFRIED, T. KINOSHITA, K. D. LANE, AND T-M. YAN. Charmonium: The model. *Phys. Rev.*, **D17**:3090, 1978. Erratum-ibid.D21:313,1980.

[59] E. EICHTEN, K. GOTTFRIED, T. KINOSHITA, K. D. LANE, AND T-M. YAN. Charmonium: Comparison with experiment. *Phys. Rev.*, **D21**:203, 1980.

[60] M. PETER. The static quark-antiquark potential in QCD to three loops. *Phys. Rev. Lett.*, **78**:602–605, 1997.

[61] S. EIDELMAN ET AL. Review of particle physics. *Phys. Lett.*, **B592**:1, 2004. 49

[62] R. P. FEYNMAN. *Photon-hadron interactions*. Benjamin, New York, 1972. 49

[63] R. P. FEYNMAN. Very high-energy collisions of hadrons. *Phys. Rev. Lett.*, **23**:1415–1417, 1969. 49

[64] J. B. KOGUT AND L. SUSSKIND. The parton picture of elementary particles. *Phys. Rept.*, **8**:75–172, 1973. 49

[65] K. G. WILSON. Confinement of quarks. *Phys. Rev.*, **D10**:2445–2459, 1974. see also K. G. Wilson, in *C76-01-19.14* CLNS-327, *presented at Coral Gables Conf., Miami, Fla, Jan 19-22, 1976*; and references therein. 50

[66] R. BALIAN, J. M. DROUFFE, AND C. ITZYKSON. Gauge fields on a lattice. 1. General outlook. *Phys. Rev.*, **D10**:3376, 1974.

[67] B. A. THACKER AND G. P. LEPAGE. Heavy quark bound states in lattice QCD. *Phys. Rev.*, **D43**:196–208, 1991.

[68] G. BODWIN (ED.) ET AL. Lattice field theory. Proceedings, 22nd International Symposium, Lattice 2004, Batavia, USA, June 21-26, 2004.

[69] K. G. WILSON. The origins of lattice gauge theory. *Nucl. Phys. Proc. Suppl.*, **140**:3–19, 2005.

[70] M. A. SHIFMAN, A. I. VAINSHTEIN, AND V. I. ZAKHAROV. QCD and resonance physics: Applications. *Nucl. Phys.*, **B147**:448–518, 1979. ; and references therein. 50

[71] S. D. GŁAZEK AND K. G. WILSON. Asymptotic freedom and bound states in Hamiltonian dynamics. *Phys. Rev.*, **D57**:3558–3566, 1998. 52

[72] F. WEGNER. *Ann. Phys. (Leipzig)*, **3**:77, 1994. 52

[73] F. WEGNER. *Phys. Rep.*, **358**:77, 2001. 52

[74] S. D. GŁAZEK. Dynamics of effective gluons. *Phys. Rev.*, **D63**:116006, 2001. 52, 54, 59, 82, 128

[75] M. M. BRISUDOVÁ. Small x divergences in a heavy quark-antiquark state. *Nucl. Phys. Proc. Suppl.*, **108**:173–175, 2002. 52

[76] M. M. BRISUDOVÁ. Small x divergences in the similarity RG approach to LF QCD. *Mod. Phys. Lett.*, **A17**:59–82, 2002. 52

[77] S. D. GŁAZEK. Harmonic oscillator force between heavy quarks. *Phys. Rev.*, **D69**:065002, 2004. 52

[78] S. GŁAZEK, A. HARINDRANATH, S. PINSKY, J. SHIGEMITSU, AND K. G. WILSON. On the relativistic bound state problem in the light front Yukawa model. *Phys. Rev.*, **D47**:1599–1619, 1993. 53

[79] M. M. BRISUDOVÁ AND R. J. PERRY. Note on restoring manifest rotational symmetry in hyperfine and fine structure in light-front QED. *Phys. Rev.*, **D54**:6453–6458, 1996.

[80] M. M. BRISUDOVÁ AND R. J. PERRY. Initial bound state studies in light-front QCD. *Phys. Rev.*, **D54**:1831–1843, 1996. 54

[81] M. M. BRISUDOVÁ, R. J. PERRY, AND K. G. WILSON. Quarkonia in Hamiltonian light-front QCD. *Phys. Rev. Lett.*, **78**:1227–1230, 1997. 54

[82] B. D. JONES, R. J. PERRY, AND S. D. GŁAZEK. Nonperturbative QED: An analytical treatment on the light front. *Phys. Rev.*, **D55**:6561–6583, 1997.

[83] R. J. PERRY. *in Hadron Physics 94: Topics on the Structure and Interaction of Hadron Systems, Proceedings, Gramado, Brazil, edited by V. Herscovitz et al.* Word Scientific, Singapore, 1995. , and revised version hep-th/9411037.

[84] R. J. PERRY AND K. G. WILSON. Perturbative renormalizability with an infinite number of relevant and marginal operators. *Nucl. Phys.*, **B403**:587–604, 1993.

[85] R. J. PERRY. A renormalization group approach to Hamiltonian light front field theory. *Ann. Phys.*, **232**:116–222, 1994.

[86] R. J. PERRY. Using Wilson's renormalization group to repair symmetries. *Phys. Rept.*, **348**:33–75, 2001. ; and references therein. 54

[87] S. D. GŁAZEK. Weinberg equation with spin. Master's thesis, Warsaw University, Institute of Theoretical Physics, 1980. See also [113]. 60

[88] S. D. GŁAZEK AND K. G. WILSON. Renormalization of overlapping transverse divergences in a model light front Hamiltonian. *Phys. Rev.*, **D47**:4657–4669, 1993. 66

[89] D. J. GROSS AND F. WILCZEK. Ultraviolet behavior of non-abelian gauge theories. *Phys. Rev. Lett.*, **30**:1343–1346, 1973. 72

[90] H. D. POLITZER. Reliable perturbative results for strong interactions? *Phys. Rev. Lett.*, **30**:1346–1349, 1973. 72

[91] G. 'T HOOFT AND M. J. G. VELTMAN. Regularization and renormalization of gauge fields. *Nucl. Phys.*, **B44**:189–213, 1972. 73

[92] J. C. COLLINS. *Renormalization. An introduction to renormalization, the renormalization group, and the operator product expansion*. Cambridge University Press, 1984. 73

[93] G. LEIBBRANDT. Introduction to the technique of dimensional regularization. *Rev. Mod. Phys.*, **47**:849, 1975. 73

[94] J. D. BJORKEN. Elements of quantum chromodynamics. 1979. Presented at the SLAC Summer Institute on Particle Physics, Stanford, Calif., Jul 9-20, 1979. 73, 88

[95] S. WEINBERG. Dynamics at infinite momentum. *Phys. Rev.*, **150**:1313–1318, 1966. 74, 81

[96] J. D. BJORKEN AND S. D. DRELL. *Relativistic Quantum Fields*. McGraw-Hill, 1965. 79, 80, 81, 175, 180, 182

[97] S. WEINBERG. *Teoria pól kwantowych (The Quantum Theory of Fields)*. Wydawnictwo Naukowe PWN, 1999.

[98] M. E. PESKIN AND D. V. SCHROEDER. *An Introduction to Quantum Field Theory*. Addison-Wesley Publishing Company, 1995. 80, 169

[99] S. J. BRODSKY, R. ROSKIES, AND R. SUAYA. Quantum electrodynamics and renormalization theory in the infinite momentum frame. *Phys. Rev.*, **D8**:4574, 1973. 76

[100] I. BIAŁYNICKI-BIRULA. *Wstęp do teorii pól kwantowych*. Państwowe Wydawnictwo Naukowe, 1971. 77, 177

[101] H. LEHMANN. On the properties of propagation functions and renormalization contants of quantized fields. *Nuovo Cim.*, **11**:342–357, 1954. 79

[102] G. KALLEN. *Helv. Phys. Acta*, **25**:417, 1952. 79

[103] F. LOW. Boson-fermion scattering in the Heisenberg representation. *Phys. Rev.*, **97**:1392–1398, 1955. 80

[104] H. LEHMANN, K. SYMANZIK, AND W. ZIMMERMANN. On the formulation of quantized field theories. *Nuovo Cim.*, **1**:205–225, 1955. 80

[105] N. E. LIGTERINK AND B. L. G. BAKKER. Equivalence of light front and covariant field theory. *Phys. Rev.*, **D52**:5954–5979, 1995. 82

[106] N. SCHOONDERWOERD. Light-front Hamiltonian field theory: Towards a relativistic description of bound states. 1998. Ph.D. Thesis, Vrije Universiteit Amsterdam, hep-ph/9811317. 82

[107] G. LEIBBRANDT. The light cone gauge in Yang-Mills theory. *Phys. Rev.*, **D29**:1699, 1984. 106

[108] K. BARDAKCI AND C. B. THORN. A worldsheet description of large N(c) quantum field theory. *Nucl. Phys.*, **B626**:287–306, 2002. 109

[109] C. B. THORN. Renormalization of quantum fields on the lightcone worldsheet. I: Scalar fields. *Nucl. Phys.*, **B699**:427–452, 2004. 109

[110] M. BURKARDT AND A. LANGNAU. Rotational invariance in light cone quantization. *Phys. Rev.*, **D44**:3857–3867, 1991. 119

[111] R. P. FEYNMAN. Space-time approach to nonrelativistic quantum mechanics. *Rev. Mod. Phys.*, **20**:367–387, 1948. 119

[112] S. D. GŁAZEK AND J. MŁYNIK. Accuracy estimate for a relativistic Hamiltonian approach to bound-state problems in theories with asymptotic freedom. *Acta Phys. Polon.*, **B35**:723–744, 2004. 121

[113] S. D. GŁAZEK. Deuteron structure in the elastic electron deuteron scattering. *Acta Phys. Polon.*, **B14**:893–902, 1983. 124

[114] P. DANIELEWICZ AND J. M. NAMYSŁOWSKI. Relativistic repulsive effects in nonrelativistic systems. *Phys. Lett.*, **B81**:110, 1979. 133

[115] P. P. SRIVASTAVA AND S. J. BRODSKY. Light-front quantized QCD in light-cone gauge. *Phys. Rev.*, **D64**:045006, 2001. 161

[116] D. E. SOPER. Massive quantum electrodynamics in the infinite momentum frame. *Phys. Rev.*, **D4**:1620, 1971.

[117] G. P. LEPAGE, S. J. BRODSKY, T. HUANG, AND P. B. MACKENZIE. Hadronic wave functions in QCD. Invited talk given at Banff Summer Inst. on Particle Physics, Banff, Alberta, Canada, Aug 16-28, 1981; Published in Banff Summer Inst.1981:0083.

[118] A. BASSETTO, G. NARDELLI, AND R. SOLDATI. Yang-Mills theories in algebraic noncovariant gauges: Canonical quantization and renormalization. Singapore, Singapore: World Scientific (1991).

[119] G. 'T HOOFT AND M. J. G. VELTMAN. Scalar one loop integrals. *Nucl. Phys.*, **B153**:365–401, 1979. 191

Appendix A

The notation used in this thesis

In the equations := means “is defined as”. As in the Pascal programming language, the quantity on the side of the colon sign is the one that is defined. Thus, $a := b$ is read as "a is defined as a quantity equal to b", while $a =: b$ is read as “b is defined as a quantity equal to a.

A.1 ET vs LF

When referring to the equal-time (ET) formulation of QFT, a Hamiltonian (H) means energy (zero component of four momentum) P^0. When referring to the light-front (LF) formulation of QFT, a Hamiltonian (H) means minus component of four momentum P^-.

Standard invariant integration measures and momentum-conservation delta functions have different meaning in ET and LF expressions. These are summarized in the following table.

Notation		Explicit meaning	
		in ET	in LF
Hamiltonian	$: H$	P^0	P^-
Integration measures	$: [k]$	$[k] := \dfrac{\mathrm{d}^3k}{(2\pi)^3 2k^0}$	$[k] := \dfrac{\mathrm{d}^2k^\perp \mathrm{d}k^+}{(2\pi)^3 2k^+}$
3-momentum	$: \vec{k}$	$\vec{k} = (k^1, k^2, k^3)$	$\vec{k} = (k^+, k^1, k^2)$
Space 3-vector	$: \vec{x}$	$\vec{x} = (x^1, x^2, x^3)$	$\vec{x} = (x^-, x^1, x^2)$
Scalar product	$: \vec{x}\vec{k}$	$\vec{x}\vec{k} = \sum_{i=1}^{3} x^i k^i = = -x^\mu k_\mu\big\vert_{x^0=0}$	$\vec{x}\vec{k} = -\frac{1}{2}x^- k^+ + x^1 k^1 + x^2 k^2 = = -x^\mu k_\mu\big\vert_{x^+=0}$
3-momentum conservation δ-s	$: \widetilde{\delta}(k)$	$\widetilde{\delta}(k) := (2\pi)^3 \delta^3(\vec{k})$	$\widetilde{\delta}(k) := 2(2\pi)^3 \delta^2(k^\perp)\delta(k^+) = =: 2(2\pi)^3 \delta^3(\vec{k})$

In dimensions other than 3+1, corresponding powers of (2π) and dimensions of $\delta(k^\perp)$ are modified accordingly.

A.2 Special notation used for RGPEP

The curly brackets $\{\quad\}_0$ denote the additional free energy denominator. For any operator A

$$A = \int [k_{cre1} \ldots k_{ani1} \ldots] A(k_{cre1} \ldots k_{ani1} \ldots)\, a^\dagger_{cre1} \ldots a_{ani1} \ldots \tag{A.1}$$

$\{A\}_0$ is

$$\left\{A\right\}_0 = \int [k_{cre1}\dots k_{ani1}\dots] \frac{1}{\left(\sum_{i\in ani} k^-_{0,i}\right) - \left(\sum_{j\in cre} k^-_j\right)} A(k_{cre1}\dots k_{ani1}\dots)\, a^\dagger_{cre1}\dots a_{ani1}\dots \tag{A.2}$$

The same denominator with a difference of k^- is indicated in Hamiltonian diagrams by dashed underlining, and a black arrow which indicates which k^- is subtracted from which one. For example,

$$\left\{A\right\}_0 = \quad \boxed{A} \quad = (-1) * \quad \boxed{A} \tag{A.3}$$

Similar figures without the dashed underlining and without the black arrow (such as in Eq. (5.111) on the page 100) are used to label intermediate states, and do not indicate energy denominators.

For the operator A of Eq. (A.1) and function of momenta f_λ, the symbol $f_\lambda A$ means

$$f_\lambda A = \int [k_{cre1}\dots k_{ani1}\dots]\, f_\lambda(k_{cre1}\dots k_{ani1}\dots) A(k_{cre1}\dots k_{ani1}\dots)\, a^\dagger_{cre1}\dots a_{ani1}\dots\,. \tag{A.4}$$

Likewise, all momentum dependent factors written in front of an operator should be understood as part of the integration kernel. For example, if the set of particles annihilated by A is denoted b, and the set of created particles a, then (A.2) could also be written as

$$\left\{A\right\}_0 = \frac{P^+_{ba}}{ba} A_{ba} = \zeta_{ba} A_{ba}\,. \tag{A.5}$$

When talking about canonical and effective theories, a plain letter H denotes the canonical Hamiltonian, and a calligraphic $\mathcal{H}_\lambda$ the effective Hamiltonian (usually followed by a letter λ denoting the width of the effective Hamiltonian). All terms of the effective Hamiltonian $\mathcal{H}_\lambda$ have a limiting form factor f_λ. Elements of $\mathcal{H}_\lambda$ *without the common factor* f_λ are denoted $\mathcal{G}$:

$$\mathcal{H}_\lambda =: f_\lambda \mathcal{G}\,. \tag{A.6}$$

A shorthand notation for momentum combinations appearing in the expressions for $\mathcal{H}_\lambda$ are defined as follows:

$$\pi_{abc} := P^+_{ba} ba + P^+_{bc} bc \tag{A.7}$$

$$\Pi_{abc} := \frac{\pi_{abc}}{ba^2 + bc^2} \tag{A.8}$$

$$\zeta_{ab} := \frac{P^+_{ab}}{ab}\,. \tag{A.9}$$

A.3 Shorthand notation of momentum differences

For the introduction of similarity form factors f_λ, a notation of two letters (e.g. ab) marking invariant mass differences is used in the following way [74] (see also Fig.4.1). For each interaction term with intermediate states a and b:

1. P^+_{ab} is the sum of p^+ momenta of all the particles of state a that were involved in any interaction (due to three-momentum conservation, this is the same as a similar sum of p^+ for the state b)

2. ab is defined as
$$ab = \left(p^-_{0,a} - p^-_{0,b}\right) P^+_{ab} \,. \tag{A.10}$$

ab defined this way does not depend on spectator particles (because their energies p^- cancel out in the difference of p^- in (A.10) and do not contribute to P^+_{ab}). Also, when one introduces child-parent relative momenta as in Appendix B.2.1, ab depends on these relative momenta only and not on the total momentum.

For each interaction term built of connected Hamiltonian expressions, the intermediate "states" are ordered: it is thus clear which is the leftmost, which is the next, and so on. These states are labeled "a,b,c..." from left to right. See for example Fig.4.1.

A.4 Other conventions

In Appendices E and F, color matrices T are used. They fulfill:

$$(T^a)_{ij}(T^a)_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \tag{A.11}$$

$$(T^a)_{ik}(T^a)_{kl} = \frac{4}{3}\delta_{il} \tag{A.12}$$

A.5 Dimensions of fields and couplings in various theories

The dimensions of quantities involved a Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 \tag{A.13}$$

depend on number of space-time dimensions in the following way:

	1+1	3+1	5+1
$\mathcal{L}$	m^2	m^4	m^6
∂^2, m^2	m^2	m^2	m^2
ϕ	1	m	m^2
g	m^2	m	1

Appendix B

Light-front coordinates

B.1 Coordinates; momenta; scalar product

I use the so-called Brodsky-Lepage convention for defining light-front coordinates. (This differs from the Kogut-Soper convention in its numerical factors only; see [25].) The light-front time is defined as

$$x^+ = x^0 + x^3 \tag{B.1}$$

and the spatial coordinate is

$$x^- = x^0 - x^3 \tag{B.2}$$

Note, that the Jacobian of change of variables is:

$$\left| \frac{\partial(x^+, x^-, x^1, x^2)}{\partial(x^0, x^1, x^2, x^3)} \right| = 2 \tag{B.3}$$

Momenta $p^-, p^+, p^\perp$ and the scalar product is given in Section 3.3.2.

B.2 Light-front momenta

B.2.1 Relative momenta

For two particles of momenta p_1 and p_2, the total momentum P and the relative momenta x, $\kappa^\perp$ are defined as follows.

$$\begin{aligned} p_1^+ &= xP^+ , && \text{(B.4)} \\ p_2^+ &= (1-x)P^+ , && \text{(B.5)} \\ p_1^\perp &= xP^\perp + \kappa^\perp , && \text{(B.6)} \\ p_2^\perp &= (1-x)P^\perp - \kappa^\perp . && \text{(B.7)} \end{aligned}$$

When there are more than two particles, for any particle of momentum $(p^+, p^\perp)$ involved in an interaction, one can define its momentum $(x, \kappa^\perp)$ relative to the total momentum $(P^+, P^\perp)$:

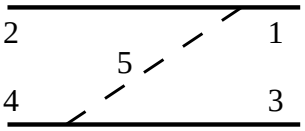
$$\begin{aligned} p^+ &=: xP^+ && \text{(B.8)} \\ p^\perp &=: xP^\perp + \kappa^\perp && \text{(B.9)} \end{aligned}$$

(in a shorthand notation $\vec{p} = x\vec{P} + \kappa^\perp$, where the vector sign denotes $(+,1,2)$ components and $\kappa^+ := 0$). Alternatively, one can look at the interaction vertex and define relative child-parent momentum $(x_{c/p}, \kappa_{c/p})$:

$$p_c^+ =: x_{c/p} P_p^+ \tag{B.10}$$
$$p_c^\perp =: x_{c/p} P_p^\perp + \kappa_{c/p}^\perp \,. \tag{B.11}$$

(in shorthand notation $\vec{p}_c =: x_{c/p}\vec{P}_p + \kappa_{c/p}^\perp$).

For example, in a term



(B.12)

momenta $x_1, \kappa_{13}, x_2, \kappa_{24}$ relative to the total momentum are defined by:

$$\vec{p}_1 = x_1\vec{P} + \kappa_{13}^\perp \tag{B.13}$$
$$\vec{p}_3 = (1-x_1)\vec{P} - \kappa_{13}^\perp \quad \text{i.e. } x_3 = 1-x_2,\ \kappa_{31}^\perp = -\kappa_{13}^\perp \tag{B.14}$$
$$\vec{p}_2 = x_2\vec{P} + \kappa_{24}^\perp \tag{B.15}$$
$$\vec{p}_4 = (1-x_2)\vec{P} - \kappa_{24}^\perp \tag{B.16}$$
$$\vec{p}_5 = (x_1-x_2)\vec{P} + \left(\kappa_{13}^\perp - \kappa_{24}^\perp\right) \,. \tag{B.17}$$

Note, that in each state with particles $\{i\}$:

$$\sum_i x_i = 1 \tag{B.18}$$
$$\sum_i \kappa_i^\perp = 0 \,. \tag{B.19}$$

The positiveness of all p^+ momenta means that all x-es are positive.

For regularization, the child-parent relative momenta are used. For the diagram (B.12) they are defined by:

$$\begin{aligned} \vec{p}_2 = x_{2/1}\vec{p}_1 + \kappa_{2/1}^\perp \quad &\Rightarrow \quad x_{2/1} = \frac{x_2}{x_1}, \qquad \kappa_{2/1}^\perp = \kappa_{24}^\perp - \frac{x_2}{x_1}\kappa_{13}^\perp \\ \vec{p}_5 = x_{5/1}\vec{p}_1 + \kappa_{5/1}^\perp \quad &\Rightarrow \quad x_{5/1} = \frac{x_1-x_2}{x_1}, \quad \kappa_{5/1}^\perp = \frac{x_2}{x_1}\kappa_{13}^\perp - \kappa_{24}^\perp \,, \end{aligned} \tag{B.20}$$

and in analogous way for $x_{5/4}$, $\kappa_{5/4}^\perp$, $x_{2/4}$ and $\kappa_{2/4}$. Note that for child particles $\{d_i\}$ in each vertex:

$$\sum_{c_i} x_{c_i/p} = 1 \tag{B.21}$$
$$\sum_{c_i} \kappa_{c_i/p}^\perp = 0 \,. \tag{B.22}$$

An integration $\int [p_1 p_2] \widetilde{\delta}(p_1 + p_2 - p_3)$ can be re-expressed in terms of the momenta relative to the total momentum:

$$\int [p_1 p_2] \widetilde{\delta}(p_1 + p_2 - p_3) \cdots = \frac{1}{2(2\pi)^3 P^+} \int \frac{\mathrm{d}^2\kappa^\perp \mathrm{d}x \theta(x)\theta(x_3 - x)}{x(x_3 - x)} \ldots \tag{B.23}$$

or by the relative child-parent momenta:

$$\int [p_1 p_2]\widetilde{\delta}(p_1+p_2-p_3)\cdots = \frac{1}{2(2\pi)^3 p_3^+}\int \frac{\mathrm{d}^2\kappa^\perp \mathrm{d}x\theta(x)\theta(1-x)}{x(1-x)}\ldots \tag{B.24}$$

Note different integration ranges (θ-functions), different denominators in front of, and under the integrals.

B.2.2 Relative pseudo-equal-time momenta

When describing relative motion of two particles of the same mass m, one can introduce a different set of relative momenta [114], especially convenient when one looks at the nonrelativistic region:

$$k^\perp := \kappa^\perp \tag{B.25}$$

$$k_3 := \sqrt{\frac{\kappa^{\perp 2}+m^2}{x(1-x)}}\left(x-\frac{1}{2}\right), \tag{B.26}$$

or, equivalently:

$$x = \frac{1}{2}\left(1+\frac{k_3}{\sqrt{\vec{k}^2+m^2}}\right). \tag{B.27}$$

The definition of momentum k^3 is chosen in such a way that the free invariant mass of the pair can be written as:

$$M^2 = \frac{\kappa^{\perp 2}+m^2}{x(1-x)} = 4\left(k^{\perp 2}+(k^3)^2+m^2\right), \tag{B.28}$$

and is equal to the equal-time expression for the invariant mass in CMS:

$$M^2 = P^{02}-\vec{P}^2 = |CMS| = (2E)^2 = 4\left(\vec{k}^2+m^2\right). \tag{B.29}$$

When changing integration variables, the Jacobian is:

$$\frac{\partial(\kappa^1\kappa^2\kappa^3)}{\partial(k^1k^2k^3)} = \frac{1}{2}\frac{k^{\perp 2}+m^2}{\sqrt{\vec{k}^2+m^2}^3} \tag{B.30}$$

and the common denominator in the integration measure, $x(1-x)$, is:

$$x(1-x) = \frac{1}{4}\frac{k^{\perp 2}+m^2}{\vec{k}^2+m^2}. \tag{B.31}$$

Hence,

$$\frac{dx d^2\kappa^\perp}{x(1-x)} = 2\frac{d^3k}{\sqrt{\vec{k}^2+m^2}}. \tag{B.32}$$

B.3 Free scalar fields

The Fourier expansion of a free scalar field $\phi(x)$ at the light-front time $x^+ = 0$ is:

$$\phi(x) = \int [k](a_k^\dagger e^{ik_\mu x^\mu} + a_k e^{-ik_\mu x^\mu})_{x^+=0} \tag{B.33}$$

$$k_\mu x^\mu\big|_{x^+=0} = \frac{1}{2}k^+x^- - k^\perp x^\perp \,. \tag{B.34}$$

Creation and annihilation operators fulfill the commutation relation:

$$\left[a_p, a_k^\dagger\right] = 2(2\pi)^3 k^+ \delta^3(k-p) \,. \tag{B.35}$$

where $\delta^3(p) = \delta^2(p^\perp)\delta(p^+)$.

A state created by $a_p^\dagger$ is denoted $|p\rangle$:

$$|p\rangle = a_p^\dagger|0\rangle \,. \tag{B.36}$$

B.4 Free fermion fields; notation of spinors

The fermion field operator fulfilling the free Euler-Lagrange equations (the Dirac equation), $\psi_m(x)$, is expanded for light-front "time" $x^+ = 0$ in terms of creation and annihilation operators as:

$$\psi_m(x)|_{x^+=0} = \sum_\sigma \int [p] \left[b_{p\sigma} u_{mp\sigma} e^{-ip_\mu x^\mu} + d_{p\sigma}^\dagger v_{mp\sigma} e^{ip_\mu x^\mu}\right]_{x^+=0} \,. \tag{B.37}$$

Creation and annihilation operators fulfill:

$$\{b_{p\sigma}, b_{k\lambda}^\dagger\} = 2(2\pi)^3 k^+ \delta^3(k-p)\delta_{\sigma\lambda} \tag{B.38}$$

$$\{d_{p\sigma}, d_{k\lambda}^\dagger\} = 2(2\pi)^3 k^+ \delta^3(k-p)\delta_{\sigma\lambda} \tag{B.39}$$

The spinors of particles of zero velocity are equal:

$$\begin{aligned} u_\uparrow &= \sqrt{2m}\begin{pmatrix}\chi_1\\0\end{pmatrix} \quad , \quad u_\downarrow = \sqrt{2m}\begin{pmatrix}\chi_{-1}\\0\end{pmatrix} , \\ v_\uparrow &= \sqrt{2m}\begin{pmatrix}0\\\chi_{-1}\end{pmatrix} \quad , \quad v_\downarrow = \sqrt{2m}\begin{pmatrix}0\\-\chi_1\end{pmatrix} , \end{aligned} \tag{B.40}$$

where the two-component spinors χ are:

$$\chi_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad , \quad \chi_{-1} = \begin{pmatrix}0\\1\end{pmatrix} . \tag{B.41}$$

Spinors of particles at motion are obtained by acting on above particle-rest-frame spinors with matrix representation of Lorentz transformations belonging to the small group (transformations which do not change the $x^+ = 0$ hyperplane). The result for fermions is:

$$u_{mp\sigma} = \frac{1}{\sqrt{mp^+}}\left[p^+\Lambda_+ + \Lambda_-(m + \alpha^\perp p^\perp)\right] u_\sigma \,, \tag{B.42}$$

and for anti-fermions:

$$v_{mp\sigma} = \frac{1}{\sqrt{mp^+}}\left[p^+\Lambda_+ + \Lambda_-(m+\alpha^\perp p^\perp)\right]v_\sigma\,. \tag{B.43}$$

Note that $\sqrt{m}$ in the denominators of (B.40) and in (B.42)-(B.43) cancels, and there is a well-defined, finite limit $\lim_{m\to 0} u_{mp\sigma}$.

Equation (B.42)-(B.43) use projection operators $\Lambda_\pm$. These are defined as follows:

$$\Lambda_\pm = \frac{1}{2}\gamma^0\left(\gamma^0 \pm \gamma^3\right) = \frac{1}{2}\gamma^0\gamma^\pm = \frac{1}{2}(1\pm\alpha^3)\,. \tag{B.44}$$

They are indeed projection operators:

$$\Lambda_+\Lambda_+ = \Lambda_+ \tag{B.45}$$
$$\Lambda_+\Lambda_- = 0\,. \tag{B.46}$$

Using:

$$\gamma^+\gamma^+ = 0 \tag{B.47}$$
$$\gamma^+\gamma^- = 4\Lambda_- \tag{B.48}$$
$$\gamma^+\gamma^\perp = -\gamma^\perp\gamma^+ \tag{B.49}$$

one sees that products of $\Lambda_\pm$ and Dirac matrices are:

$$\Lambda_+\gamma^+ = 0 \tag{B.50}$$
$$\Lambda_+\gamma^- = \gamma^- = \gamma^-\Lambda_- \tag{B.51}$$
$$\Lambda_+\gamma^\perp = \gamma^\perp\Lambda_+\,. \tag{B.52}$$

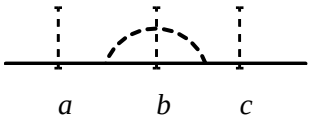
In Dirac representation $\Lambda_\pm$ fulfill:

$$\Lambda_+ = \frac{1+\alpha^3}{2} = \frac{1}{2}\begin{bmatrix} 1 & \sigma^3 \\ \sigma^3 & 1 \end{bmatrix} \tag{B.53}$$
$$\Lambda_-\alpha^\perp = \frac{1}{2}\begin{bmatrix} -\sigma^3\sigma^\perp & \sigma^\perp \\ \sigma^\perp & -\sigma^3\sigma^\perp \end{bmatrix}\,. \tag{B.54}$$

Products of spinors u or v for equal momenta $\vec{p}$ are:

$$\bar{u}_{p\sigma}\gamma^+ u_{p\sigma'} = 2p^+\delta_{\sigma\sigma'} \tag{B.55}$$
$$\bar{v}_{p\sigma}\gamma^+ v_{p\sigma'} = 2p^+\delta_{\sigma\sigma'} \tag{B.56}$$

A sum over polarizations of intermediate fermion in a self-interaction loop



gives:

$$\sum_b \bar{u}_{P,a}u_b\bar{u}_b u_{P,c} = \delta_{ac}\frac{m^2(1+x)^2+\kappa^2}{x}\,, \tag{B.57}$$

where x is a relative child-parent momentum of the intermediate fermion.

In practice, it is convenient to express standard matrix elements of spinors u in terms of relative momenta and two-component spinors χ. For two fermions of momenta p_1, p_2:

$$\vec{p}_1 = x_1\vec{P} + \kappa_1^\perp \tag{B.58}$$
$$\vec{p}_2 = x_2\vec{P} + \kappa_2^\perp \,, \tag{B.59}$$

involved in a Yukawa vertex, the corresponding product of spinors can be written as:

$$\bar{u_1}u_2 = \frac{1}{\sqrt{x_1x_2}}\chi_1^\dagger\left[m(x_1+x_2) - \sigma^3\sigma^\perp\left(x_1\kappa_2^\perp - \kappa_1^\perp x_2\right)\right]\chi_2 \,. \tag{B.60}$$

Appendix E presents similar products for QED and QCD vertices.

B.5 Free vector fields; polarization vectors

In the light-front gauge $A^+ = 0$, the vector potential is:

$$A^\mu = \left(A^+ = 0, A^-, A^\perp\right) \,. \tag{B.61}$$

The independent variables are $A^\perp$, while A^- in free theory is constrained to be:

$$A^- = \frac{2i\partial^\perp A^\perp}{i\partial^+} \,. \tag{B.62}$$

The Fourier transform at $x^+ = 0$ is:

$$A^\mu(x) = \sum_\sigma \int [k]\left(a_{k\sigma}\varepsilon_{k\sigma}^\mu e^{-ikx} + a_{k\sigma}^\dagger \varepsilon_{k\sigma}^{*\mu} e^{ikx}\right)_{x^+=0} \,. \tag{B.63}$$

The creation operators fulfill:

$$\left[a_{k\sigma'}, a_{p\sigma}^\dagger\right] = 2(2\pi)^3 k^+ \delta^3(k-p)\delta_{\sigma'\sigma} \tag{B.64}$$

(σ denotes polarization) and polarization vectors have to be:

$$\varepsilon^- = \frac{2k^\perp\varepsilon^\perp}{k^+} \tag{B.65}$$
$$\varepsilon^+ = 0 \tag{B.66}$$

with $\varepsilon^\perp$ being two independent parameters characterizing the polarization of the field. They can be chosen as follows:

$$\varepsilon_1^\perp = \begin{pmatrix}1\\0\end{pmatrix} \tag{B.67}$$
$$\varepsilon_{-1}^\perp = \begin{pmatrix}0\\1\end{pmatrix} \,. \tag{B.68}$$

They are normalized as follows:

$$\varepsilon_\sigma^i \varepsilon_{\sigma'}^{*i} = \delta_{\sigma\sigma'} \,, \tag{B.69}$$

(with summation over repeated superscript i), and sum over polarizations gives:

$$\varepsilon_\sigma^i \varepsilon_\sigma^{*j} = \varepsilon_1^i \varepsilon_1^{*j} + \varepsilon_{-1}^i \varepsilon_{-1}^{*j} = \delta^{ij} \,. \tag{B.70}$$

Appendix C

Details of RGPEP

C.1 Effective particles

C.1.1 Definition of the effective particles creation operators

Effective particles creation and annihilation operators q_λ are defined as operators unitarily equivalent to the initial, bare operators q_∞:

$$q_\infty = U_\lambda^\dagger q_\lambda U_\lambda \tag{C.1}$$

U_λ is an operator acting in the Fock space. It can be expressed by any complete set of creation and annihilation operators q_β. Functional dependence of U_λ on q_∞ and q_λ could theoretically have been different. It is interesting to notice that this is not, in fact, the case: the rotation operator expressed in both bases has exactly the same form. Let U_λ and $\mathcal{V}_\lambda$ be functional dependencies of the same unitary-rotation operator expressed in a different operator basis:

$$U_\lambda(q_\lambda) = \mathcal{V}_\lambda(q_\infty) = \hat{A}_\lambda \,. \tag{C.2}$$

But,

$$\underline{\mathcal{V}_\lambda(q_\infty)} = U_\lambda(q_\lambda) = \hat{A}\cdot U_\lambda(q_\infty)\cdot\hat{A}^\dagger \,. \tag{C.3}$$

$\hat{A}$ can be, for example, written as $\mathcal{V}_\lambda(q_\infty)$, and canceled with the underlined $\mathcal{V}$. This leads to:

$$U_\lambda(q_\infty)\cdot\mathcal{V}_\lambda^\dagger(q_\infty) = 1 \tag{C.4}$$

$$U_\lambda(q_\infty) = \mathcal{V}_\lambda(q_\infty) \tag{C.5}$$

which means that the form of dependence of this operator on q_λ (denoted U_λ) is exactly the same as dependence on q_∞ (denoted $\mathcal{V}_\lambda$):

$$U_\lambda(q_\lambda) = U_\lambda(q_\infty) \tag{C.6}$$

Therefore, one does not have to indicate in terms of which of these operators (q_λ or q_∞) U_λ is expressed. Note, however, that this is true only for q_λ and q_∞: in terms of operators q_{λ_2} corresponding to a different width λ_2, U_λ would have a different form $U_\lambda^{\lambda_2}$:

$$U_\lambda(q_\lambda) = U_\lambda(q_\infty) = U_\lambda^{\lambda_2}(q_{\lambda_2}) \,, \tag{C.7}$$

where $U_\lambda^{\lambda_2}$ is a function different than U_λ.

C.1.2 Consequences of unitarity of U_λ

U_λ has to be unitary:

$$U_\lambda^\dagger U_\lambda = 1 . \tag{C.8}$$

Using perturbative expansion:

$$U_\lambda = \left(1 + u^{(1)} + u^{(2)} + \ldots\right) \tag{C.9}$$

one gets:

$$\left(u^{(1)\dagger} + u^{(1)}\right) + \left(u^{(2)\dagger} + u^{(2)} + u^{(1)\dagger}u^{(1)}\right) + \ldots = 0 . \tag{C.10}$$

Therefore, the first order has to be anti-hermitian ($u^{(1)\dagger} = -u^{(1)}$).

The second order $u^{(2)}$ is slightly more complicated. One can split $u^{(2)}$ into a hermitian ($h^{(2)}$) and anti-hermitian ($a^{(2)}$) parts:

$$u^{(2)} = a^{(2)} + h^{(2)} \tag{C.11}$$

where

$$a^{(2)\dagger} = -a^{(2)} \tag{C.12}$$
$$h^{(2)\dagger} = h^{(2)} . \tag{C.13}$$

The second-order unitarity (C.10) leads to:

$$h^{(2)} = \frac{1}{2}u^{(1)}u^{(1)} , \tag{C.14}$$

while the anti-hermitian part, $a^{(2)}$ is arbitrary.

To summarize, if U_λ is expanded into perturbative series in coupling constant, $U_\lambda = 1 + u^{(1)} + u^{(2)} + \ldots$, then the following apply:

- The first order term has to be anti-hermitian $u^{(1)\dagger} = -u^{(1)}$.
- The hermitian part of the second-order term has to be $h^{(2)} = \frac{1}{2}u^{(1)}u^{(1)}$.

C.1.3 RGPEP equations

The Hamiltonian operator can be expressed in terms of operators creating bare particles, $q_\infty^\dagger$ (the form of the Hamiltonian in this basis is denoted H), or in terms of operators creating effective particles, $q_\lambda^\dagger$ (denoted $\mathcal{H}_\lambda$). Thus, H and $\mathcal{H}_\lambda$ denote the same operator in terms of different creation and annihilation operator basis.

$$\mathcal{H}_\lambda(q_\lambda) = \mathrm{H}(q_\infty) . \tag{C.15}$$

In terms of one basis, $q_\infty^\dagger$, one can write:

$$\mathcal{H}_\lambda(q_\infty) = U_\lambda^\dagger \mathcal{H}_\lambda(q_\lambda) U_\lambda = U_\lambda^\dagger \mathrm{H}(q_\infty) U_\lambda . \tag{C.16}$$

Henceforth, all operators are expressed in terms of q_∞, which will not be marked explicitly. Differentiating (C.16) over λ leads to:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{H}_\lambda \;=\; [\mathcal{H}_\lambda, T]\;, \tag{C.17}$$

where

$$T \;:=\; -\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}U_\lambda^\dagger\right)U_\lambda = U_\lambda^\dagger\frac{\mathrm{d}}{\mathrm{d}\lambda}U_\lambda\;. \tag{C.18}$$

Note, that $T(g=0)=0$, and therefore its expansion in the coupling constant starts at g^1.

Let us denote:

$$\mathcal{H}_\lambda = f_\lambda\mathcal{G}_\lambda \tag{C.19}$$

and split $\mathcal{G}_\lambda = \mathrm{H}_0 + \mathcal{G}_I$. Since $f_\lambda\mathrm{H}_0 = \mathrm{H}_0$ and $(1-f_\lambda)\mathrm{H}_0 = f_\lambda'\mathrm{H}_0 = 0$, differentiating (C.19) with respect to λ leads to:

$$f_\lambda'\mathcal{G} + f_\lambda\mathcal{G}' = [\mathrm{H}_0, T] + [f_\lambda\mathcal{G}_I, T] \tag{C.20}$$

In RGPEP, this equation is split into two separate conditions, which is an arbitrary step (i.e. the following equations make stronger requirements than the above):

$$\begin{cases} f_\lambda\mathcal{G}' &=& f_\lambda[f_\lambda\mathcal{G}_I, T] \\ [T,\mathrm{H}_0] &=& (1-f_\lambda)[f_\lambda\mathcal{G}_I, T] - f_\lambda'\mathcal{G} \end{cases} \tag{C.21}$$

If f_λ is non-zero for all arguments, the first equation can be written as:

$$\mathcal{G}' = [f_\lambda\mathcal{G}_I, T]\;, \tag{C.22}$$

and using this the second equation can be simplified:

$$\begin{cases} T &=& \left\{((1-f_\lambda)\,\mathcal{G}_I)'\right\}_0 \\ \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{G} &=& [f_\lambda\mathcal{G}_I, T] \end{cases} \tag{C.23}$$

These are the RGPEP equations from which one can calculate the effective Hamiltonian in perturbation theory. Both expansions of T and $\mathcal{G}$ start at g^1. The first of these equations enables calculating T based on a given order of the effective Hamiltonian. The second equation enables an evaluation of the next order effective Hamiltonian $\mathcal{H}_\lambda^{(n)}$ based on $T^{(n-1)}$. $\mathcal{H}_\lambda^{(n)}$ can then be used to evaluate $T^{(n)}$, and so on.

Once one knows T, one can also extract U_λ:

$$U_\lambda^\dagger\frac{\mathrm{d}}{\mathrm{d}\lambda}U_\lambda = T\;. \tag{C.24}$$

By multiplying this equation by U_λ on the left, this can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}U_\lambda = U_\lambda T \tag{C.25}$$

Thus, in subsequent orders, one gets:

$$U_\lambda \;=\; 1 + u^{(1)} + u^{(2)} + \ldots \tag{C.26}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}u^{(1)} \;=\; T^{(1)} \tag{C.27}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}u^{(2)} \;=\; u^{(1)}T^{(1)} + T^{(2)}\;, \tag{C.28}$$

with boundary conditions:

$$U_\infty = 1\;. \tag{C.29}$$

C.2 The effective Hamiltonian

C.2.1 The zeroth and first orders

In the zeroth order:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{G}^{(0)} = 0 \tag{C.30}$$

$$\mathcal{G}^{(0)} = H^{(0)} \tag{C.31}$$

$$\mathcal{H}_\lambda^{(0)} = H^{(0)}\,. \tag{C.32}$$

Since both the expansion of $\mathcal{G}$ and of T start at the first order, the first oder of the right-hand-side of the first of Eqs. (C.23) is zero:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{G}^{(1)} = 0 \tag{C.33}$$

$$\mathcal{G}_\lambda^{(1)} = \mathcal{G}_\infty^{(1)} = \mathrm{H}^{(1)} \tag{C.34}$$

$$\mathcal{H}_\lambda^{1} = f_\lambda \mathcal{G}^{(1)} = f_\lambda \mathrm{H}^{(1)} \tag{C.35}$$

$$T^{(1)} = \Big\{(1-f_\lambda)\mathrm{H}^{(1)}\Big\}_0' = \Big\{-f_\lambda'\mathrm{H}^{(1)}\Big\}_0 = u^{(1)\prime} \tag{C.36}$$

$$u^{(1)} = \Big\{(c-f_\lambda)\mathrm{H}^{(1)}\Big\}_0 \tag{C.37}$$

For $\lambda\to\infty$ (i.e. $f_\lambda\to 1$) $U_\lambda\to 1$, so $u^{(1)}\to 0$, and therefore $c=1$.

$$u^{(1)} = \Big\{(1-f_\lambda)\mathrm{H}^{(1)}\Big\}_0\,, \tag{C.38}$$

which can also be written as $u^{(1)}_{ab} = \dfrac{P^+_{ab}}{ba}(1-f_{ab})\mathrm{H}^{(1)}_{ab}$.

Note that, as an additional requirement, T cannot have diagonal elements $\sim a_k^\dagger a_k$. However, since for $ab\to 0$ the factor $(1-f_{ab})\sim ab^2$ goes to zero faster than the denominators, not only do the diagonal elements not get rotated, but also terms close to the diagonal change only infinitely. One may consider the lack of rotation of the diagonal elements as a result of the limit of the equation (C.38), rather than as an additional requirement.

C.2.2 The second order

Formulae not based on specific form of f_λ

$$\mathcal{G}^{(2)\prime} = \Big[f_\lambda \mathrm{H}^{(1)}, T^{(1)}\Big] = \Big[f_\lambda \mathrm{H}^{(1)}, u^{(1)\prime}\Big] \tag{C.39}$$

$$\mathcal{G}^{(2)} = \mathcal{G}_{HH} + \mathcal{G}_{H2} \tag{C.40}$$

$$\mathcal{G}_{HH} := \int_\infty^\lambda \mathrm{d}z \Big[f_z \mathrm{H}^{(1)}, u_z^{(1)\prime}\Big] =: \mathcal{F}_{abc} H^{(1)} H^{(1)}\big|_{connected} \tag{C.41}$$

$$\mathcal{G}_{H2} := H^{(2)}, \tag{C.42}$$

which defines the second-order effective Hamiltonian $\mathcal{H}_\lambda^{(2)} = f_\lambda \mathcal{G}^{(2)}$.

The second-order rotation operator $U^{(2)}$ is:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} u^{(2)} &= T^{(2)} + u^{(1)} u^{(1)\prime} = \left\{ (1-f_\lambda)\, \mathcal{G}^{(2)} \right\}_0' + \frac{1}{2}(u^{(1)} u^{(1)})' + \\ &\quad + \frac{1}{2}(u^{(1)} u^{(1)\prime} - u^{(1)\prime} u^{(1)}) \qquad \text{(C.43)} \\ u^{(2)} &= \left\{ (1-f_\lambda)\, \mathcal{G}^{(2)} \right\}_0 + \frac{1}{2} u^{(1)} u^{(1)} + \frac{1}{2} \int_\infty^\lambda \mathrm{d}z \left[u_{1z}, u_{1z}' \right] \qquad \text{(C.44)} \end{aligned}$$

Therefore $u^{(2)}$ has a hermitian part, $\frac{1}{2} u^{(1)} u^{(1)}$, which is in agreement with unitarity condition. What is more, $u^{(2)}$ has parts (the hermitian parts) that are products, and not commutators, of operators. It thus has disconnected parts. Nevertheless, when calculating $\mathcal{H}_\lambda$ these parts together with $u^{(1)} H u^{(1)}$ become a commutator, and the effective Hamiltonian does not have unconnected parts.

Formulae for $f_{ab} = \exp(-ab^2/\lambda^4)$

$f_{ab} = \exp(-ab^2/\lambda^4)$ is the similarity form factor used throughout Chapters 3,4 and 5. The rest of this Appendix is thus written for this choice of the similarity form factor.

Let:

$$\begin{aligned} t &:= \frac{1}{\lambda^4} \qquad \text{(C.45)} \\ f_{\lambda,ab} &= \exp\left(-\frac{ab^2}{\lambda^4} \right) = \exp\left(-ab^2 t \right) \qquad \text{(C.46)} \\ \frac{\mathrm{d}}{\mathrm{d}\lambda} f_{\lambda,ab} &= -ab^2 f_{\lambda,ab} \frac{dt}{d\lambda} . \qquad \text{(C.47)} \end{aligned}$$

This leads to:

$$\mathcal{F}_{abc}^{(2)} = \frac{P_{ba}^+ ba + P_{bc}^+ bc}{ba^2 + bc^2} \left(f_{\lambda,ba} f_{\lambda,bc} - 1 \right) \qquad \text{(C.48)}$$

The characteristic combination of momenta will be also present in the following sections. To simplify expressions I introduce notation:

$$\begin{aligned} \pi_{abc} &:= P_{ba}^+ ba + P_{bc}^+ bc \qquad \text{(C.49)} \\ \Pi_{abc} &:= \frac{\pi_{abc}}{ba^2 + bc^2} \qquad \text{(C.50)} \\ \zeta_{ab} &:= \frac{P_{ab}^+}{ab} . \qquad \text{(C.51)} \end{aligned}$$

Using this notation $\mathcal{F}_{abc}^{(2)}$ can be written as:

$$\mathcal{F}_{abc}^{(2)} = \Pi_{abc} \left(f_{\lambda,ba} f_{\lambda,bc} - 1 \right) . \qquad \text{(C.52)}$$

C.2.3 The third order

$$T^{(2)} = \left\{(1-f_\lambda)\mathcal{G}^{(2)}\right\}'_{0^{\text{no }a^\dagger a}} = \zeta_{ca}\left((1-f_{ca})\mathcal{F}^{(2)}_{abc}\right)'\cdot H^{(1)}H^{(1)}\big|_{connected,\text{no }a^\dagger a} \quad \text{(C.53)}$$

$$\mathcal{G}^{(3)\prime} = [f_\lambda\mathcal{G},T]^{(3)} = \left[f_\lambda H^{(1)},T^{(2)}\right] + \left[f_\lambda\mathcal{G}^{(2)},T^{(1)}\right] \quad \text{(C.54)}$$

There are three kinds of terms.

- Terms coming from three $H^{(1)}$ without self-interaction loops; these will be denoted:

$$\mathcal{G}^{(3)}_{\text{no }\text{–O–}} =: \mathcal{F}^{(3)}_{abcd}H^{(1)}H^{(1)}H^{(1)}\big|_{\text{no }\text{–O–}}\,. \quad \text{(C.55)}$$

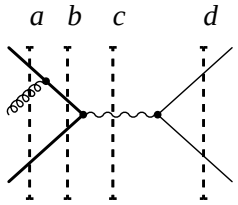
- Terms coming from three $H^{(1)}$ with a mass-kind loop on one of the external lines; these will be denoted:

$$\mathcal{G}^{(3)}_{\text{–O–}\!>} =: \mathcal{F}^{(3)}_{abad}H^{(1)}H^{(1)}H^{(1)}\big|_{\text{–O–}\!>} \quad \text{(C.56)}$$

  for the loop on the left-hand-side lines and similarly for the right-hand-side loop; note the repeated a subscript of the $\mathcal{F}^{(3)}_{abad}$.

- Terms coming from one $H^{(1)}$ and one $H^{(2)}$; for the scalar theory the only $H^{(2)}$ contributing would be the mass counterterm, and therefore this term would be combined with $\mathcal{G}^{(3)}_{\text{–O–}\!>}$; in other theories there are also seagull terms, for example.

The term without self-interaction loops, $\mathcal{F}^{(3)}_{abcd}$

It may be helpful to think of these effective Hamiltonian terms as being of a type 

.

$$\begin{aligned}\mathcal{F}^{(3)\prime}_{abcd} &= f_{ab}\zeta_{db}\left[(1-f_{bd})\mathcal{F}^{(2)}_{bcd}\right]' - \zeta_{ca}\left[(1-f_{\text{ca}})\mathcal{F}^{(2)}_{dcb}\right]' f_{cd} + \\ &\quad + f_{ca}\mathcal{F}^{(2)}_{abc}\left(-\zeta_{dc}f_{dc}\right)' + \left(\zeta_{ba}f_{ba}\right)'\mathcal{F}^{(2)}_{bcd}f_{bd}\,. \end{aligned} \quad \text{(C.57)}$$

I combine the first and the last term into a new symbol, $F^{(3)}_{abcd}$, while the second and the third will be called $F^{(3)}_{dcba}$:

$$\begin{aligned}F^{(3)\prime}_{abcd}\left(-\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right) &= \Pi_{bcd}\left[\left(f_{cb}f_{cd}-1\right)f_{ab}f_{bd}\pi_{abd} + \right. \\ &\quad \left. + \zeta_{db}(cb^2+cd^2)(1-f_{db})f_{ab}f_{cb}f_{cd}\right]\,. \end{aligned} \quad \text{(C.58)}$$

Thus:

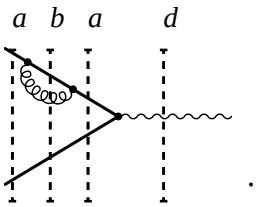
$$\begin{aligned}F^{(3)}_{abcd} &= \Pi_{bcd}\Big\{\pi_{abd}\left[\frac{f_{ab}f_{bc}f_{cd}f_{bd}-1}{ab^2+bc^2+cd^2+bd^2} - \frac{f_{ab}f_{bd}-1}{ab^2+bd^2}\right] + \\ &\quad + \zeta_{db}\left(bc^2+cd^2\right)\left[\frac{f_{ab}f_{bc}f_{cd}-1}{ab^2+bc^2+cd^2} - \frac{f_{ab}f_{bc}f_{cd}f_{bd}-1}{ab^2+bc^2+cd^2+bd^2}\right]\Big\}\,. \end{aligned} \quad \text{(C.59)}$$

Likewise, the second contribution is:

$$F^{(3)}_{dcba} = \Pi_{abc}\Big\{\pi_{acd}\left[\frac{f_{ab}f_{bc}f_{cd}f_{ac}-1}{ab^2+bc^2+cd^2+ac^2}-\frac{f_{ac}f_{cd}-1}{ac^2+cd^2}\right]+ \\ +\zeta_{ac}\left(ba^2+bc^2\right)\left[\frac{f_{ab}f_{bc}f_{cd}-1}{ab^2+bc^2+cd^2}-\frac{f_{ab}f_{bc}f_{cd}f_{ca}-1}{ab^2+bc^2+cd^2+ca^2}\right]\Big\}. \quad \text{(C.60)}$$

This agrees with [40].

The term with a self-interaction loop on the left-hand-side, $\mathcal{F}^{(3)}_{abad}$

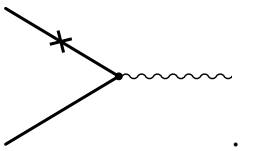


It may be helpful to think of these effective Hamiltonian terms of being a type .
There are two differences compared to $\mathcal{F}^{(3)}_{abcd}$: the c state has exactly the same momenta as a (therefore, for example, $ac=0$) and there is no contribution from $T^{(2)}$ that would be proportional to $a^\dagger a$. This leads to:

$$\mathcal{F}^{(3)\,\prime}_{abad} = f_{ab}\zeta_{db}\left[\left(1-f_{bd}\right)\mathcal{F}^{(2)}_{bad}\right]' + 0 + \mathcal{F}^{(2)}_{aba}\left(-\zeta_{da}f_{da}\right)' + \left(\zeta_{ba}f_{ba}\right)'\mathcal{F}^{(2)}_{bad}f_{bd} \quad \text{(C.61)}$$

$$\mathcal{F}^{(3)}_{abad} = \Pi_{bad}\Bigg\{\pi_{abd}\left[\frac{f_{ab}^2f_{ad}f_{bd}-1}{2ab^2+ad^2+bd^2}-\frac{f_{ab}f_{bd}-1}{ab^2+bd^2}\right]+ \\ \zeta_{db}\left(ba^2+ad^2\right)\left[\frac{f_{ab}^2f_{ad}-1}{2ab^2+ad^2}-\frac{f_{ab}^2f_{ad}f_{bd}-1}{2ab^2+ad^2+bd^2}\right]\Bigg\}+ \\ +\zeta_{ba}P^+_{da}da\left[\frac{f_{da}-1}{da^2}-\frac{f_{da}f_{ab}^2-1}{2ab^2+ad^2}\right] \quad \text{(C.62)}$$

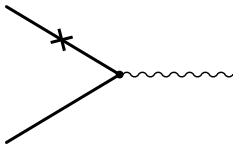
Mass-counterterm contribution to $\mathcal{G}^{(3)}$



It may be helpful to think of these effective Hamiltonian terms of being a type .
Again, the mass counterterm does not contribute directly to $T^{(2)}$. Therefore only the second term of Eq. (C.54) leads to this kind of terms, namely:

$$\mathcal{G}'_{+\!\!>\!\!-}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right) = X_{\delta m}\frac{P^+_{da}}{da}da^2f_{da}H_{>\!\!-}\big|_{connected} \quad \text{(C.63)}$$

$$\mathcal{G}^{(3)}_{+\!\!>\!\!-} = (-1)\,\zeta_{da}\left(f_{da}-1\right)\cdot \quad \text{(C.64)}$$



C.3 Other operators in terms of effective particles

The equation:

$$q_\infty = U_\lambda^\dagger q_\lambda U_\lambda = \tag{C.65}$$

$$= q_\lambda - u^{(1)} q_\lambda + q_\lambda u^{(1)} - u^{(1)} q_\lambda u^{(1)} + u^{(2)\dagger} q_\lambda + q_\lambda u^{(2)} \tag{C.66}$$

expresses creation and annihilation operators q_∞ in terms of q_λ (or the other way around). This means, that if one has any operator A expressed in terms of one type of the creation operators, it can be re-expressed in terms of the other set.

One may take the example of changing the creation and annihilation operator basis of the quarks, according to the formulae (i.e., U_λ) based on strong interaction Hamiltonian. This Hamiltonian will be denoted:

$$H_{QCD} = H_{0q} + H_{0g} + H^{(g)} + H^{(g^2)} \tag{C.67}$$

Note that, in this thesis, I analyze scalar and Yukawa interactions only. Yet, the scalar theory of Chapter 5 is designed in such a way as to produce a scattering amplitude analogous to that of QCD coupled to QED. I thus refer in this Appendix to a strongly coupled part of a Hamiltonian as H_{QCD} even for the scalar theory. In the case of the scalar Hamiltonian of Chapter 5, $H^{(g^2)}$ would just consist of quark and gluon mass counterterms; in the case of real QCD, it would be more complicated (it would have the seagull terms, for example). Now, let us re-express full Hamiltonian:

$$H = H_{QCD} + H_{0e} + H_{0\gamma} + H^{(e)} + H^{(e^2)} \tag{C.68}$$

in terms of effective quarks and gluons as defined by their strong interactions.

Interactions of effective quarks and gluons

Because the strong interaction Hamiltonian is defined in terms of quark and gluon creation and annihilation operators only, when re-expressed in terms of the effective particles it is such an effective Hamiltonian as defined in Appendix C.2.

In other words, if one rotates quark and gluon creation operators as required by the strong Hamiltonian, its form in the new basis will be exactly as demanded by the RGPEP equations. Note, that the free quark and gluon parts also contribute to the effective interaction $\mathcal{H}_{\lambda\, I}$ through terms of $u^{(i)}$, $i \geq 1$.

$$H_{\lambda,QCD} = f_\lambda \mathcal{G}_{\lambda,QCD} \tag{C.69}$$

The question is, therefore, how changing of basis of quarks and gluons would affect the electromagnetic part of the Hamiltonian.

Interactions of effective electrons and photons

By contrast, the parts of the Hamiltonian that did not have any quark or gluon creation or annihilation operators will not notice the change of the basis. These parts of the effective Hamiltonian are exactly equal to the corresponding terms in the bare Hamiltonian:

$$H_{\lambda,e,\gamma} = H_{\infty,e,\gamma}\,. \tag{C.70}$$

Interactions of effective quarks and gluons with photons

The electromagnetic interactions would thus only notice the change of the basis if they involved quarks. Note that because the Hamiltonian is expressed in terms of the products of quark and gluon creation operators, rather than rotating each creation operator separately one can perform a unitary rotation of the whole term of the Hamiltonian.

Since the rotation is expressed in terms of commutators, only connected parts will appear. As an example, I will consider hence a rotation of a term which creates two quarks from a photon. There is a corresponding hermitian conjugated term that annihilates a pair of quarks, terms with photon emission from a quark, etc., but neither of these contribute to the g^2e^2 order scattering amplitude for $e^+e^- \to hadrons$ analyzed in Chapter 5.

$$\left(1-u^{(1)}+u^{(2)\dagger}\right) H_E|_{con.} = H_E - \left\{(1-f_\lambda)H^{(1)}_{QCD}\right\}_0 H_E|_{con.} - \\ - \left\{(1-f_\lambda)H^{(2)}_{QCD}\right\}_0 H_E|_{con.} + \left(\tilde{\mathcal{F}}_{abc}H^{(1)}_{QCD}H^{(1)}_{QCD}\right) H_E|_{con.} \quad \text{(C.71)}$$

(*con.* means that there is only the connected part of corresponding expressions). The factor $\tilde{\mathcal{F}}$ in the last term is:

$$\begin{aligned}\tilde{\mathcal{F}}_{abc} &= -\frac{P^+_{ca}}{ca}\frac{P^+_{ba}ba+P^+_{bc}bc}{ba^2+bc^2} f_{ca}\,(1-f_{ba}f_{bc}) + \\ &\quad + \frac{P^+_{ba}P^+_{ca}}{ba\,ca} - \frac{P^+_{bc}}{bc}\frac{P^+_{ca}}{ca} f_{ab}f_{bc} + \frac{P^+_{ba}}{ba}\frac{P^+_{bc}}{bc} f_{ba}\end{aligned} \quad \text{(C.72)}$$

A similar term for $ac = 0$ has to be analyzed separately. The only contributions of this type may come only from terms that come partially from two separate order g rotations of each quark. This leads to

$$\tilde{\mathcal{F}}_{aba} = -\frac{1}{2}\left(\frac{P^+_{ba}}{ba}\right)^2 (1-f_{ba})^2 \,. \quad \text{(C.73)}$$

This result is analyzed in details in Section 5.6.2, and also used in F.

C.4 Notation used for the simple model

Below I give details of the notation and calculations used in Chapter 2. This is based directly on [1].

The simple model considered in Chapter 2 uses a slightly different notation from the rest of the thesis since it is limited to a subspace of two sectors of the Fock space and uses an algebraic version of the Hamiltonian similarity renormalization procedure with sharp similarity form factors f_λ ("diagonal proximum" operators) [3].

For any operator A,

$$A = \int |1\rangle\langle 2| A_{12}\,, \quad \text{(C.74)}$$

where $|i\rangle$ are eigenstates of a free Hamiltonian H_0, $\{A\}_0$ is defined as:

$$\{A\}_0 = \int |1\rangle\langle 2| \frac{1}{E_2-E_1} A_{12}\,, \quad \text{(C.75)}$$

where E_2 and E_1 are eigenvalues of H_0, corresponding to the right and left projection states respectively. This means that $\{A\}_0$ has a form similar to A with a simple addition of an extra denominator, with difference of energies (P^-) of all created and all annihilated particles. $\{A\}_0$ is the solution of the equation:

$$\left[\{A\}_0, H_0\right] = A\,. \tag{C.76}$$

The action of the diagonal proximum operator f_λ is defined as:

$$f_\lambda A = \int |1\rangle\langle 2|\, \tilde{f}_\lambda(1,2) A_{12}\,. \tag{C.77}$$

In the algebraic version of the similarity renormalization procedure [3], one has to choose sharp f_λ. For the calculation in Chapter 2, the following was chosen:

$$\tilde{f}_\lambda(1,2) = \theta(\lambda^2 - |\mathcal{M}_1^2 - \mathcal{M}_2^2|)\,, \tag{C.78}$$

where $\mathcal{M}_i$ is the free invariant mass of a state i. This means that, when f_λ acts on any operator, it chops off all far off-diagonal terms (multiplying them by zeros) and leaves only terms which do not change the free energy of a state excessively (i.e., by more than λ). Note that, to simplify the notation, I omit the tilde symbol in formulae such as (C.77) and (C.78) and use the symbol f_λ for both the operator f_λ and the function $\tilde{f}_\lambda$, except where this might cause confusion.

The functions $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ are defined as follows:

$$\alpha(s) := -\frac{1}{16\pi^2}\int \mathrm{d}\mathcal{M}^2\,\mathrm{d}x\,\theta(\Delta^2-\mathcal{M}^2)\,\frac{x}{\mathcal{M}^2-s+i\varepsilon}\,, \tag{C.79}$$

$$\beta(s) := -\frac{1}{16\pi^2}\int \mathrm{d}\mathcal{M}^2\,\mathrm{d}x\,\theta(\Delta^2-\mathcal{M}^2)\,\frac{1}{\mathcal{M}^2-s+i\varepsilon}\,, \tag{C.80}$$

$$\gamma(s) := \int \mathrm{d}\mathcal{M}^2\,\mathrm{d}x\,\theta(\Delta^2-\mathcal{M}^2)\,\frac{(1-x)\mathcal{M}^2-\mu^2+(1-x)m^2}{\mathcal{M}^2-s+i\varepsilon}\,, \tag{C.81}$$

where x is integrated over the whole kinematically allowed region, i.e., x has to be between the two solutions x_B of the equation:

$$\mathcal{M}^2 = \frac{m^2}{x_B} + \frac{\mu^2}{1-x_B}\,, \tag{C.82}$$

and $\mathcal{M}^2 \in \left((m+\mu)^2;\infty\right)$. Also, a limit $\varepsilon \to 0$ (after performing the integration in (C.79)-(C.81)) is implicit. α, β and γ are divergent functions of the cutoff Δ. Their finite parts are defined as follows:

$$\alpha_f(s) = \lim_{\Delta\to\infty}\left[\alpha(s) + \frac{1}{2\cdot 16\pi^2}\log\frac{\Delta^2}{m^2}\right]\,, \tag{C.83}$$

$$\beta_f(s) = \lim_{\Delta\to\infty}\left[\beta(s) + \frac{1}{16\pi^2}\log\frac{\Delta^2}{m^2}\right]\,, \tag{C.84}$$

$$\gamma_f(s) = \lim_{\Delta\to\infty}\left[\gamma(s) - \frac{1}{2}\Delta^2 - \frac{1}{2}(s-m^2-2\mu^2)\log\frac{\Delta^2}{m^2}\right]\,. \tag{C.85}$$

Appendix D

Yukawa theory

D.1 Canonical theory

D.1.1 Hamiltonian in terms of the fields

The starting point for the procedures in Chapter 4 is a Lagrangian:

$$\mathcal{L} = \overline{\psi}_I \left(i\partial\!\!\!/ - m - g\phi_I\right)\psi_I + \frac{1}{2}\left(\partial_\mu\phi_I\partial^\mu\phi_I - \mu^2\phi_I^2\right) \ , \tag{D.1}$$

where ψ_I denotes a doublet of fields, $\overline{\psi}_I = (\overline{\psi}_1, \overline{\psi}_2)$, and the fields with a subscript I are full interacting fields fulfilling Euler-Lagrange equations for the above Lagrangian:

$$(\partial^\mu\partial_\mu + \mu^2)\phi_I = -g\overline{\psi}_I\psi_I \tag{D.2}$$

$$\left(i\partial\!\!\!/ - m - g\phi_I\right)\psi_I = 0 \ . \tag{D.3}$$

Fields without subscript I will denote free fields, fulfilling Euler-Lagrange equations for the above Lagrangian with $g = 0$. Using projection operators $\Lambda_\pm$ (B.44), one can define the two components of the field ψ as:

$$\psi_+ := \Lambda_+\psi_I \tag{D.4}$$

$$\psi_- := \Lambda_-\psi_I \ . \tag{D.5}$$

Multiplication of (D.3) by Λ_+ leads to:

$$\psi_- = \frac{1}{i\partial^+}\left(i\partial^\perp\alpha^\perp + m\beta\right)\psi_+ + \frac{1}{i\partial^+}\beta g\phi_I\psi_+ \ , \tag{D.6}$$

which is a constraints equation: when one provides initial conditions, this equation has to be fulfilled explicitly. One introduces fields ψ fulfilling free Euler-Lagrange equations as follows:

$$\psi := \tilde{\psi}_- + \psi_+ \tag{D.7}$$

$$\tilde{\psi}_- := \psi_-(g=0) = \frac{1}{i\partial^+}\left(i\partial^\perp\alpha^\perp + m\beta\right)\psi_+ \ . \tag{D.8}$$

The full interacting field can be expressed in terms of the free field by:

$$\psi_I = \psi + \frac{1}{i\partial^+}\beta g\phi_I\psi_+ \ . \tag{D.9}$$

Likewise, multiplication of (D.3) by Λ_- leads to a dynamical equation:

$$\begin{aligned} i\partial^- \psi_+ &= \frac{1}{i\partial^+}\left(-\partial^{\perp 2}+m^2\right)\psi_+ + \\ &+ (g\phi_I\beta)\,\tilde{\psi}_- + \left(i\partial^\perp\alpha^\perp + m\beta\right)\frac{1}{i\partial^+}\,(\beta g\phi_I)\,\psi_+ + \\ &g^2\phi_I\frac{1}{i\partial^+}\phi_I\psi_+ \,. \end{aligned} \tag{D.10}$$

The canonical Hamiltonian of this theory is:

$$H = \frac{1}{2}\int d^3x\,\bar{\psi}_I\left(i\gamma^+\right)\partial^-\psi_I + \phi_I\left(-\partial^{\perp 2}+\mu^2\right)\phi_I \,, \tag{D.11}$$

which, when expressed in terms of the free fields ψ, becomes:

$$H = H_0 + H_Y + H_+ \tag{D.12}$$

$$H_0 = \frac{1}{2}\int d^3x : \left[\phi\left(-\partial^{\perp 2}+\mu^2\right)\phi + \bar{\psi}\gamma^+\frac{-\partial^{\perp 2}+m^2}{i\partial^+}\psi\right] : \tag{D.13}$$

$$H_Y = g\int d^3x : [\bar{\psi}\psi\phi] : \tag{D.14}$$

$$H_+ = g^2\int d^3x : \left[\bar{\psi}\phi\frac{\gamma^+}{2i\partial^+}\phi\psi\right] : \,. \tag{D.15}$$

D.1.2 Hamiltonian in terms of the creation/annihilation operators

Substituting the free fields Fourier expansion (see Appendices B.3 and B.4) and normal-ordering leads to the Hamiltonian expressed in terms of creation and annihilation operators. In this step, one drops all terms that appear during normal-ordering due to delta functions.

$$H^\Delta = H_0 + H_Y^\Delta + H_+^\Delta + X^\Delta \tag{D.16}$$

The free part of the Hamiltonian is:

$$H_0 = \int [k]\frac{k^{\perp 2}+\mu^2}{k^+}a_k^\dagger a_k + \int [p]\sum_{\sigma i}\frac{p^{\perp 2}+m^2}{p^+}\left(b_{p\sigma}^{(i)\dagger}b_{p\sigma}^{(i)} + d_{p\sigma}^{(i)\dagger}d_{p\sigma}^{(i)}\right) \,. \tag{D.17}$$

The part of order g is:

$$\begin{aligned} H_Y^\Delta &= g\sum_{\sigma\eta i}\int [pkl]2(2\pi)^3\delta^3(p_{created}-p_{annihilated})\times \\ &\times\left(a_k^\dagger b_{p\sigma}^{(i)\dagger}b_{l\eta}^{(i)}\bar{u}_{p\sigma}u_{l\eta} + a_k^\dagger d_{p\sigma}^{(i)}b_{l\eta}^{(i)}\bar{v}_{p\sigma}u_{l\eta} - a_k^\dagger d_{l\eta}^{(i)\dagger}d_{p\sigma}^{(i)}\bar{v}_{p\sigma}v_{l\eta} + \right. \\ &\left. + b_{p\sigma}^{(i)\dagger}b_{l\eta}^{(i)}a_k\bar{u}_{p\sigma}u_{l\eta} + b_{p\sigma}^{(i)\dagger}d_{l\eta}^{(i)\dagger}a_k\bar{u}_{p\sigma}v_{l\eta} - d_{l\eta}^{(i)\dagger}d_{p\sigma}^{(i)}a_k\bar{v}_{p\sigma}v_{l\eta}\right)r_{\Delta\delta}r_{\Delta\delta} \,. \end{aligned} \tag{D.18}$$

In this term, a regularization has been introduced. I choose a regularization on relative child-parent momenta: namely, for each particle there is a term:

$$r_{\Delta\delta} = \exp\left(\frac{-\kappa_{c/p}^2}{\Delta^2}\right)r_\delta\left(x_{c/p}\right) \,. \tag{D.19}$$

A specific form of the small-x cutoff r_δ will be left unspecified here, except from requiring that it makes integrations like the one in Eq. (D.28) finite. In each term of $r_{\Delta\delta}$ there are two such regulating factors, one for each of the child particles. However, the small-x regulator r_δ will be included only for fermions.

The seagull term is not involved in the calculation of Chapter 4.

D.2 Effective theory to the order g^2

In this part the boson mass μ is chosen to be zero.

D.2.1 General formulae to order g^2

$$\mathcal{H}_{\lambda\,0} = H_0 \tag{D.20}$$
$$\mathcal{H}_\lambda^{(1)} = f_\lambda H_Y \tag{D.21}$$
$$\mathcal{H}_\lambda^{(2)} = f_\lambda H_+ + f_\lambda X_\Delta^{(2)} + f_\lambda \left(\mathcal{F}_{abc}^{(2)}\right) H_Y H_Y|_{connected} \tag{D.22}$$

For:

$$f_{ab} = \exp\left(-\frac{ab^2}{\lambda^4}\right) \tag{D.23}$$

$\mathcal{F}^{(2)}$ is:

$$\mathcal{F}_{abc}^{(2)} = \frac{P_{bc}^+ bc + P_{ba}^+ ba}{bc^2 + ba^2}\left(f_{ab}f_{bc} - 1\right)\,. \tag{D.24}$$

D.2.2 Second order effective fermion mass term

For each family of fermions:

$$f_\lambda \mathcal{F}^{(2)} H_Y H_Y|_{\frown} = \sum_\sigma \int [p] b_{p\sigma}^\dagger b_{p\sigma} \frac{1}{p^+} \delta m_{\Delta,\lambda}^2\,. \tag{D.25}$$

Using (B.57), $\delta m_{\Delta,\lambda}^2$ can be written as:

$$\delta m_{\Delta,\lambda}^2 = \frac{1}{16\pi^3 p^+}\sum_{\sigma_2}\int \frac{dx d^2\kappa}{x(1-x)}\frac{1}{ba}(f_{ba}^2 - 1)\,(\bar{u}_{p\sigma_a} u_{p_2\sigma_2}\bar{u}_{p_2\sigma_2} u_{p\sigma_c})\, r_{\Delta\delta}^2 = \tag{D.26}$$
$$= \frac{g^2}{16\pi^2}\int_0^1 dx \int_0^\infty dz \frac{m^2(1+x)^2 + z}{(1-x)^2 m^2 + z}\frac{1}{x}(f_{ba}^2 - 1) e^{-4z/\Delta^2} r_\delta^2(x)\,, \tag{D.27}$$

where x is the relative momentum of the fermion in the fermion-boson pair.

The part without f_{ba} is divergent, and the specific form of divergence requires the second-order counterterm to be[1] :

$$X_2 = \sum_\sigma \int [p] b_{p\sigma}^\dagger b_{p\sigma} \frac{1}{p^+}\left[\frac{g^2}{16\pi^2}\int_0^1 dx \int_0^\infty dz \frac{m^2(1+x)^2 + z}{(1-x)^2 m^2 + z}\frac{1}{x} e^{-4z/\Delta^2} r_\delta^2(x) + \delta m_{X,phys}^2\right] =$$
$$= \sum_\sigma \int [p] b_{p\sigma}^\dagger b_{p\sigma} \frac{1}{p^+}\frac{g^2}{16\pi^2}\left[\left(\int_0^1 dx r_\delta^2 \frac{1}{x}\right)\frac{\Delta^2}{4} + 4m^2 \log\frac{\Delta^2}{m^2} + const.\right]\,, \tag{D.28}$$

[1] Compared to [2] the quadratically Δ-dependent part of (D.28) has an additional factor $1/2$, because of different choice of regularization. The choice of regularization made in this appendix is consistent with the rest of the thesis.

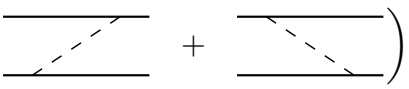
where $\delta m^2_{X,phys}$ and *const*. are constants (independent of Δ and momentum p) of dimension $mass^2$.

Similarly, the effective boson mass and the boson mass counterterm can be calculated.

D.2.3 Second order effective fermion-fermion potential

In second order there are a number of terms in $H^{(\lambda)}$. Of these, only the effective vertex (D.21) and mass correction presented above, and the fermion-fermion – fermion-fermion potential shown below (denoted $\mathcal{H}_{\lambda,V}$) are important for further calculation.

$$\mathcal{H}_{\lambda,V} \quad = \quad f_{ac}\mathcal{F}^{(2)}_{abc}\, H_Y H_Y|_{\diagup\diagup} \quad = \quad f_{ac}\mathcal{F}^{(2)}_{abc}\Bigg(\quad + \quad \Bigg) \tag{D.29}$$



These two orderings are referred to as a "slash" ($/$) and a "backslash" ($\backslash$) orderings.

D.3 Model subspace

The eigenvalue equation for a Hamiltonian in the full Fock-space can be reduced to an equation in a smaller subspace by using the R operation presented in Appendix J.2. In Chapter 4 reductions to one-particle and two-particles of limited energies spaces is described. The details of this procedure are presented below.

Derivation of the Schrödinger equation in the case of reduction to bare-particles subspaces and effective-particles subspaces follows the same path. However, the calculation for the effective particles is slightly more complicated, because of additional terms in $\mathcal{H}_\lambda$ (for example, in H^Δ there are no terms like in Eq. (D.29) for a term in $\mathcal{H}_\lambda$). Thus I give explicit expressions for effective particles. The corresponding expressions for bare particles can be calculated by following analogous steps, or by putting all f_λ equal to 1 (i.e. $\lambda \to \infty$) in the expressions for the effective particles.

D.3.1 Projection operators

The operator projecting on one fermion space is:

$$\hat{P}_1 \quad = \quad \sum_\sigma \int [k] b^\dagger_{k\sigma} |0\rangle \langle 0| b_{k\sigma} \,. \tag{D.30}$$

The projection on a subspace with two fermions of different kinds and of limited relative momenta is:

$$\hat{P}_2 \quad = \quad \sum_{\sigma\eta} \int [kp] \theta(\Omega^2 - M^2) b^{(1)\dagger}_{k\sigma} b^{(2)\dagger}_{p\eta} |0\rangle \langle 0| b^{(2)}_{p\eta} b^{(1)}_{k\sigma} \,. \tag{D.31}$$

Note, that for $\hat{P}_2$ to be a projection operator it is essential that θ would not be replaced by any smooth function. If one defined a projection on a space with two fermions of the same kind, there would be additional factor $1/2$.

D.3.2 One-fermion subspace

Reductions to one-effective-fermion subspace and one-bare-fermion subspace lead to subspace Hamiltonians:

$$H_{P_1} = \int [p] \sum_{\eta} \frac{p^{\perp 2} + m^2_{phys}}{p^+} b^\dagger_{p\eta} |0\rangle \langle 0| b_{p\eta} \, , \tag{D.32}$$

with the physical fermion mass in the eigenvalue:

$$m^2_{phys} = m^2 + m^2_{X,phys} \, , \tag{D.33}$$

where $m^2_{X,phys}$ is a finite part of the mass counterterm (cf. Eq. (D.28)). m^2_{phys} is equal to the value corresponding to the eigenvalue of the full Hamiltonian H^Δ calculated in perturbation theory.

For one-bare-particle subspace the divergent Δ-dependence of the counterterm cancels with Δ-dependent $H_{-<}H_{>-}$ loop. The result is finite, but depends on the finite part of the counterterm $m^2_{X,phys}$.

For one-effective-particle subspace, Δ-dependence already canceled out when one calculated the effective Hamiltonian. When projecting one one-effective-particle subspace, the λ-dependence of the effective mass (a term $\sim f^2_\lambda$ in (D.27)) cancels with f^2_λ in the $f_\lambda H_{-<} f_\lambda H_{>-}$ loop, and the resulting physical mass does not depend on λ. This is related to the fact that $\mathcal{H}_\lambda$ was obtained by a unitary rotation and therefore the eigenvalues of $\mathcal{H}_\lambda$ are the same as H^Δ (in particular, they do not depend on λ).

D.3.3 Two-effective-fermions subspace

Free part

$$\begin{aligned} H^0_{model} &= \hat{P}_2 H^{(0)\lambda} \hat{P}_2 = \\ &= \sum_{\sigma\eta} \int [kp] \theta(\Omega^2 - M^2) b^{(1)\dagger}_{k\sigma} b^{(2)\dagger}_{p\eta} |0\rangle \langle 0| b^{(2)}_{p\eta} b^{(1)}_{k\sigma} \left(\frac{p^{\perp 2} + m^2}{p^+} + \frac{k^{\perp 2} + m^2}{k^+} \right) \end{aligned} \tag{D.34}$$

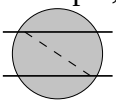
Together with a projection of $X^{(2)}$ and a mass (i.e., loop) part of $PHQHP$ in R, it gives:

$$\begin{aligned} H^{0+\delta m^2}_{model} = \sum_{\sigma\eta} \int [kp] \theta(\Omega^2 - M^2) b^{(1)\dagger}_{k\sigma} b^{(2)\dagger}_{p\eta} |0\rangle \langle 0| b^{(2)}_{p\eta} b^{(1)}_{k\sigma} \times \\ \times \left(\frac{p^{\perp 2} + m^2_{phys}}{p^+} + \frac{k^{\perp 2} + m^2_{phys}}{k^+} \right) \, , \end{aligned} \tag{D.35}$$

where m^2_{phys} is the eigenvalue of the one-particle equation (D.32).

Projection of potential term

$$H_V = \hat{P}_2 \mathcal{H}_{\lambda,V} \hat{P}_2 \tag{D.36}$$

where $\mathcal{H}_{\lambda,V}$ is defined in Eq. (D.29). Two kinds of terms in this expression are distinguishable (i.e., do not mix), since the pair of fermions of different kinds is considered; for identical fermions for each momentum, both diagrams would contribute for each value of external momenta. Thus kinematics determines the kind of diagram (for example, if k^+ of first fermion is bigger in the right than in the left state, one has the diagram  .

Taken together, the potential term is:

$$H_V = \sum_{1234} \int [1234]\widetilde{\delta}(1+3-2-4)\, v(1234)\theta_{\Omega 24}\theta_{\Omega 13}\, |13\rangle\langle 24| \tag{D.37}$$

$$v(1234) = g^2 \int [q]\widetilde{\delta} f_{ac}\mathcal{F}^{(2)}_{abc}\bar{u}_1 u_2 \bar{u}_3 u_4 r_{\Delta\delta} r_{\Delta\delta} |\_/\_ + \tag{D.38}$$

$$+g^2 \int [q]\widetilde{\delta} f_{ac}\mathcal{F}^{(2)}_{abc}\bar{u}_1 u_2 \bar{u}_3 u_4 r_{\Delta\delta} r_{\Delta\delta} |\_\backslash\_ \tag{D.39}$$

One Effective Boson Exchange (OEBE) term

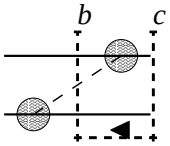
$$H_{OEBE} = \frac{1}{2}\hat{P}_2 H \hat{Q}\Big\{H\Big\}_0 \hat{P}_2 - \frac{1}{2}\hat{P}_2\Big\{H\Big\}_0 \hat{Q} H \hat{P}_2 \tag{D.40}$$

where, to get H_{OEBE} in the lowest order (i.e., g^2), one has $H = \mathcal{H}_{\lambda,Y}$ and:

$$\hat{Q} = \sum_{12}\int [12q] b_1^{(1)\dagger} b_2^{(2)\dagger} a_q^\dagger |0\rangle\langle 0| a_q b_2^{(2)} b_1^{(1)} \tag{D.41}$$

(superscripts (1) and (2) denote kind of fermion and are fixed; subscripts 1 and 2 denote momenta and polarizations and one sums and integrates over them). Each of these terms can be of the kind  or  and one has each of these diagrams with energy denominators for the left or right vertex.

The part of H_{OEBE} corresponding to the first diagram, with denominator in the right part is:



$$= \sum_{1234}\int [1234]\widetilde{\delta}(1+2-3-4)\theta_{\Omega 12}\theta_{\Omega 34}\, |12\rangle\langle 34|$$

$$\frac{1}{2}g^2 \int [q]\widetilde{\delta}(1-2-q)\, f_{ab}\bar{u}_1 u_3 \frac{P^+_{bc}}{(cb)} f_{bc}\bar{u}_2 u_4 \tag{D.42}$$

Because of a minus sign before the term with the other denominator, it will be $-ba = ab$. All terms of H_{OEBE} are thus:

$$H_{OEBE} = \sum_{1234}\int [1234]\widetilde{\delta}(1+2-3-4)\theta_{\Omega 12}\theta_{\Omega 34}\, |12\rangle\langle 34|\, v_{OEBE}(1234) \tag{D.43}$$

$$v_{OEBE}(1234) = \frac{1}{2}g^2 \bar{u}_1 u_3 \bar{u}_2 u_4 \left[\int [q]\widetilde{\delta} f_{ab} f_{bc}\left(\frac{P^+_{bc}}{cb} + \frac{P^+_{ab}}{ab}\right)_{\_/\_} + \right.$$

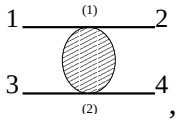
$$\left. + \int [q]\widetilde{\delta} f_{ab} f_{bc}\left(\frac{P^+_{bc}}{cb} + \frac{P^+_{ab}}{ab}\right)_{\_\backslash\_}\right] \tag{D.44}$$

D.4 Eigenvalue equation in two-effective-fermions subspace

D.4.1 The equation

The model Hamiltonian resulting from operations presented above is:

$$H_{model} = \sum_{12} \int [12] \theta_{\Omega 12} |12\rangle \langle 12| \left(\frac{k_1^{\perp 2} + m_{phys}^2}{k_1^+} + \frac{k_2^{\perp 2} + m_{phys}^2}{k_2^+} \right) + \\ + \sum_{1234} \int [1234] 4 m_{phys}^2 \widetilde{\delta}(1+2-3-4) \theta_{\Omega 12} \theta_{\Omega 34} |12\rangle \langle 34| \; v_{model}(1234) \quad \text{(D.45)}$$

where, for momenta labeled according to the figure: 

, $v_{model}(1234)$ is:

$$v_{model}(1234) = g^2 \frac{\bar{u}_1 u_2 \bar{u}_3 u_4}{4 m_{phys}^2} \left(\left[\frac{1}{q^+} f_{ac} \mathcal{F}^{(2)} r_{\Delta\delta} r_{\Delta\delta} + \frac{1}{2} \frac{1}{q^+} f_{ab} f_{bc} \left(\frac{P_{bc}^+}{cb} + \frac{P_{ab}^+}{ab} \right) r_{\Delta\delta} r_{\Delta\delta} \right]_{\sqsupset\sqsubset} + \right. \\ \left. + \left[\frac{1}{q^+} f_{ac} \mathcal{F}^{(2)} r_{\Delta\delta} r_{\Delta\delta} + \frac{1}{2} \frac{1}{q^+} f_{ab} f_{bc} \left(\frac{P_{bc}^+}{cb} + \frac{P_{ab}^+}{ab} \right) r_{\Delta\delta} r_{\Delta\delta} \right]_{\sqsupset\sqsupset} \right) \quad \text{(D.46)}$$

A factor $4m_{phys}^2$ has been introduced into v_{model} for convenience later in the calculation.

This Hamiltonian commutes with total momentum $P^\perp$, P^+ and with J_z defined as for a free theory.

One can search for an eigenstate of this Hamiltonian:

$$H_{model} |\Psi_P\rangle = P^- |\Psi_P\rangle \quad \text{(D.47)}$$

in the form:

$$|\Psi_P\rangle \;=\; \sum_{\sigma,\sigma'} \int [p,p'] \theta_\Omega \widetilde{\delta}(P-p-p') \psi_{\sigma\sigma'}(x,\kappa) b_{p\sigma}^{(1)\dagger} b_{p'\sigma'}^{(2)\dagger} |0\rangle \; . \quad \text{(D.48)}$$

After projecting on bra $\langle k_1, k_2|$ and introducing relative momenta $x, \kappa^\perp$ (and $y, \eta^\perp$ for the integrated momenta), equation (D.47) takes the following form:

$$\frac{\kappa^2 + m_{phys}^2}{x(1-x)} \theta_\kappa \psi_{\sigma\sigma'}(x,\kappa) + \sum_{\sigma,\sigma'} \frac{1}{16\pi^3} \int \frac{d^2\eta dy}{y(1-y)} \theta_\kappa \theta_\eta \, 4 m_{phys}^2 v_{model} \psi_{\zeta\zeta'}(y,\eta) = \\ = M^2 \theta_\kappa \psi_{\sigma\sigma'}(x,\kappa) \quad \text{(D.49)}$$

D.4.2 Elements of v_{model}

Spinors products

Relative momenta are introduced as follows:

$$\vec{p}_1 \;=\; x\vec{P} + \kappa^\perp \quad \text{(D.50)}$$
$$\vec{p}_3 \;=\; (1-x)\vec{P} - \kappa^\perp \quad \text{(D.51)}$$
$$\vec{p}_2 \;=\; y\vec{P} + \eta^\perp \quad \text{(D.52)}$$
$$\vec{p}_4 \;=\; (1-y)\vec{P} - \eta^\perp, \quad \text{(D.53)}$$

for the external momenta and for the one integrates over. In terms of these, the spinor products appearing in v_{model} can be written as:

$$\bar{u_1}u_2 = \frac{1}{\sqrt{xy}}\chi_1^\dagger\left[m(x+y)-\sigma^3\sigma^\perp(x\eta^\perp - y\kappa^\perp)\right]\chi_2 \tag{D.54}$$

$$\begin{aligned}\bar{u_3}u_4 = &\frac{1}{\sqrt{(1-x)(1-y)}}\chi_3^\dagger\left[m(2-x-y)+\right.\\ &\left.+\sigma^3\sigma^\perp\left((1-x)\eta^\perp-(1-y)\kappa^\perp\right)\right]\chi_4\,.\end{aligned} \tag{D.55}$$

Momentum factors for the slash-type potential

$$[]_{\not{-}\not{-}} = \left[\frac{1}{q^+}f_{ac}\mathcal{F}^{(2)}r_{\Delta\delta}r_{\Delta\delta}+\frac{1}{2}\frac{1}{q^+}f_{ab}f_{bc}\left(\frac{P^+_{bc}}{cb}+\frac{P^+_{ab}}{ab}\right)r_{\Delta\delta}r_{\Delta\delta}\right]_{\not{-}\not{-}} = \tag{D.56}$$

$$\begin{aligned}&= \frac{1}{y-x}\theta(y-x)r_{\Delta\delta}r_{\Delta\delta}\times\\ &\quad\times\left[f_{ac}\frac{(1-x)ba+ybc}{ba^2+bc^2}\left(f_{ab}f_{bc}-1\right)-\frac{1}{2}f_{ab}f_{bc}\left(\frac{y}{bc}+\frac{1-x}{ba}\right)\right]\end{aligned} \tag{D.57}$$

$$ba = \left(\frac{\eta^2+m^2}{1-y}+\frac{(\eta-\kappa)^2}{y-x}\right)(1-x)-\kappa^2-m^2 \tag{D.58}$$

$$bc = \left(\frac{\kappa^2+m^2}{x}+\frac{(\eta-\kappa)^2}{y-x}\right)y-\eta^2-m^2 \tag{D.59}$$

$$ac = \frac{\kappa^2+m^2}{x(1-x)}-\frac{\eta^2+m^2}{y(1-y)} \tag{D.60}$$

Momentum factors for the backslash-type potential

$$\begin{aligned}[]_{\not{-}\not{-}} &= \frac{1}{x-y}\theta(x-y)r_{\Delta\delta}r_{\Delta\delta}\times\\ &\quad\times\left[f_{ac}\frac{x\,ba+(1-y)bc}{ba^2+bc^2}\left(f_{ab}f_{bc}-1\right)-\frac{1}{2}f_{ab}f_{bc}\left(\frac{1-y}{bc}+\frac{x}{ba}\right)\right]\end{aligned} \tag{D.61}$$

$$ba = \left(\frac{\eta^2+m^2}{y}+\frac{(\kappa-\eta)^2}{x-y}\right)x-\kappa^2-m^2 \tag{D.62}$$

$$bc = \left(\frac{\kappa^2+m^3}{1-x}+\frac{(\kappa-\eta)^2}{x-y}\right)(1-y)-\eta^2-m^2 \tag{D.63}$$

$$ac = \text{as in}\,(D.60) \tag{D.64}$$

D.4.3 The equation in terms of pseudo-equal-time momenta

Instead of the relative momenta $x,\kappa^\perp$ one can use the pseudo-equal-time momenta introduced in Appendix B.2.2. However, the equation obtained is not symmetric and to symmetrize it one introduces new, rescaled wave function ϕ:

$$\phi(\vec{p}) := \frac{1}{(p^2+m^2_{phys})^{1/4}}\psi_{\sigma\sigma'}(\vec{p})\,. \tag{D.65}$$

This way Eq. (D.49) takes the form:

$$\frac{\vec{k}^2}{m_{phys}}\theta_k\phi_{\sigma\sigma'}(\vec{k})+\sum_{\sigma,\sigma'}\frac{1}{8\pi^3}\int\frac{d^3p}{\Big((k^2+m_{phys}^2)(p^2+m_{phys}^2)\Big)^{1/4}/m_{phys}}\theta_k\theta_p\,v_{model}\phi_{\zeta\zeta'}(\vec{p})= \\ =\frac{M^2-4m_{phys}^2}{4m_{phys}}\theta_k\phi_{\sigma\sigma'}(\vec{k})\,. \quad \text{(D.66)}$$

Note that, when using momenta $\vec{k}$, the nonrelativistic kinetic energy appears even without any nonrelativistic approximation. Also, if one introduces a binding energy B by subtracting two rest-frame masses of fermions from an equal-time center of mass bound state energy E_{cms}:

$$B:=E_{cms}-2m_{phys}\,, \quad \text{(D.67)}$$

then, for weak binding ($|B|\ll m_{phys}$), one can approximate:

$$M^2=E_{cms}^2=(B+2m_{phys})^2\approx 4Bm_{phys}+4m_{phys}^2\,, \quad \text{(D.68)}$$

and the eigenvalue expression on the right-hand-side becomes:

$$\frac{M^2-4m_{phys}^2}{4m_{phys}}\approx B\,. \quad \text{(D.69)}$$

For $B\ll m$ (or $M^2\approx 4m_{phys}^2$), the eigenvalue on the right-hand side of Eq. (D.66) can thus be interpreted as a binding energy. When this condition is not fulfilled (i.e., for strong binding) this eigenvalue has to be interpreted in terms of M^2, without the approximations above.

D.4.4 The leading nonrelativistic approximation

If one assumes that only nonrelativistic momenta are important (which can be forced by introducing a small Ω parameter for the reduction procedure), the potential is greatly simplified. If one keeps only leading parts in k/m series expansion, the equation (D.66) takes the following form:

$$\frac{\vec{k}^2}{m_{phys}}\theta_k\phi_{\sigma\sigma'}(\vec{k})+\sum_{\sigma,\sigma'}\frac{1}{8\pi^3}\int d^3p\,\theta_k\theta_p\left(\frac{-g^2}{(\vec{k}-\vec{p})^2}\right)\times \\ \times r_{\Delta\delta}r_{\Delta\delta}\left[f_{ac}+f_{ab}f_{bc}(1-f_{ac})\right]\delta_{\sigma\zeta}\delta_{\sigma'\zeta'}\phi_{\zeta\zeta'}(\vec{p})=B\theta_k\phi_{\sigma\sigma'}(\vec{k})\,, \quad \text{(D.70)}$$

where the factor in the parenthesis is the Coulomb potential in the momentum space:

$$v_{Coulomb}(\vec{k}-\vec{p}) \;:=\; -4\pi\alpha\frac{1}{(\vec{k}-\vec{p})^2}=-g^2\frac{1}{(\vec{k}-\vec{p})^2}\,. \quad \text{(D.71)}$$

This eigenvalue equation is diagonal in spin indexes: one can take $\phi_{\sigma\sigma'}(\vec{k})$ as a product of momentum independent spin factor, and a spin-independent momentum factor:

$$\phi_{\sigma\sigma'}(\vec{k}) \;=:\; f_{\sigma\sigma'}\phi(\vec{k})\,. \quad \text{(D.72)}$$

Then $f_{\sigma\sigma'}$ is arbitrary, and $\phi(\vec{k})$ fulfills:

$$\frac{\vec{k}^2}{m_{phys}}\theta_k\phi(\vec{k}) + \int \frac{d^3p}{(2\pi)^3}\theta_k\theta_p\, v_{Coulomb}(\vec{k}-\vec{p})r_{\Delta\delta}r_{\Delta\delta}\left[f_{ac} + f_{ab}\,f_{bc}(1-f_{ac})\right]\phi(\vec{p}) = B\,\theta_k\phi(\vec{k})\,. \tag{D.73}$$

Thus, in the leading nonrelativistic approximation one gets a Schrödinger equation with a standard Coulomb potential multiplied by the combination of the similarity factors f_λ, and by factors θ limiting the range of integration.

Whether the similarity form factors matter depends on the width of the form factors compared to the physical mass m_{phys}. For wide form factors (big lambdas), all f_λ are approximately equal to 1 for any fixed value of momenta, and there should be no difference between the above nonrelativistic equation and the equation with the standard Coulomb potential, without the similarity factors.

Three related questions present themselves: is the nonrelativistic approximation justified? are relativistic momenta important in Eq. (D.66) (i.e., before the nonrelativistic approximation)? And what is the role of the similarity form factors in (D.66)? These questions can, in fact, be reduced to one question: can Eq. (D.66) be approximated by a nonrelativistic equation with a Coulomb potential?

Since in the final equation only the physical mass enters, to simplify the notation I will henceforth drop the subscript *phys*, i.e.:

$$m := m_{phys} \tag{D.74}$$

and not the bare mass.

D.5 Numerical comparison 1: Tamm-Dancoff vs. Coulomb

D.5.1 Tamm-Dancoff two particles bound-state equation

Trying to solve initial ("bare") QFT by searching for an eigenstate dominated by two bare fermion states leads to an eigenvalue equation:

$$\frac{\vec{k}^2}{m}\theta_k\phi_{\sigma\sigma'}(\vec{k}) + \sum_{\sigma,\sigma'}\int \frac{d^3p}{(2\pi)^3}\theta_k\theta_p\, v_{TD}\phi_{\zeta\zeta'}(\vec{p}) \;=\; B\theta_k\phi_{\sigma\sigma'}(\vec{k})\,, \tag{D.75}$$

or, if one does not impose any momentum limitations:

$$\frac{\vec{k}^2}{m}\phi_{\sigma\sigma'}(\vec{k}) + \sum_{\sigma,\sigma'}\int \frac{d^3p}{(2\pi)^3}v_{TD}\phi_{\zeta\zeta'}(\vec{p}) \;=\; B\phi_{\sigma\sigma'}(\vec{k})\,. \tag{D.76}$$

Instead of keeping complicated regulators (each vertex is regulated separately), I drop $r_{\Delta\delta}$ and introduce again a momentum cutoff $k < k_{max}$, this time in potential part only. Therefore:

$$v_{TD} \;=\; \frac{g^2\bar{u}_1u_2\bar{u}_3u_4}{4m\left((k^2+m^2)(p^2+m^2)\right)^{1/4}}\left([]_{\diagup\diagdown} + []_{\diagdown\diagup}\right)\theta(k-k_{max})\theta(p-k_{max})\,, \tag{D.77}$$

with:

$$[]_{\diagup\diagup} \;=\; \frac{1}{y-x}\theta(y-x)(-\frac{1}{2})\left(\frac{y}{bc}+\frac{1-x}{ba}\right) \tag{D.78}$$

$$[]_{\diagdown\diagdown} \;=\; \frac{1}{x-y}\theta(x-y)(-\frac{1}{2})\left(\frac{1-y}{bc}+\frac{x}{ba}\right)\,. \tag{D.79}$$

The spinor factors and *ba*, *bc* are given in Appendix D.4.2.

This equation can be compared with the Schrödinger equation with Coulomb potential in perturbation theory. The unperturbed Hamiltonian with the Coulomb potential is:

$$\begin{aligned} H_0 &= \sum_{\sigma_{k1}\sigma_{k2}} \int d^3k\,|k\rangle\,\frac{k^2}{m}\,\langle k| + \\ &+ \sum_{\sigma_{k1}\sigma_{p1}\sigma_{k2}\sigma_{p2}} \int d^3k d^3p\,|k\rangle\,\delta_{\sigma_{k1}\sigma_{p1}}\delta_{\sigma_{k2}\sigma_{p2}}\frac{(-g^2)}{(2\pi)^3}\frac{1}{(\vec{k}-\vec{p})^2}\,\langle p|\ , \end{aligned} \tag{D.80}$$

where $|k\rangle$ denotes a state of two particles of relative momenta $\vec{k}$, with implicit spin labels (i.e., $|k\rangle := |k\sigma_{k1}\sigma_{k2}\rangle$), and normalized as follows:

$$\langle k \mid p\rangle = \delta^3(k-p)\delta_{\sigma_{k1}\sigma_{p1}}\delta_{\sigma_{k2}\sigma_{p2}}\ . \tag{D.81}$$

The lowest-energy eigenvalue of H_0 is:

$$B_0 = -g^4 m/64\pi^2 \tag{D.82}$$

and the corresponding ground-state is (cf. Appendix J.1):

$$\psi^{(0)}_{\sigma\sigma'} = N\int d^3k \frac{1}{\left(k^2+a^2\right)^2} f_{\sigma\sigma'}\left|k\sigma\sigma'\right\rangle\ . \tag{D.83}$$

The potential in which bound-state perturbation theory is performed is:

$$\begin{aligned} V_{TD-C} &= \sum_{\sigma_{k1}\sigma_{p1}\sigma_{k2}\sigma_{p2}} \int d^3k d^3p\,|k\rangle\,\frac{g^2}{(2\pi)^3}\left[\delta_{\sigma_{k1}\sigma_{p1}}\delta_{\sigma_{k2}\sigma_{p2}}\frac{1}{(\vec{k}-\vec{p})^2} - \right. \\ &\left. -\theta_k\theta_p\frac{\bar{u}_{\sigma_{k1}}u_{\sigma_{p1}}\bar{u}_{\sigma_{k2}}u_{\sigma_{p2}}}{4m\left((k^2+m^2)(p^2+m^2)\right)^{1/4}}\left(\square_{\diagup\diagdown} + \square_{\diagdown\diagup}\right)\right]\langle p| \end{aligned} \tag{D.84}$$

To simplify the notation, I denote spin subscripts of the four degenerate $\psi^{(0)}$ as follows: $1 = \uparrow\downarrow$, $2 = \downarrow\uparrow$, $3 = \uparrow\uparrow$, $4 = \downarrow\downarrow$.

D.5.2 First order correction

The general structure

Since the above potential preserves J_3 component of angular momentum, the only non-vanishing first-order corrections to the Coulomb ground bound state energy from V_{TD-C} are:

$$\Delta E^{(1)}_{ij} = \begin{bmatrix} \langle 1|V|1\rangle & \langle 1|V|2\rangle & 0 & 0 \\ \langle 2|V|1\rangle & \langle 2|V|2\rangle & 0 & 0 \\ 0 & 0 & \langle 3|V|3\rangle & 0 \\ 0 & 0 & 0 & \langle 4|V|4\rangle \end{bmatrix}\ . \tag{D.85}$$

The upper-left corner of this matrix is diagonalized by the symmetric/antisymmetric basis choice: $\psi = |\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle$.

Below, I calculate the correction to energy relative to the zeroth-order eigenvalue: $\Delta E/E_0$. Note that here $E_0 = |B_0| > 0$.

The diagonal matrix elements $\langle i|V|i\rangle/E_0$ (equal for each $i \in \{1\ldots4\}$) are real and denoted $d^{(1)}$, the off-diagonal $\langle 1|V|2\rangle/E_0$ are also real and denoted $b^{(1)}$. The correction to the energy is a sum or a difference of the two. The state $\psi_+ := \psi_0(k)\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ has energy $E^0(-1+d^{(1)}+b^{(1)})$, the state $\psi_- := \psi_0(k)\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ has energy $E^0(-1+d^{(1)}-b^{(1)})$. Depending on the sign of $b^{(1)}$, either one can have the lower energy. The energies of states $\psi_{\uparrow\uparrow}$ and $\psi_{\downarrow\downarrow}$ are equal $E_0(-1+d^{(1)})$.

Results

α	$\Delta E_+/E_0$	$\Delta E_-/E_0$	error	$b^{(1)}$
0.01	8.7 E-6	-5.8 E-5	1.2 E-6	3.3 E-5
0.1	1.3 E-3	-5.2 E-3	2.6 E-4	3.3 E-3
0.3	2.0 E-2	-3.5 E-2	1.1 E-3	2.7 E-2
0.6	8.0 E-2	-0.11	3.2 E-3	9.7 E-2

Summary

For all α examined above, the first-order corrections to the energies are small. However but for $\alpha = 0.6$ they are considerable (of the order of 10%). In all cases, the bulk of the correction comes from off-diagonal (spin-changing) matrix elements $b^{(1)}$. This is because the spin-diagonal part of the potential resembles Coulomb very closely.

In all cases $b^{(1)} > 0$ and therefore the state $\psi_0(k)\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ has lower energy. For this state I calculated the second order correction.

D.5.3 Second order correction

Since degeneration was removed in first order, in the second order one gets the following correction to $\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$ state:

$$\Delta E^{(2)} = \frac{1}{2}\sum_{i \text{ such that } E_i \neq E_0} (\langle 1| - \langle 2|)V|i\rangle\langle i|V(|1\rangle - |2\rangle)\frac{1}{E_0 - E_i} \tag{D.86}$$

I calculated only part of this correction, which is sensitive to k_{max} (essentially, I dropped terms with lower powers of momenta, e.g., terms without double-spin-flips; see the description in Chapter 4)

$$\Delta E^{(2)} = \frac{1}{2}\sum_{i \text{ such that } E_i \neq E_0} (\langle 1|V|i,2\rangle\langle i,2|V|1\rangle + \langle 2|V|i,1\rangle\langle i,1|V|2\rangle)\frac{1}{E_0 - E_i} = \tag{D.87}$$

$$= \frac{1}{2}2\sum_{i \text{ such that } E_i \neq E_0} \langle 1|V|i,2\rangle\langle i,2|V|1\rangle\frac{1}{E_0 - E_i} \tag{D.88}$$

$$\Delta\tilde{E}^{(2)} := \int_0^{4\alpha m} d^3p \int_m^{k_{max}} d^3k \int_0^{4\alpha m} d^3p' \langle \vec{p},1|V\left|\vec{k},2\right\rangle\left\langle \vec{k},2\right|V|\vec{p}',1\rangle\frac{(+1)}{k^2/m}\psi(p)\psi(p') \tag{D.89}$$

The limits of the integrals and simplifications to the denominators are described in Chapter 4.

Note that $|1\rangle$ in Eqs. (D.86)-(D.88) has a Coulomb wave function, while $\left|\vec{k},1\right\rangle$ is a free momentum eigenstate. For visualization purposes, I also changed the sign of this expression to positive.

The result for the second-order correction is presented in Figure 4.3 on page 63 and described in Section 4.5.2.

D.6 Numerical comparison 2: two-effective-particles bound-state equation versus Coulomb

One can analyze the equation (D.66), which describes a bound state of two effective fermions, following steps described above for two bare fermions bound-state equation (D.75) (Appendix D.5). The results of this are analyzed in Section 4.6.

Appendix E

Canonical Hamiltonian of QED and QCD

E.1 Lagrangian

For description of light-front quantization in the light-front gauge, see e.g. [115] (see also [116-118]).

The starting point is the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}Tr\left(F^{\mu\nu}_{gl.}F_{gl.\mu\nu}\right) - \frac{1}{4}F^{\mu\nu}_{ph.}F_{ph.\mu\nu} + \sum_{fl.e}\bar{\psi}_e\left(i\not{D}_{QED} - m_e\right)\psi_e + \tag{E.1}$$

$$+\sum_{fl.q}\bar{\psi}_q\left(i\not{D}_{QED+QCD} - m_q\right)\psi_q\,, \tag{E.2}$$

where:

$$F^{\mu\nu}_{gl.} := \partial^\mu A^\nu - \partial^\nu A^\mu + ig\left[A^\mu, A^\nu\right] \tag{E.3}$$

$$F^{\mu\nu}_{ph.} := \partial^\mu A^\nu_E - \partial^\nu A^\mu_E \tag{E.4}$$

$$D^\mu_{QED} := \partial^\mu + ieA^\mu_E \tag{E.5}$$

$$D^\mu_{QED+QCD} := \partial^\mu + ieA^\mu_E + igA^\mu \tag{E.6}$$

$$A^\mu := A^{\mu a}T^a \tag{E.7}$$

$$\left[T^a, T^b\right] = if^{abc}T^c \tag{E.8}$$

$$Tr\left(T^aT^b\right) = \frac{1}{2}\delta^{ab} \tag{E.9}$$

Note that f^{abc} is antisymmetric under exchange of any of the indexes. Writing the summations explicitly:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu a}_{gl.}F^a_{gl.\mu\nu} - \frac{1}{4}F^{\mu\nu}_{ph.}F_{ph.\mu\nu} + \bar{\psi}_e\left(i\not{\partial} - e\not{A}_E - m_e\right)\psi_e + \tag{E.10}$$

$$\bar{\psi}_q\left(i\not{\partial} - e_q\not{A}_E - g\not{A}^aT^a - m_q\right)\psi_q \tag{E.11}$$

$$F^{\mu\nu a} = \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} - gf^{abc}A^{\mu b}A^{\nu c} \tag{E.12}$$

E.2 Equations of motion

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} \tag{E.13}$$

For fields $\bar{\psi}_e$ and $\bar{\psi}_q$:

$$(i\partial\!\!\!/ - eA\!\!\!/_E - m_e)\,\psi_e = 0 \tag{E.14}$$
$$\left(i\partial\!\!\!/ - e_q A\!\!\!/_E - g A\!\!\!/ - m_q\right)\psi_q = 0 \tag{E.15}$$

(For fields ψ_e and ψ_q, after hermitian conjugation, one gets the same equations.

For A^μ:

$$D^\nu F_{\nu\mu} = g\bar{\psi}_q \gamma_\mu T^a \psi_q T^a \tag{E.16}$$
$$D^\nu = \partial + ig\,[A^\nu, .] \tag{E.17}$$

For A^μ_E:

$$\partial^\nu F_{ph.\nu\mu} = e\bar{\psi}_e \gamma_\mu \psi_e + e_q \bar{\psi}_q \gamma_\mu \psi_q \tag{E.18}$$

E.3 Physical degrees of freedom

E.3.1 Fermions

Projecting Eqs. (E.14) and (E.15) on Λ_- leads to a constraint equation. In the light-front gauge:

$$A^+ = A^+_E = 0 \tag{E.19}$$

this leads to the following expressions for the constraint components of fermion fields:

$$\psi_{q-} = \frac{1}{i\partial^+}\left(i\alpha^\perp\partial^\perp - e_q\alpha^\perp A^\perp_E - g\alpha^\perp A^\perp + m_q\beta\right)\psi_{q+} \tag{E.20}$$
$$\psi_{e-} = \frac{1}{i\partial^+}\left(i\alpha^\perp\partial^\perp - e\alpha^\perp A^\perp_E + m_e\beta\right)\psi_{e+}\,. \tag{E.21}$$

One can introduce free fields:

$$\psi_{qm} := \psi_{q+} + \psi_{qm-} \tag{E.22}$$
$$\psi_{qm-} := \frac{1}{i\partial^+}\left(i\alpha^\perp\partial^\perp + m_q\beta\right)\psi_{q+} \tag{E.23}$$
$$\psi_{em} := \psi_{e+} + \psi_{em-} \tag{E.24}$$
$$\psi_{em-} := \frac{1}{i\partial^+}\left(i\alpha^\perp\partial^\perp + m_e\beta\right)\psi_{e+}\,. \tag{E.25}$$

E.3.2 Bosons

There is a similar situation for gluon fields. In the light-front gauge

$$A^\mu = \left(A^+ = 0, A^-, A^\perp\right) \tag{E.26}$$

the minus component of A^μ can be expressed in terms of other components:

$$A^- = \frac{2i\partial^\perp A^\perp}{i\partial^+} - \frac{2g}{(i\partial^+)^2}\left[A^\perp, i\partial^+ A^\perp\right] + \frac{2g}{(i\partial^+)^2}\bar{\psi}_{qm}\gamma^+ T^a \psi_{qm} T^a \,. \tag{E.27}$$

One introduces free gluon fields by putting $g = 0$ in the above equation:

$$\tilde{A}^\mu = \left(A^+ = 0, \tilde{A}^-, A^\perp\right) \tag{E.28}$$

$$\tilde{A}^- = \frac{2i\partial^\perp A^\perp}{i\partial^+} \tag{E.29}$$

Likewise, for electromagnetic field one has:

$$A_E^\mu = \left(A^+ = 0, A^-, A^\perp\right) \tag{E.30}$$

$$A_E^- = \frac{2i\partial^\perp A_E^\perp}{i\partial^+} + \frac{2e_q}{(i\partial^+)^2}\bar{\psi}_{qm}\gamma^+\psi_{qm} + \frac{2e}{(i\partial^+)^2}\bar{\psi}_{em}\gamma^+\psi_{em} \tag{E.31}$$

$$\tilde{A}_E^\mu = \left(A^+ = 0, \tilde{A}_E^-, A_E^\perp\right) \tag{E.32}$$

$$\tilde{A}_E^- = \frac{2i\partial^\perp A_E^\perp}{i\partial^+} \tag{E.33}$$

E.4 Energy momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)}\partial^\nu \phi_i - g^{\mu\nu}\mathcal{L} \tag{E.34}$$

This defines the Hamiltonian

$$H = P^- = \frac{1}{2}\int d^2x^\perp dx^- : T^{+-} : \tag{E.35}$$

Using the equations of motion and re-expressing constraint components of full interacting fields

by their free counterparts, one gets:

$$\frac{1}{2}T^{+-} = \frac{1}{2}\bar{\psi}_{qm}\gamma^{+}\frac{-\partial^{\perp 2}+m_q^2}{i\partial^{+}}\psi_{qm}+\frac{1}{2}\bar{\psi}_{em}\gamma^{+}\frac{-\partial^{\perp 2}+m_e^2}{i\partial^{+}}\psi_{em}+ \tag{E.36}$$

$$+\frac{1}{2}\partial^k A^{ia}\partial^k A^{ia}+\frac{1}{2}\partial^k A_E^i\partial^k A_E^i+ \tag{E.37}$$

$$+\bar{\psi}_{qm}\left(e_q\tilde{\not{A}}_E+g\tilde{\not{A}}\right)\psi_{qm}+\bar{\psi}_{em}\left(e\tilde{\not{A}}_E\right)\psi_{em}+ \tag{E.38}$$

$$+j_E^{+}\frac{1}{2\left(i\partial^{+}\right)^2}j_E^{+}+j_s^{+a}\frac{1}{2\left(i\partial^{+}\right)^2}j_s^{+a}+ \tag{E.39}$$

$$+\bar{\psi}_{qm}\left(e_q\tilde{\not{A}}_E+g\tilde{\not{A}}\right)\frac{\gamma^{+}}{2i\partial^{+}}\left(e_q\tilde{\not{A}}_E+g\tilde{\not{A}}\right)\psi_{qm}+\bar{\psi}_{em}\left(e\tilde{\not{A}}_E\right)\frac{\gamma^{+}}{2i\partial^{+}}\left(e\tilde{\not{A}}_E\right)\psi_{em}+ \tag{E.40}$$

$$+2ig\,Tr\left(\partial^{\mu}\tilde{A}^{\nu}\left[\tilde{A}_{\mu},\tilde{A}_{\nu}\right]\right)-\frac{g^2}{2}Tr\left(\left[\tilde{A}^{\mu},\tilde{A}^{\nu}\right]\left[\tilde{A}_{\mu},\tilde{A}_{\nu}\right]\right) \tag{E.41}$$

where the currents j^{μ} are defined as:

$$j^{+} := g\bar{\psi}_{qm}\gamma^{+}T^{a}\psi_{qm}T^{a} \tag{E.42}$$

$$j_A^{+} := g\left[i\partial^{+}A^{\perp},A^{\perp}\right]=-ig\left[\partial^{+}\tilde{A}^{\mu},\tilde{A}_{\mu}\right] \tag{E.43}$$

$$j_s^{+} := j^{+}+j_A^{+} \tag{E.44}$$

$$j_E^{+} := e_q\bar{\psi}_{qm}\gamma^{+}\psi_{qm}+e\bar{\psi}_{em}\gamma^{+}\psi_{em} \tag{E.45}$$

E.5 Expansion into creation and annihilation operators

E.5.1 Fields expanded in creation and annihilation operators

For quark and electron fields, the Fourier transform defines creation and annihilation operators:

$$\psi_m(x) = \sum_{\lambda}\int[k]\left(b_{k\lambda}u_{k\lambda}e^{-ikx}+d_{k\lambda}^{\dagger}v_{k\lambda}e^{ikx}\right)_{x^{+}=0}, \tag{E.46}$$

where for electrons the index λ indicates polarization, and for quarks polarization and color.

Fourier expansion of the unconstrained components of the gluon field is:

$$A^{\perp a}(x) = \sum_{\lambda}\int[k]\left(a_{k\lambda a}\varepsilon_{\lambda}^{\perp}e^{-ikx}+a_{k\lambda a}^{\dagger}\varepsilon_{\lambda}^{*\perp}e^{ikx}\right)_{x^{+}=0} \tag{E.47}$$

The remaining components of the free gluon field are:

$$A^{+}(x) = 0 \tag{E.48}$$

$$A^{-}(x) = \frac{2i\partial^{\perp}A^{\perp}}{i\partial^{+}}. \tag{E.49}$$

Taken together, A^μ can be written as:

$$A^\mu(x) = \sum_{\lambda a}\int [k]\left(a_{k\lambda a}T^a\varepsilon^\mu_{k\lambda}e^{-ikx}+a^\dagger_{k\lambda a}T^a\varepsilon^{*\mu}_{k\lambda}e^{ikx}\right)_{x^+=0} \tag{E.50}$$

with polarization vectors:

$$\varepsilon^- = \frac{2k^\perp\varepsilon^\perp}{k^+},\quad \varepsilon^+=0 \tag{E.51}$$

Creation and annihilation operators fulfill:

$$\left\{b_{k\lambda a},b^\dagger_{p\sigma b}\right\} = \left\{d_{k\lambda a},d^\dagger_{p\sigma b}\right\} = \left[a_{k\lambda a},a^\dagger_{p\sigma b}\right] = 2(2\pi)^3k^+\delta^3(k-p)\delta_{\lambda\sigma}\delta_{ab}\,, \tag{E.52}$$

where λ, σ denote polarizations and $a,\ b$ denote color.

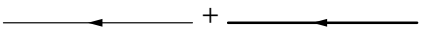
In all terms, integrals over x give:

$$\int d^2x^\perp dx^- e^{ix(k_{cre}-k_{ani})} = 2(2\pi)^3\delta^2(k^\perp_{cre}-k^\perp_{ani})\delta(k^+_{cre}-k^+_{ani}) =: \tag{E.53}$$

$$=: 2(2\pi)^3\delta^3(k_{cre}-k_{ani}) = \widetilde{\delta}(k_{cre}-k_{ani}) \tag{E.54}$$

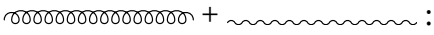
Below, I list terms of the Hamiltonian density (E.41) which contribute to calculating $e^+e^-\to$ *hadrons* scattering amplitude. These terms are expressed in terms of the creation and annihilation operators, and with spinor products expressed in terms of two-dimensional spinors.

E.5.2 Free Hamiltonian



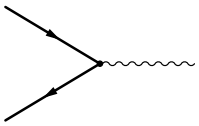
$$H_{0\,fermions} = \sum_\lambda\int [k]\frac{k^{\perp2}+m_e^2}{k^+}\left(b^\dagger_{k\lambda}b_{k\lambda}+d^\dagger_{k\lambda}d_{k\lambda}\right)+$$
$$+\sum_{\lambda c}\int [k]\frac{k^{\perp2}+m_q^2}{k^+}\left(b^\dagger_{k\lambda c}b_{k\lambda c}+d^\dagger_{k\lambda c}d_{k\lambda c}\right) \tag{E.55}$$

λ denotes polarization and c is a color and flavor index of quarks.



$$H_{0bosons} = \sum_{\lambda a}\int [p]\frac{p^{\perp2}}{p^+}a^\dagger_{gl-p\lambda a}a_{gl-p\lambda a}+\sum_\lambda\int [p]\frac{p^{\perp2}}{p^+}a^\dagger_{ph-p\lambda}a_{ph-p\lambda} \tag{E.56}$$

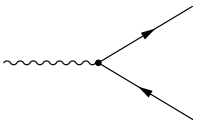
E.5.3 Electrodynamics of electrons and quarks



For the ultraviolet calculation I assume $m_e=0$ and $m_q=0$ (see also Appendix F.1).

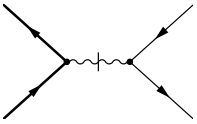
$$H_{>\!-} = e_q\sum_{123}\int [123]\,\bar{u}_1\not{\varepsilon}_3v_2\,\delta_{color\,12}\,\widetilde{\delta}(1+2-3)\,b^\dagger_{q1}d^\dagger_{q2}a_{p3} \tag{E.57}$$

$$\bar{u}_1\not{\varepsilon}_3v_2 = \frac{1}{\sqrt{x(1-x)}}\chi_1\left(-(1-2x)\kappa^k\varepsilon^k_3\sigma^3+(-i)\,(\kappa\times\varepsilon_3)^3\right)\xi_{-2} \tag{E.58}$$



$$H_{-\prec} = e\sum_{123}\int [123]\,\bar{v}_2 \not{\varepsilon}^*_3 u_1\,\widetilde{\delta}(1+2-3)\,a^\dagger_{p3} d_{e2} b_{e1} \tag{E.59}$$

$$\bar{v}_2 \not{\varepsilon}^*_3 u_1 = \frac{1}{\sqrt{x(1-x)}}\xi_{-2}\left(-(1-2x)\kappa^k\varepsilon^k_3\sigma^3 + i\,(\kappa\times\varepsilon_3)^3\right)\chi_1 \tag{E.60}$$



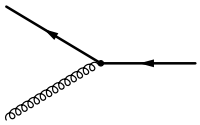
$$H_{\succ\!\!+\!\!\prec} = e_q e\int [1234]\frac{\bar{u}_{q1}\gamma^+ v_{q2}}{\left(k_3^+ + k_4^+\right)^2}\,\bar{v}_{e3}\gamma^+ u_{e4}\widetilde{\delta}(1+2-3-4)\,b^\dagger_{q1} d^\dagger_{q2} d_{e3} b_{e4} \tag{E.61}$$

$$\bar{u}_1\gamma^+ v_2 = 2\sqrt{p_1^+ p_2^+}\,\chi_1^\dagger\sigma^3\xi_{-2} \tag{E.62}$$

$$\bar{u}_1\gamma^+ u_2 = 2\sqrt{p_1^+ p_2^+}\,\chi_1^\dagger\sigma^3\chi_2 \tag{E.63}$$

$$\bar{v}_1\gamma^+ v_2 = 2\sqrt{p_1^+ p_2^+}\,\xi_{-1}^\dagger\sigma^3\xi_{-2} \tag{E.64}$$

E.5.4 Chromodynamics

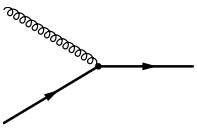


$$H_{Y1} = g\sum_{123}\int [123]\bar{u}_1 \not{\varepsilon}^*_2 u_3\,(T^a)_{13}\,\widetilde{\delta}\,b^\dagger_1 a^\dagger_{2a} b_3 \tag{E.65}$$

$$\bar{u}_1 \not{\varepsilon}^*_2 u_3 = \sqrt{\frac{x_3}{x_1}}\chi_1^\dagger\left(-\frac{x_1+x_3}{x_2}\kappa_{1/3}\varepsilon^{*\perp}_2 - i\left(\kappa_{1/3}\times\varepsilon^*_2\right)^3\sigma^3\right)\chi_3 \tag{E.66}$$

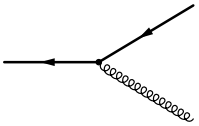
$$x_i = \frac{p_i^+}{P^+} \tag{E.67}$$

$$\kappa^\perp_{1/3} = p_1^\perp - \frac{x_1}{x_3}p_3^\perp \tag{E.68}$$



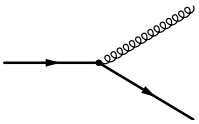
$$H_{Y2} = -g\sum_{123}\int [123]\bar{v}_3 \not{\varepsilon}^*_2 v_1\,(T^a)_{31}\,\widetilde{\delta}\,a^\dagger_{2a} d^\dagger_1 d_3 \tag{E.69}$$

$$\bar{v}_3 \not{\varepsilon}^*_2 v_1 = \sqrt{\frac{x_3}{x_1}}\xi_{-3}^\dagger\left(-\frac{x_1+x_3}{x_2}\kappa_{1/3}\varepsilon^{*\perp}_2 + i\left(\kappa_{1/3}\times\varepsilon^*_2\right)^3\sigma^3\right)\xi_{-1} \tag{E.70}$$



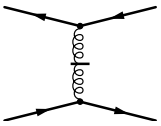
$$H_{Y3} = g\sum_{123}\int [123]\bar{u}_3 \not{\varepsilon}_2 u_1 \,(T^a)_{31}\,\widetilde{\delta}\, b_3^\dagger a_{2a} b_1 \tag{E.71}$$

$$\bar{u}_3 \not{\varepsilon}_2 u_1 = \sqrt{\frac{x_3}{x_1}}\chi_3^\dagger\left(-\frac{x_1+x_3}{x_2}\kappa_{1/3}\varepsilon_2^\perp + i\left(\kappa_{1/3}\times\varepsilon_2\right)^3\sigma^3\right)\chi_1 \tag{E.72}$$



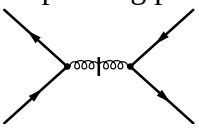
$$H_{Y4} = -g\sum_{123}\int [123]\bar{v}_1 \not{\varepsilon}_2 v_3 \,(T^a)_{13}\,\widetilde{\delta}\, d_3^\dagger d_1 a_{2a} \tag{E.73}$$

$$\bar{v}_1 \not{\varepsilon}_2 v_3 = \sqrt{\frac{x_3}{x_1}}\xi_{-1}^\dagger\left(-\frac{x_1+x_3}{x_2}\kappa_{1/3}\varepsilon_2^\perp - i\left(\kappa_{1/3}\times\varepsilon_2\right)^3\sigma^3\right)\xi_{-3} \tag{E.74}$$



$$H_\Xi = -g^2\sum_{1234a}\int [1234]\,\bar{u}_1\gamma^+ u_2\frac{1}{\left(k_3^+ + k_4^+\right)^2}\bar{v}_3\gamma^+ v_4\, b_{q1}^\dagger d_{q4}^\dagger d_{q3} b_{q2}\,\widetilde{\delta}\,(T^a)_{12}\,(T^a)_{34} \tag{E.75}$$

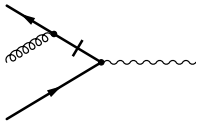
Corresponding product of spinors is given in Eq. (E.62).



$$H_{>\!+\!<g} = g^2\sum_{1234a}\int [1234]\frac{\bar{u}_{q1}\gamma^+ v_{q2}\;\;\bar{v}_{q3}\gamma^+ u_{q4}}{\left(k_3^+ + k_4^+\right)^2}\,\widetilde{\delta}\,(T^a)_{12}\,(T^a)_{34}\, b_{q1}^\dagger d_{q2}^\dagger d_{e3} b_{e4} \tag{E.76}$$

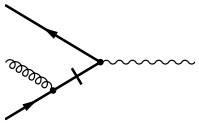
Corresponding product of spinors is given in Eq. (E.62).

E.5.5 Mixed QED-QCD terms



$$H_{>+q>} = ge_q\sum_{1234a}\int [1234]\,\bar{u}_1 \not{\varepsilon}_3^* \frac{\gamma^+}{2\left(k_4^+ - k_2^+\right)} \not{\varepsilon}_4 v_2 \widetilde{\delta}\,(T^a)_{12}\; b_1^\dagger a_{3a}^\dagger d_2^\dagger a_4 \tag{E.77}$$

$$\bar{u}_1 \not{\varepsilon}_{g3}^* \frac{1}{2}\gamma^+ \not{\varepsilon}_{ph4} v_2 = \sqrt{p_1^+ p_2^+}\;\chi_1\left[\varepsilon_3^{*\perp}\varepsilon_4^\perp\sigma^3 + i\left(\varepsilon_3^*\times\varepsilon_4\right)^3\right]\xi_{-2} \tag{E.78}$$



$$H_{>+qb>} = e_q g \int [1234] \bar{u}_1 \not{\varepsilon}_4 \frac{\gamma^+}{2\left(k_4^+ - k_1^+\right)} \not{\varepsilon}_3^* v_2 \, \widetilde{\delta} \, (T^a)_{12} \, b_1^\dagger a_{3a}^\dagger d_2^\dagger a_4 \qquad \text{(E.79)}$$

$$\bar{u}_1 \not{\varepsilon}_{ph4} \frac{1}{2} \gamma^+ \not{\varepsilon}_{g3}^* v_2 = \sqrt{p_1^+ p_2^+} \, \chi_1 \left[\varepsilon_3^{*\perp} \varepsilon_4^\perp \sigma^3 - i \left(\varepsilon_3^* \times \varepsilon_4 \right)^3 \right] \xi_{-2} \qquad \text{(E.80)}$$

Appendix F

Effective Hamiltonian of QCD and QED

F.1 Effective QED and QCD Hamiltonian: list of terms contributing to $e^+e^- \to hadrons$

In this appendix I give some of the terms in the effective Hamiltonian $\mathcal{H}_\lambda$ of QCD coupled to QED which would contribute to calculating a scattering amplitude for the process $e^+e^- \to$ *hadrons*. Only some terms have already been calculated and are given below. This preliminary analysis is presented here to give a general picture of how much more complicated the analysis in QCD coupled to QED is, compared to the scalar model analyzed in Section 5.6.

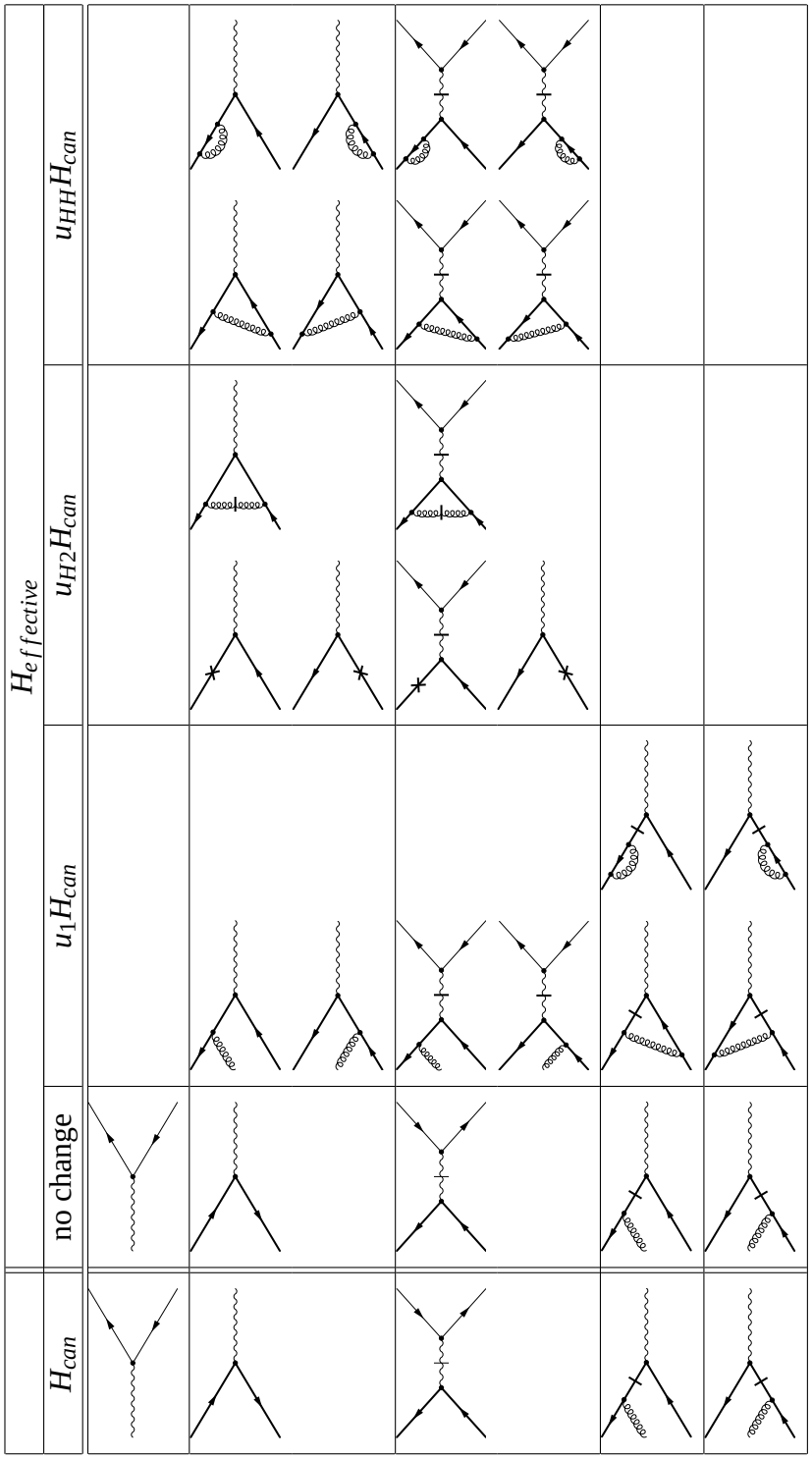
Based on standard way of simplifying the calculation of this scattering amplitude [98], I consider massless quarks and add a small gluon mass μ_g which is to go to zero in the end of calculation.

Calculation of the effective Hamiltonian $\mathcal{H}_\lambda$ and counterterms in the canonical Hamiltonian H^Δ is based on the simplified procedure described in Section 5.6.2, starting from the canonical Hamiltonian of QCD coupled to QED, presented in Appendix E.

F.1.1 Electromagnetic interactions of electrons only

Parts of $\mathcal{H}_\lambda$ coming from H for electrons and photons only do not change.

F.1.2 Electromagnetic interactions of quarks

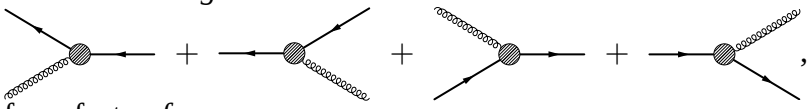


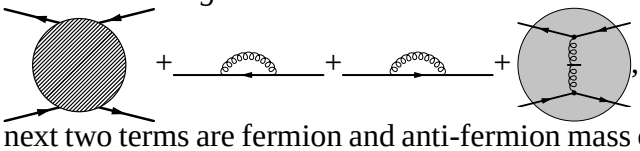
Out of the terms in the above table, I have only calculated the terms with quark self-interactions and corresponding counterterms. All terms with triangle-type strong-interactions

corrections to electromagnetic interactions of quarks remain to be calculated in future.

F.1.3 Strong interactions of quarks

The effective Hamiltonian of strong interactions comes not only from $U^\dagger H_{con.}$ terms, but also from all the combinations in $U^\dagger HU$. The result is full H_{effQCD}, as follows:

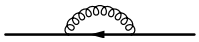
- Terms of order g^0: $\mathcal{H}_{\lambda\,0QCD} = H_{0QCD}$.
- Terms of order g:
  
  , where the bulb denotes a form factor f_λ.

  The counterterms of a similar structure are of the order g^3 and therefore do not contribute to the amplitude $e^+e^- \to hadrons$ in the order e^2g^2.
- Terms of order g^2:
  
  , where the first term comes from $\mathcal{F}_2 H_1 H_1$, next two terms are fermion and anti-fermion mass corrections, and the last one is the seagull term with a form factor f_λ.

  There are also mass counterterms of order g^2

F.2 Terms with self-interaction loops

F.2.1 Effective quark mass and mass counterterm



$$\mathcal{H}_{\lambda\,\delta m} = \mathcal{F}^{(2)}_{aba} H^{(g)}_{<} H^{(g)}_{>} \tag{F.1}$$

I calculate denominators *ba* as if gluons had a small mass $\mu_g \to 0$.

$$\mathcal{H}_{\lambda\,\delta m} = \sum_1 \int [1] b_1^\dagger b_1 \frac{\delta m_\lambda^2 + \delta m_\Delta^2}{p_1^+} \tag{F.2}$$

$$\delta m_\lambda^2 = \frac{g^2}{6\pi^3} \int \frac{dx d^2\kappa}{x} \frac{1}{\kappa^2 + x\mu_g^2} \frac{1+x^2}{(1-x)^2} \kappa^2 r_\delta^2 f_{ba}^2 \tag{F.3}$$

$$\delta m_\Delta^2 = -\frac{g^2}{6\pi^3} \int \frac{dx d^2\kappa}{x} \frac{1}{\kappa^2 + x\mu_g^2} \frac{1+x^2}{(1-x)^2} \kappa^2 r_{\Delta\delta}^2 \tag{F.4}$$

$$r_{\Delta\delta} = e^{-2\kappa^2/\Delta} r_\delta \tag{F.5}$$

$$r_\delta = r_\delta(x) r_\delta(1-x) \tag{F.6}$$

$$\delta m_\Delta^2 = -\frac{g^2}{6\pi^2}\frac{\Delta^2}{4}\int_0^1 \frac{dx}{x}\frac{1+x^2}{(1-x)^2}r_\delta^2 + \frac{g^2}{3\pi^2}\log\Delta^2\int_0^1 \frac{dx}{x}\frac{1+x^2}{(1-x)^2}r_\delta^2 x\mu_g^2 + \tag{F.7}$$

$$-\frac{g^2}{6\pi^2}\int_0^1 \frac{dx}{x}\frac{1+x^2}{(1-x)^2}r_\delta^2\left[x\mu_g^2\gamma_E + x\mu_g^2\log 4x\mu_g^2\right] \tag{F.8}$$

The limit $\mu_g \to 0$ is well defined. The first part:

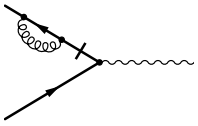
$$\delta m_\Delta^2 = -\frac{g^2}{6\pi^2}\frac{\Delta^2}{4}\left(\int_0^1 \frac{dx}{x}\frac{1+x^2}{(1-x)^2}r_\delta^2\right) \tag{F.9}$$

is divergent and determines the form of quark-mass counterterm in H^Δ. The other, finite and λ-dependent part is:

$$\delta m_\lambda^2 = \frac{g^2}{6\pi^3}\int \frac{dxd^2\kappa}{x}\frac{1+x^2}{(1-x)^2}\kappa^2 r_\delta^2\frac{1}{ba}f_{ba}^2 \tag{F.10}$$

where x is x of the quark in the loop.

F.2.2 Seagull with quark self-interaction loop

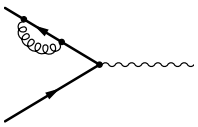


$$\mathcal{H}_{\lambda\,\searrow\!\!-+} = g^2 e_q\frac{4}{3}\sum_{1'14}\int[11'234]\frac{x_1}{x_4-x_{1'}}\chi_1^\dagger\Big(-\frac{x_2+x_1}{x_3}\kappa_{2/1}^k\varepsilon_4^k\sigma^3 + i\varepsilon^{3sk}\kappa_{2/1}^s\varepsilon_4^k - \\ -\frac{x_2+x_1}{x_3}\kappa_{2/1}^k i\varepsilon^{3ks}\varepsilon_4^s - \varepsilon^{3sk}\kappa_{2/1}^s\sigma^3\varepsilon^{3kz}\varepsilon_4^z\Big)\xi_{-1}\,\delta_{11'col}\widetilde{\delta\delta}\,b_1^\dagger d_{1'}^\dagger a_4 \tag{F.11}$$

Integration over $x_{2/1}$ is symmetric ($r_{\Delta\delta}$ does not depend in this term on any complicated combination of momenta), therefore this term is zero.

$$\mathcal{H}_{\lambda\,\searrow\!\!-+} = 0 \tag{F.12}$$

F.2.3 QED vertex with quark self-interaction loop



In any self-interaction loop

$$\sum_{12}\bar{u}_3\not{\varepsilon}_2 u_1\bar{u}_1\not{\varepsilon}_2^* u_{3"} = \frac{x_3}{x_1}\left(\left(\frac{x_1+x_3}{x_2}\right)^2+1\right)\kappa_{1/3}^2\delta_{33"}\,. \tag{F.13}$$

Therefore, the  term in $\mathcal{H}_\lambda$ is

$$\mathcal{H}_{\lambda}\ = g^2 \sum_{1'2'3} \int [2'1'] \frac{1}{p^+_{1"}} \tilde{\mathcal{F}}_{aba}\, \bar{u}_3 \not{\varepsilon}_{2'} u_{1'} \bar{u}_{1'} \not{\varepsilon}^*_{2'} u_{1"} \frac{4}{3} \delta_{col 31'} \widetilde{\delta} \tag{F.14}$$

$$e_q \sum_{1"2"3"} \int [1"2"3"]\, \bar{u}_{1"} \not{\varepsilon}_{3"} v_{2"}\, \delta_{color\, 1"2"}\, \widetilde{\delta}\, b^\dagger_3 d^\dagger_{q2"} a_{p3"} = \tag{F.15}$$

$$= e_q \sum_{1"2"3"} \int [1"2"3"]\, \bar{u}_{1"} \not{\varepsilon}_{3"} v_{2"}\, \delta_{color\, 1"2"}\, \widetilde{\delta}\, b^\dagger_{1"} d^\dagger_{q2"} a_{p3"} \cdot (*) \tag{F.16}$$

where $(*)$ is:

$$(*) = \frac{4}{3} g^2 \frac{1}{2(2\pi)^3} \int \frac{dx d^2\kappa}{x_{1'} x_{2'}} (-\frac{1}{2}) \left(\frac{1}{ba}\right)^2 (1-f_{ba})^2\, \frac{x_3}{x_{1'}} \left(\left(\frac{x_{1'}+x_3}{x_{2'}} \right)^2 + 1 \right) \kappa^2_{1'/3} r^2_{\Delta\delta} \tag{F.17}$$

A divergent part of $(*)$ is:

$$(*)_\Delta\ = -\frac{1}{3}\frac{g^2}{8\pi^2} \int_0^1 \frac{dx}{(1-x)} x_3 \left[(x+x_3)^2 + (1-x)^2 \right] r^2_\delta \left(-1-\gamma - \log\left(4x\mu_g^2\right) \right) - \tag{F.18}$$

$$-\frac{1}{3}\frac{g^2}{8\pi^2} \left(\int_0^1 \frac{dx}{(1-x)} x_3 \left[(x+x_3)^2 + (1-x)^2 \right] r^2_\delta \right) \log \Delta^2 \tag{F.19}$$

The last line requires a counterterm. A similar anti-quark self-interaction loop gives the contribution:

$$(\bar{*})_\Delta\ = -\frac{1}{3}\frac{g^2}{8\pi^2} \left(\int_0^1 \frac{dx}{(1-x)} (1-x_3) \left[(x+1-x_3)^2 + (1-x)^2 \right] r^2_\delta \right) \log \Delta^2 \,. \tag{F.20}$$

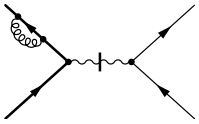
Taken together, the divergent parts require a counterterm:

$$X_{>\!-}\ = e_q \sum_{1"2"3"} \int [1"2"3"]\, \bar{u}_{1"} \not{\varepsilon}_{3"} v_{2"}\, \delta_{color\, 1"2"}\, \widetilde{\delta}\, b^\dagger_{1"} d^\dagger_{q2"} a_{p3"} \tag{F.21}$$

$$-\frac{1}{3}\frac{g^2}{8\pi^2} \left(\int_0^1 \frac{dx}{(1-x)} \left[2(1+x^2) - (4x+3) x_3 (1-x_3) \right] r^2_\delta \right) \log \frac{\Delta^2}{C} \tag{F.22}$$

where C is a constant of dimension $mass^2$.

F.2.4 Photon seagull quark mass loop



$$H_{>+<}\ = e_q e \int [1234]\, \bar{u}_{q1} \gamma^+ v_{q2} \frac{1}{\left(k^+_3 + k^+_4\right)^2} \bar{v}_{e3} \gamma^+ u_{e4} \widetilde{\delta}(1+2-3-4)\, b^\dagger_{q1} d^\dagger_{q2} d_{e3} b_{e4} \tag{F.23}$$

$$\mathcal{H}_{\lambda\,>\!\!<}\ = (\text{loop factor as in Eq.(F.13)}) \cdot H_{>+<} \tag{F.24}$$

After including both fermion and anti-fermion loops:

$$X_{>\!\!<} = e_q e \int [1234]\, \bar{u}_{q1} \gamma^+ v_{q2} \frac{1}{\left(k^+_3 + k^+_4\right)^2} \bar{v}_{e3} \gamma^+ u_{e4} \widetilde{\delta}(1+2-3-4)\, b^\dagger_{q1} d^\dagger_{q2} d_{e3} b_{e4} \times$$

$$\times (-1) \frac{1}{3}\frac{g^2}{8\pi^2} \left(\int_0^1 \frac{dx}{(1-x)} \left[2(1+x^2) - (4x+3) x_1 (1-x_1) \right] r^2_\delta \right) \log \frac{\Delta^2}{C} \tag{F.25}$$

F.3 Terms in $\mathcal{H}_\lambda$ due to mass counterterm in H^Δ

The only way the quark mass counterterm in H^Δ could enter $\mathcal{H}_\lambda$ in order g^2e is through $u_{H2}H_{QED}$ terms of the type  . However, $u_{H2} = \{(1-f_\lambda)H_2\}$, and in the case of such a term, $a = b$ and $f_\lambda \equiv 1$, and therefore $u_{H2} \equiv 0$. Thus there are no such terms in $\mathcal{H}_\lambda$ (see also comments on p.97).

Appendix G

Scattering amplitude: LSZ formula

Below I derive the expression for the S matrix following closely the Bjorken and Drell textbook [96]. I point out key differences and quote their equations when necessary; the result is discussed in Chapter 5.

G.1 *In* and *out* fields

G.1.1 Free quantum fields

A free quantum field $\phi_0(x^\mu)$ fulfills the requirements:

$$[P^\mu, \phi_0(x)] = i\frac{\partial \phi_0(x)}{\partial x_\mu} \tag{G.1}$$

$$\left(\Box + m^2\right)\phi_0(x) = 0 \tag{G.2}$$

– the free Klein-Gordon equation[1]. One can look for solutions of these equations in terms of their spatial $(x^\perp, x^-)$ Fourier transform:

$$\phi_0(x) = \int \frac{\mathrm{d}^2k^\perp \mathrm{d}k^+}{2(2\pi)^3 k^+} a_{0\vec{k}}(x^+) e^{i\vec{k}\vec{x}}\,, \tag{G.3}$$

where $\vec{k} := \left(k^+, k^\perp\right)$, $\vec{k}\vec{x} := -\frac{1}{2}k^+x^- + k^\perp x^\perp$, and the integration measure is written for the 3+1 dimensions case. From this, $a_{0\vec{k}}(x^+)$ fulfills:

$$\left(-i\partial^- k^+ + k^{\perp 2} + m^2\right) a_{0\vec{k}}(x^+) = 0\,, \tag{G.4}$$

which can be written as:

$$i\partial^- a_{0\vec{k}}(x^+) = \frac{k^{\perp 2} + m^2}{k^+} a_{0\vec{k}}(x^+)\,. \tag{G.5}$$

By introducing a symbol k_m^- for the eigenvalue of this equation:

$$k_m^- = \frac{k^{\perp 2} + m^2}{k^+} \tag{G.6}$$

[1] $\Box = \partial_\mu \partial^\mu = \partial^+\partial^- - \left(\partial^\perp\right)^2$

the dependence of a on x^+ can be written as:

$$a_{0\vec{k}}(x^+) = \exp\left(-ik_m^- x_-\right) a_{0\vec{k}}(0)\,. \tag{G.7}$$

Thus one particular set of solutions of the Klein-Gordon equation (G.2) is:

$$\phi_0(x) = \int \frac{\mathrm{d}^2k^\perp \mathrm{d}k^+}{2(2\pi)^3 k^+} a_{0\vec{k}} e^{-ik_m^\mu x_\mu}\,, \tag{G.8}$$

where $a_{0\vec{k}} := a_{0\vec{k}}(x^+ = 0)$. Note that all physical particles have $k^+ \geq 0$. When acting on any state, the part of (G.8) with $k^+ < 0$ lowers k^+, which can be interpreted as an annihilation of particles. Therefore one splits the positive and negative k^+:

$$\phi_0(x) = \int [k] \left[a_{0\vec{k}} e^{-ik_m^\mu x_\mu} + a_{0\vec{k}}^\dagger e^{ik_m^\mu x_\mu}\right]\,, \tag{G.9}$$

where $[k] := \dfrac{\mathrm{d}^2k^\perp \mathrm{d}k^+}{2(2\pi)^3 k^+}\theta(k^+)$. Commutation relations imposed on fields $\phi_0(x)$ and their canonical momenta $\pi_0(x)$ (see Section 3.3.3) lead to the equal-x^+ commutation relations for a_0:

$$\left[a_{0\vec{k}}, a_{0\vec{p}}^\dagger\right] = 2(2\pi)^3 k^+ \delta^2(k^\perp - p^\perp)\delta(k^+ - p^+) \tag{G.10}$$

and all other commutators are zero.

Using the commutation relations and replacing in (G.5) the x^+ derivative by a commutator with the Hamiltonian operator P_0^-, (G.1), one gets

$$\left[P_0^- - \int [k] \frac{k^{\perp 2} + m^2}{k^+} a_{0\vec{k}}^\dagger a_{0\vec{k}}, a_{0\vec{k}}\right] = 0 \tag{G.11}$$

and the same for a commutator with $a_{0\vec{k}}^\dagger$. The assumption that the set of operators $a_{0\vec{k}}$ and $a_{0\vec{k}}^\dagger$ is a complete set means that the expression in the left part of the commutator in Eq. (G.11) has to be a c-number c:

$$P_0^- = \int [k] \frac{k^{\perp 2} + m^2}{k^+} a_{0\vec{k}}^\dagger a_{0\vec{k}} + c\,. \tag{G.12}$$

The vacuum state $|0\rangle$ (i.e., the ground state of P_0^-) is the one that is annihilated by all the annihilation operators:

$$a_{0\vec{k}}|0\rangle = 0\,, \tag{G.13}$$

that is, the one with no particles. It is an eigenstate of P_0^-, with an eigenvalue c, i.e., the vacuum energy is c:

$$P_0^-|0\rangle = c|0\rangle\,. \tag{G.14}$$

In fact, the c constant contributes exactly the same amount to the energy of any state. Thus, measuring the energy of any state relative to the vacuum energy is equivalent to assigning the constant c a zero value.

An interesting feature of the light-front construction presented above is that the creation and annihilation operators are components of the Fourier transform of the filed $\phi_0(x)$ corresponding

to the minus or plus sign in $\exp(\mp ik^+x^-/2)$ exponents. Both a and $a^\dagger$ can be extracted from one Fourier transform:

$$a^\dagger_{0\vec{p}} = p^+ \int \mathrm{d}^2x^\perp \mathrm{d}x^- e^{-ip_m x}\phi_0(x^\mu) \tag{G.15}$$

$$a_{0\vec{p}} = p^+ \int \mathrm{d}^2x^\perp \mathrm{d}x^- e^{+ip_m x}\phi_0(x^\mu)\,, \tag{G.16}$$

where p^+ can be replaced by a spatial derivative ∂^+ of the fields. This contrasts with the equal-time quantization, where to extract $a^\dagger$ one needs both the field and the corresponding momentum, i.e., its time derivative.

G.1.2 Interacting quantum fields

For an interacting theory, one can introduce fields $\phi(x)$. They are to fulfill the equations:

$$\begin{cases} [P^\mu, \phi(x)] = i\dfrac{\partial\phi(x)}{\partial x_\mu} & \text{(G.17)} \\ (\Box + m^2)\phi(x) = j(x)\,. & \text{(G.18)} \end{cases}$$

This time one can also introduce three-dimensional Fourier transforms. If one splits the creating part (i.e., those parts increasing k^+) and the annihilating part, the field ϕ can be written as:

$$\phi(x) = \int [k] \left[a^\dagger_{\vec{k}}(x^+) + a_{\vec{k}}(x^+) \right]\,, \tag{G.19}$$

but the equation (G.18) cannot be rewritten as a simple expression for the $\partial^-\phi(x)$: the dependence of the fields and corresponding creation and annihilation operators on the light-front time x^+ is complicated. Nevertheless, the last equation can be inverted:

$$a^\dagger_{\vec{k}}(x^+) = p^+ \int \mathrm{d}^2x\mathrm{d}x^- e^{+i\vec{p}\vec{x}}\phi(\vec{x}, x^+) \tag{G.20}$$

where $\vec{x}\vec{p} := -\frac{1}{2}x^- p^+ + x^\perp p^\perp$.

G.1.3 Asymptotic condition

In a scattering process, one first considers a number of particles (usually two) in normalizable wave-packets, separated well enough not to be able to interact with each other. The final particles emerging from the interaction region are assumed not to interact with each other either.

In order to describe the well-separated wave-packets of particles in the language of QFT one can introduce so-called "smeared" field operators [100]. For this, we let $f(x^\mu)$ be any (c-number) normalizable solution of the free Klein-Gordon equation

$$(\Box + m^2) f(x) = 0\,. \tag{G.21}$$

One can introduce smeared creation operators $a_f(x^+)$ according to the following relation:

$$a_f(x^+) = \int \mathrm{d}^2x\mathrm{d}x^- \left[(i\partial^+) f(x)\right] \phi(\vec{x}, x^+) \tag{G.22}$$

Note that, for the plain-wave solutions of Eq. (G.21), this equation reduces to (G.20). It is assumed that the packets can have a reasonably small momentum width. Below, I generally omit the wave-packet smearing subscript f.

It is assumed that matrix elements of smeared packets of the fields $\phi(x)$ between any normalizable states $|\alpha\rangle$ and $|\beta\rangle$ behave for the time $x^+ \to -\infty$ like matrix elements of the similar packets of certain free fields ϕ_{in}:

$$\lim_{x^+\to-\infty} \langle\beta|\,\phi(x^\mu)\,|\alpha\rangle = \sqrt{Z} \lim_{x^+\to-\infty} \langle\beta|\,\phi_{in}(x^\mu)\,|\alpha\rangle \ . \tag{G.23}$$

This means that the *in* creation operators – which, as free operators evolve with simply a change of a phase (G.7) – create physical states as the time x^+ approaches $-\infty$.

G.2 Reduction formula for scalar fields

Let us consider the scattering amplitude with a particle of momentum $\vec{p} = (p^+, p^\perp)$ and some other particles (denoted α) in the initial state, and some particles (β) in the final state:

$$\begin{aligned}
S_{\beta,\vec{p}\alpha} &= {}_{out}\langle\beta \mid \alpha, \vec{p}\rangle_{in} = && \text{(G.24)}\\
&= {}_{out}\langle\beta|a^\dagger_{in}(\vec{p})\,|\alpha\rangle_{in} = && \text{(G.25)}\\
&= {}_{out}\langle\beta - \vec{p} \mid \alpha\rangle_{in} + {}_{out}\langle\beta|a^\dagger_{in}(\vec{p}) - a^\dagger_{out}(\vec{p})\,|\alpha\rangle_{in} = && \text{(G.26)}\\
&= {}_{out}\langle\beta - \vec{p} \mid \alpha\rangle_{in} + {}_{out}\langle\beta| \int \mathrm{d}^2x^\perp \mathrm{d}x^- e^{-ip_m x} p^+ \left(\phi_{in}(x^\mu) - \phi_{out}(x^\mu)\right)|\alpha\rangle_{in} \ , && \text{(G.27)}
\end{aligned}$$

where $|\beta - \vec{p}\rangle_{out} := a_{out,\vec{p}}\,|\beta\rangle_{out}$ is a state with one *out* particle of momentum $\vec{p}$ removed. Note that, although x^+ does appear in (G.27), it is x^+ independent. As arbitrary values of x^+ can be used, they can be chosen to approach $\pm\infty$. In this limit, one can use the asymptotic conditions:

$$\lim_{x^+\to\pm\infty} \langle\beta|\,\phi_{out/in}(x^\mu)\,|\alpha\rangle = \frac{1}{\sqrt{Z}} \lim_{x^+\to\pm\infty} \langle\beta|\,\phi(x^\mu)\,|\alpha\rangle \ . \tag{G.28}$$

$$\begin{aligned}
S_{\beta,\vec{p}\alpha} &= {}_{out}\langle\beta - \vec{p} \mid \alpha\rangle_{in} - \\
&\quad - \frac{1}{\sqrt{Z}} \left(\lim_{x^+\to\infty} - \lim_{x^+\to-\infty} \right) \int \mathrm{d}^2x^\perp \mathrm{d}x^- e^{-ip_m x} p^+ \, {}_{out}\langle\beta|\phi(x^\mu)\,|\alpha\rangle_{in} = && \text{(G.29)}\\
&= {}_{out}\langle\beta - \vec{p} \mid \alpha\rangle_{in} - \frac{1}{\sqrt{Z}} \int \mathrm{d}^4x^\mu \frac{\partial}{\partial x^+} \left[e^{-ip_m x} p^+ \, {}_{out}\langle\beta|\phi(x^\mu)\,|\alpha\rangle_{in} \right] \ , && \text{(G.30)}
\end{aligned}$$

where $\mathrm{d}^4x^\mu = \mathrm{d}^2x^\perp \mathrm{d}x^+ \mathrm{d}x^- = 2\mathrm{d}^2x^\perp \mathrm{d}x^+ \mathrm{d}x_+ = 2\mathrm{d}x^0 \mathrm{d}x^3 d^2x^\perp$.[2] Differentiating the exponent leads to $p^-_m p^+ = p^{\perp 2} + m^2$. Thus:

$$\begin{aligned}
S_{\beta,\vec{p}\alpha} &= {}_{out}\langle\beta - \vec{p} \mid \alpha\rangle_{in} - \\
&\quad - \frac{i}{\sqrt{Z}} \int \mathrm{d}^4x^\mu e^{-ip_m x} \frac{1}{2} \left[-p^{\perp 2} - m^2 + 2p^+(-i)\frac{\partial}{\partial x^+} \right] {}_{out}\langle\beta|\phi(x^\mu)\,|\alpha\rangle_{in} = && \text{(G.31)}\\
&\quad \text{note: } p^+ = i\overleftarrow{\partial}^+ = -i\overrightarrow{\partial}^+,\ p^\perp = -i\overleftarrow{\partial}^\perp = +i\overrightarrow{\partial}^\perp \ ;\\
&= {}_{out}\langle\beta - \vec{p} \mid \alpha\rangle_{in} + \frac{i}{\sqrt{Z}} \frac{1}{2} \int \mathrm{d}^4x^\mu e^{-ip_m x} \left[\overrightarrow{\Box} + m^2 \right] {}_{out}\langle\beta|\phi(x^\mu)\,|\alpha\rangle_{in} \ . && \text{(G.32)}
\end{aligned}$$

[2] Note that when I break equations at a minus sign, I put minus sign both at the end of the first line and at the beginning of the second line (which is usually clearer than putting one $+$ and one $-$).

In a similar way, if the state $|\beta\rangle_{out}$ has a particle of momentum $\vec{p}'$ we can remove it from the state in favor of an extra field in the matrix element:

$$\begin{aligned}
{}_{out}\langle\beta|\phi(x)\,|\alpha\rangle_{in} &= {}_{out}\langle\beta-\vec{p}'\,|a_{\vec{p}'out}\phi(x)\,|\alpha\rangle_{in} + {}_{out}\langle\beta-\vec{p}'\,|\phi(x)a_{\vec{p}'in}\,|\alpha\rangle_{in} - \\
&\quad - {}_{out}\langle\beta-\vec{p}'\,|\phi(x)a_{\vec{p}'in}\,|\alpha\rangle_{in} = \qquad \text{(G.33)}\\
&= {}_{out}\langle\beta-\vec{p}'\,|\phi(x)\,|\alpha-\vec{p}'\rangle_{in} + \\
&\quad + {}_{out}\langle\beta-\vec{p}'\,|\left[a_{\vec{p}'out}\phi(x)-\phi(x)a_{\vec{p}'in}\right]|\alpha\rangle_{in}\ . \qquad \text{(G.34)}\\
a_{\vec{p}'out} &= \int \mathrm{d}^2y^\perp \mathrm{d}y^- e^{ip'_m y} p'^+ \phi_{out}(y) = \qquad \text{(G.35)}\\
&= \frac{1}{\sqrt{Z}} \lim_{y^+\to+\infty} \int \mathrm{d}^2y^\perp \mathrm{d}y^- e^{ip'_m y} p'^+ \phi(y), \qquad \text{(G.36)}
\end{aligned}$$

where the last equation is understood to be a part of a smeared matrix element. We can proceed in a similar way for the *in* field. Note that, because a_{in} stands to the right of $\phi(x)$ and a_{out} stands to the left, the limits:

$$\lim_{y^+\to\infty}\phi(y)\phi(x) - \lim_{y^-\to\infty}\phi(x)\phi(y) = \left(\lim_{y^+\to\infty} - \lim_{y^-\to\infty}\right) T_{x^+}\left[\phi(x)\phi(y)\right]\ . \qquad \text{(G.37)}$$

"Time-ordering" in x^+ will be denoted here as $T_{(+)}$ (contrast ordering in x^0 denoted as T).

One can now repeat steps (G.29) to (G.32) for all other particles in $|\alpha\rangle$ and $|\beta\rangle$. The final result for all $p_i \neq q_j$ is:

$$\begin{aligned}
{}_{out}\langle p_1\dots p_m\,|\,q_1\dots q_n\rangle_{in} &= \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \prod_{i=1}^{m}\int \mathrm{d}^4x_i \prod_{j=1}^{n}\int \mathrm{d}^4y_j e^{-iq_{im}x_i}\left(\overrightarrow{\Box}_{xi}+m^2\right)\times \\
&\quad \times\langle 0|\,T_{(+)}\left[\phi(y_1)\dots\phi(y_n)\phi(x_1)\dots\phi(x_m)\right]|0\rangle\times \\
&\quad \times\left(\overleftarrow{\Box}_{yj}+m^2\right)e^{ip_{mj}y_j}\ . \qquad \text{(G.38)}
\end{aligned}$$

When any p are equal to any q, there are forward-scattering terms (compare the first terms of the equations (G.32) and (G.34)).

G.3 Perturbation expansion of the tau functions and the S matrix

One may assume that the operators $a_{\vec{k}}(x^+)$ and a complete set of free operators $a_{0\vec{k}}(x^+)$ are unitarily equivalent, that is, that there exists an operator $U(x^+)$ such that

$$a_{\vec{k}}(x^+) = U^{-1}(x^+)a_{0\vec{k}}(x^+)U(x^+) \qquad \text{(G.39)}$$

and the same for $a^\dagger$. Note that this is consistent with the fact that, for a given time x^+, operators $a_{\vec{k}}(x^+)$ and $a_{0\vec{k}}(x^+)$ fulfill the same commutation relations. Moreover, they have the same quantum numbers – here, for scalar particles, the three-momentum.

The dependence of $a_{0\vec{k}}(x^+)$ on x^+ is known, hence:

$$a_{0\vec{k}}e^{ik_m^- x^+/2} = UaU^{-1} \qquad \text{(G.40)}$$

By differentiating this equation with respect to x_- one gets:

$$a_{0\vec{k}}(x^+)(ik_m^-) = \dot{U}aU^{-1} + U\dot{a}U^{-1} + Ua\dot{U}^{-1} = \tag{G.41}$$
$$= \dot{U}U^{-1}a_0(x^+) + Ui\left[H, a(x^+)\right]U^{-1} + a_0U\dot{U}^{-1}\,, \tag{G.42}$$

where a dot indicates differentiation over x_-. Note that

$$\dot{U}U^{-1} + U\dot{U}^{-1} = 0\,. \tag{G.43}$$

Moreover, from the fact that H is a product of $a(x^+)$ for arbitrary time it follows that

$$UH(a)U^{-1} = H(a_0)\,, \tag{G.44}$$

where $H(a_0)$ has the same form as $H(a)$, but with all the creation operators a replaced by their free counterparts. Substituting Eq. (G.43) to the last part of Eq. (G.42), and (G.44) to the middle part, one gets:

$$0 = -ik_m a_0(x^+) + \left[\dot{U}U^{-1} + iH(a_0), a_0\right]\,, \tag{G.45}$$

If one denotes:

$$H_I(x^+) := H_0(a_0) - H(a_0) \tag{G.46}$$
$$H_0(a_0) := \int [k] k_m^- a^\dagger_{0\vec{k}} a_{0\vec{k}}\,. \tag{G.47}$$

(G.45) can be written as:

$$\left[\dot{U}U^{-1} + iH_I, a_0\right] = 0\,, \tag{G.48}$$

which is fulfilled for all a_0 operators and arbitrary times. Together with the completeness of the a_0 operators set, this means that the combination on the left side of the commutator is a c-number, $E_0(x^+)$. One can introduce a convenient combination:

$$H_I' := H_I(x^+) + E_0(x^+)\,, \tag{G.49}$$

and a new operator:

$$U(x^+, x'^+) := U(x^+)U^{-1}(x'^+)\,. \tag{G.50}$$

From the definition of $E_0(x^+)$ it follows that $U(x^+, x'^+)$ fulfills a Schrödinger equation:

$$\frac{\partial}{\partial x_-}U(x^+, x'^+) = iH_I'U(x^+, x'^+) \tag{G.51}$$

with boundary conditions:

$$U(x^+, x^+) = 1\,. \tag{G.52}$$

The equation for $U(x^+, x'^+)$ can be solved in perturbation theory. Standard manipulations [96] give the following result:

$$U(x^+, x'^+) = T_{(+)}\left[\exp\left(-i\int_{x'^+}^{x^+} H_I'(\xi^+)\cdot\frac{1}{2}\mathrm{d}\xi^+\right)\right]\,. \tag{G.53}$$

In Appendix G.2 the S matrix was expressed in terms of the Green function:

$$\tau(x_1, \ldots, x_n) = \langle 0|\, T_{(+)}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle\,. \tag{G.54}$$

Using the definition of U operators above, the expression for this function can be rewritten in terms of ϕ_0:

$$\begin{aligned}\tau(x_1,\ldots,x_n) &= \langle 0|\, T_{(+)}\left[U^{-1}(x_1)\phi_0(x_1)U(x_1)\ldots U^{-1}(x_n)\phi_0(x_n)U(x_n)\right]|0\rangle = &\text{(G.55)}\\ &= \langle 0|\, T_{(+)}\left[U^{-1}(x_1^+)\phi_0(x_1)U(x_1^+,x_2^+)\ldots\right.\\ &\qquad \left.\ldots U^{-1}(x_{n-1}^+,x_n^+)\phi_0(x_n)U(x_n^+)\right]|0\rangle\ . &\text{(G.56)}\end{aligned}$$

We can use x^+_{max} to denote the latest time and x^+_{min} to denote the earliest time. For the description of a scattering process, these would go to $x^+_{min/max}\to\pm\infty$. Using Eq. (G.53) and the fact that, under the time order operator $T_{(+)}$, change of order of the expressions is allowed, one gets:

$$\begin{aligned}\tau(x_1,\ldots,x_n) &= \langle 0|U^{-1}(x^+_{max})T_{(+)}\left[\phi_0(x_1)\ldots\phi_0(x_n)\exp\left(-i\int_{x'^+}^{x^+}H'_I(\xi^+)\cdot\frac{1}{2}\mathrm{d}\xi^+\right)\right]\times\\ &\quad\times U(x^+_{min})|0\rangle\ . &\text{(G.57)}\end{aligned}$$

It is straightforward to show that the vacuum is an eigenstate of U. Let $|\vec{p},\alpha\rangle_0$ be an arbitrary state with at least one free particle of momentum $\vec{p}$. Then:

$$\begin{aligned}{}_0\langle\vec{p},\alpha|U(x^+)|0\rangle &= {}_0\langle\alpha|a_{0\vec{p}}U(x^+)|0\rangle = &\text{(G.58)}\\ &= e^{+ip_mx}\,{}_0\langle\alpha|a_{0\vec{p}}(x^+)U(x^+)|0\rangle = &\text{(G.59)}\\ &= e^{ip_mx}\,{}_0\langle\alpha|U(x^+)a_{\vec{p}}(x^+)U^{-1}(x^+)U(x^+)|0\rangle = &\text{(G.60)}\\ &= e^{ip_mx}\,{}_0\langle\alpha|U(x^+)a_{\vec{p}}(x^+)|0\rangle\ . &\text{(G.61)}\end{aligned}$$

In a corresponding expression in the equal-time theory, one was forced to take the limit $t\to-\infty$ and, by the asymptotic condition (G.23), get proportionality to:

$$\sqrt{Z}\ \ {}_{in}\langle\alpha|U(x^+)a_{\vec{p},in}|0\rangle_{in}\ , \tag{G.62}$$

which is zero. To do this, one had to choose as the free operators a_0 in Eq. (G.39) the physical *in* operators a_{in} (or *out* operators, if the limit $t\to+\infty$ had been used).

In the case of a light-front Hamiltonian, such a limiting procedure, although possible, is not necessary: in the cutoff theory the free and the physical vacua are essentially the same, and the result of Eq. (G.61) is zero for arbitrary x^+ and arbitrary choice of a_0.

It is thus clear that $U(x^+)|0\rangle$ does not have components with a number of particles greater than zero, and:

$$U(x^+)|0\rangle = \zeta(x^+)|0\rangle\ , \tag{G.63}$$

i.e., the vacuum is an eigenstate of $U(x^+)$. The c-number eigenvalue $\zeta(x^+)$ may depend on x^+.

We are concerned with a limit

$$\begin{aligned}\zeta_+\zeta_- &= \lim_{x^+\to\infty}\langle 0|U^{-1}(x^+)|0\rangle\langle 0|U(-x^+)|0\rangle = &\text{(G.64)}\\ &= \lim_{x^+\to\infty}\langle 0|U^{-1}(x^+,-x^+)|0\rangle = &\text{(G.65)}\\ &= \lim_{x^+\to\infty}\langle 0|U(-x^+,x^+)|0\rangle = &\text{(G.66)}\\ &= \lim_{x^+\to\infty}\langle 0|\exp\left(+i\int_{-x^+}^{x^+}H'_I(\xi^+)\cdot\frac{1}{2}\mathrm{d}\xi^+\right)|0\rangle\ . &\text{(G.67)}\end{aligned}$$

Note that in the cutoff theory, the normal-ordered Hamiltonian annihilates the vacuum $H_I|0\rangle = 0$, and thus only the c-number components of H_I' contribute to this expression:

$$\zeta_+\zeta_- = \lim_{x^+\to\infty} \exp\left(+i\int_{-x^+}^{x^+} E_0'(\xi^+)\cdot\frac{1}{2}\mathrm{d}\xi^+\right) . \tag{G.68}$$

Such terms also appear – with the opposite exponent sign – as a factor in internal exponent of Eq. (G.57). Once these are canceled, one is left with:

$$\tau(x_1,\ldots,x_n) \;=\; \langle 0|\,T_{(+)}\left[\phi_0(x_1)\ldots\phi_0(x_n)\exp\left(-i\int_{x'^+}^{x^+} H_I(\xi^+)\cdot\frac{1}{2}\mathrm{d}\xi^+\right)\right]|0\rangle \ . \tag{G.69}$$

This equation is a light-front analog of Equation (17.22) in the Bjorken and Drell textbook [96].

Appendix H

Description of scattering amplitude using effective particles

H.1 Interpolating, bare- and effective-particle fields

The free interpolating fields $\phi_{in}(x)$, and corresponding creation operators $a^\dagger_{in}$ correspond to the physical situation one describes (i.e., the way one prepares an experiment). It is assumed, that in a distant past ($x^+ \to -\infty$) the physical situation is described by free evolution, i.e., only by a phase shift $\exp(-ik_m^- x^+/2)$.

Two kinds of interacting fields are introduced: $\phi_\infty(x)$ (expressed as an integral of the bare-particle creation operators a_∞) and $\phi_\lambda(x)$ (expressed as an integral of the effective-particle creation operators a_λ). Evolution of both these fields is determined by the same evolution operator H:

$$H := H^\Delta(a_\infty) = \mathcal{H}_\lambda(a_\lambda) \; . \tag{H.1}$$

Namely,

$$i\frac{\partial \phi_\infty}{\partial x_-} = [H, \phi_\infty] \; , \tag{H.2}$$

$$i\frac{\partial \phi_\lambda}{\partial x_-} = [H, \phi_\lambda] \; , \tag{H.3}$$

(it may be more natural to use $H^\Delta(a_\infty)$ in the first of these equations, and $\mathcal{H}_\lambda(a_\lambda)$ in the second equation, but since these operators are equal, it does not matter).

However, both fields have different asymptotic behavior. For example, one can consider a matrix element of each field between the vacuum and one-*out*-particle state:

$$\langle p|\,\phi_\infty(a_\infty)\,|0\rangle = \sqrt{Z_\infty}\,\langle p|\,\phi_{in}(a_{in})\,|0\rangle \tag{H.4}$$

$$\langle p|\,\phi_\lambda(a_\lambda)\,|0\rangle = \sqrt{Z_\lambda}\,\langle p|\,\phi_{in}(a_{in})\,|0\rangle \; . \tag{H.5}$$

The constants Z_∞ and Z_λ correspond to a normalization of a one-physical-particle state, and are different: Z_∞ appears in expansion of the one-physical-particle state in terms of bare-particle Fock sectors, and Z_λ appears in expansion of the one-physical-particle state in terms of effective-particle Fock sectors (cf. Sec. 5.2.2).

This result can be generalized to an asymptotic condition for matrix elements of these fields between any (normalizable) *in* and *out* states, in the limit $x^+ \to \infty$:

$$\lim_{x^+\to\infty} {}_{out}\langle\beta|\phi_\infty(a_\infty)\,|\alpha\rangle_{in} = \lim_{x^+\to\infty} \sqrt{Z_\infty}\ {}_{out}\langle\beta|\phi_{in}(a_{in})\,|\alpha\rangle_{in} \tag{H.6}$$

$$\lim_{x^+\to\infty} {}_{out}\langle\beta|\phi_\lambda(a_\lambda)\,|\alpha\rangle_{in} = \lim_{x^+\to\infty} \sqrt{Z_\lambda}\ {}_{out}\langle\beta|\phi_{in}(a_{in})\,|\alpha\rangle_{in}\ . \tag{H.7}$$

H.2 Reduction formula in terms of bare and effective fields

All steps of Appendix G can be repeated using either asymptotic condition for the bare fields ϕ_∞ (Eq. (H.6)), or the effective field ϕ_λ (Eq. (H.7)). This leads to two equivalent forms of the LSZ equation, expressing the same S-matrix element in terms of time-ordered matrix element of either of the two kinds of fields:

$$\begin{aligned} {}_{out}\langle p_1\dots p_m \mid q_1\dots q_n\rangle_{in} &= \left(\frac{i}{\sqrt{Z_\infty}}\right)^{m+n} \prod_{i=1}^{m}\int \mathrm{d}^4x_i \prod_{j=1}^{n}\int \mathrm{d}^4y_j e^{-iq_{im}x_i}\left(\overrightarrow{\Box}_{xi}+m^2\right)\times \\ &\times \langle 0|\,T_{(+)}\left[\phi_\infty(y_1)\dots\phi_\infty(y_n)\phi_\infty(x_1)\dots\phi_\infty(x_m)\right]|0\rangle\times \\ &\times \left(\overleftarrow{\Box}_{yj}+m^2\right)e^{ip_{mj}y_j} = \qquad (H.8) \\ &= \left(\frac{i}{\sqrt{Z_\lambda}}\right)^{m+n} \prod_{i=1}^{m}\int \mathrm{d}^4x_i \prod_{j=1}^{n}\int \mathrm{d}^4y_j e^{-iq_{im}x_i}\left(\overrightarrow{\Box}_{xi}+m^2\right)\times \\ &\times \langle 0|\,T_{(+)}\left[\phi_\lambda(y_1)\dots\phi_\lambda(y_n)\phi_\lambda(x_1)\dots\phi_\lambda(x_m)\right]|0\rangle\times \\ &\times \left(\overleftarrow{\Box}_{yj}+m^2\right)e^{ip_{mj}y_j}\ . \qquad (H.9) \end{aligned}$$

H.3 Perturbative expansion of the S matrix in terms of bare and effective fields

Appendix G.3 above presents perturbative expansion of the time-ordered product in the equation (H.8). Let us now address the question of how the expansion of the same S-matrix element looks like when one uses effective particles, (H.9).

We can use now unitary equivalence of the bare and effective creation operators:

$$a^\dagger_{\vec{k},\infty}(x^+) = U^\dagger_\lambda(x^+)a^\dagger_{\vec{k},\lambda}(x^+)U_\lambda(x^+)\ . \tag{H.10}$$

Substituting this to unitary-equivalence of the free (*in*) and bare (∞) operators, (G.39), one gets:

$$a_{\vec{k},\lambda}(x^+) = U_\lambda(x^+)U^{-1}(x^+)a_{0\vec{k}}(x^+)U(x^+)U^\dagger_\lambda(x^+) \tag{H.11}$$

(note that U_λ is RGPEP similarity rotation, while U is the operator introduced in Appendix G.3 in the context of scattering matrix). This means that $a_0(x^+)$ and $a_\lambda(x^+)$ operators are unitarily equivalent:

$$a_{\vec{k},\lambda}(x^+) = W^{-1}_\lambda(x^+)a_{0\vec{k}}(x^+)W_\lambda(x^+). \tag{H.12}$$

This is analogous to (G.39) for the bare and *in* operators, but the unitary rotation operator is now:

$$W_\lambda(x^+) := U(x^+)U^\dagger_\lambda(x^+)\ . \tag{H.13}$$

One can now follow closely the derivation from Appendix G.3. First, one differentiates Eq.(H.12) with respect to x_-, using exact, known evolution of a_0:

$$a_{0\vec{k}}(0)\cdot e^{(ik^-x^+/2)} = W_\lambda(x^+)a_{\vec{k},\lambda}(x^+)W_\lambda^{-1}(x^+) \tag{H.14}$$

By differentiating this equation with respect to x_- one gets:

$$\begin{aligned} a_{0\vec{k}}(x^+)\cdot k_m^- &= \dot{W}_\lambda a_\lambda W_\lambda^{-1} + W_\lambda \dot{a}_\lambda W_\lambda^{-1} + W_\lambda a_\lambda \dot{W}_\lambda^{-1} = && \text{(H.15)}\\ &= \dot{W}_\lambda W_\lambda^{-1} a_0 + W_\lambda i\left[H, a_\lambda\right] W_\lambda^{-1} + a_0 W_\lambda \dot{W}_\lambda^{-1} = && \text{(H.16)}\\ &= \left[\dot{W}_\lambda W_\lambda^{-1}, a_0\right] + W_\lambda i\left[\mathcal{H}_\lambda\left(a_\lambda\right), a_\lambda\right] W_\lambda^{-1} = && \text{(H.17)}\\ &= \left[\dot{W}_\lambda W_\lambda^{-1} + i\mathcal{H}_\lambda\left(a_0\right), a_0\right] , && \text{(H.18)} \end{aligned}$$

where I used unitarity of W_λ. By defining:

$$\begin{aligned} H_0(a_0) &:= \int k_0^- a_0^\dagger a_0 && \text{(H.19)}\\ \mathcal{H}_{\lambda,I} &:= \mathcal{H}_\lambda\left(a_0\right) - H_0(a_0) , && \text{(H.20)} \end{aligned}$$

equation (H.18) can be written as:

$$\left[\dot{W}_\lambda W_\lambda^{-1} + i\mathcal{H}_{\lambda,I}, a_0\right] = 0 . \tag{H.21}$$

Following steps from Appendix G.3 with operator U replaced by W, one gets perturbative expansion of the S matrix in the form (H.9):

$$\begin{aligned} &\left\langle 0\right| T_{(+)}\left[\phi_\lambda(x_1)\ldots\phi_\lambda(x_n)\right]\left|0\right\rangle = \\ &\qquad = \left\langle 0\right| T_{(+)}\left[\phi_0(x_1)\ldots\phi_0(x_n)\exp\left(-i\int_{x'^+}^{x^+}\mathcal{H}_{\lambda,I}(\xi^+)\cdot\frac{1}{2}\mathrm{d}\xi^+\right)\right]\left|0\right\rangle , \end{aligned} \tag{H.22}$$

where the expansion is done in powers of $\mathcal{H}_{\lambda,I}$.

To summarize: The same S-matrix element ${}_{out}\langle\beta\mid\alpha\rangle_{in}$ can be calculated

- using the bare fields;
  — one gets LSZ formula (H.8) and expansion (G.69) for the x^+-ordered product of fields in terms of the bare interaction Hamiltonian H_I^Δ;
- or using effective fields;
  — one gets LSZ formula (H.9) with different wave-function renormalization factors Z, and a perturbative expansion (H.22) in powers of the effective interaction Hamiltonian $\mathcal{H}_{\lambda,I}$.

This result is analyzed further in Section 5.3.

Appendix I

Examples of scalar S-matrix elements in 1+1 dimensions

I.1 Scalar propagator in 1+1 dimensions

The perturbative formula (G.69) allows us to calculate any Green's function in perturbation theory. However, the formula needs to be reduced to a set of simpler rules. To see how the formula works in a specific example I consider here a propagator in scalar theory in 1+1 dimensions, introduced in Section 5.4. Although this theory is finite, and the calculation reviewed below does not include regularization, it can be generalized to more complicated cases.

One can use Eq. (5.15) to calculate a two-point tau function (quark propagator):

$$\tau(x_1,x_2) = \langle 0|\, T_{(+)}\left[\phi_q(x_1)\phi_q(x_2)\right]|0\rangle \ . \tag{I.1}$$

The zeroth order term in (5.15) is simply:

$$\tau^{(0)}(x_1,x_2) = \langle 0|\, T_{(+)}\left[\phi_{0q}(x_1)\phi_{0q}(x_2)\right]|0\rangle \tag{I.2}$$

which leads to free Feynman propagator:

$$\tau^{(0)}(x_1,x_2) \quad = \quad \int_0^\infty \frac{dk^+}{4\pi k^+}\left[\theta(\Delta x^+)e^{-ik_m^\mu \Delta x_\mu} + \theta(-\Delta x^+)e^{+ik_m^\mu \Delta x_\mu}\right] \ . \tag{I.3}$$

θ-function can be replaced by an integral:

$$\theta(\Delta x^+) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega+i\varepsilon}e^{-i\omega\Delta x^+/2} \ . \tag{I.4}$$

Also, one can change variables in the second integral in Eq. (I.3) from k^+ to $-k^+$. This leads to:

$$\tau^{(0)}(x_1,x_2) = \int_{-\infty}^{\infty}\frac{d^2k}{(2\pi)^2}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik\Delta x} \tag{I.5}$$

where $d^2k = \frac{1}{2}dk^+dk^-(= dk^+dk_+ = dk^0dk^1)$. Note the following:

- Although in the initial expression there was an integration over only physical momenta $k^+ > 0$ and k_m^- was fixed at the physical positive value m^2/k^+, in the final expression there is integration over the full range of the Fourier parameter k^μ: $k^+ \in (-\infty,\infty)$, $k^- \in (-\infty,\infty)$.

- The expression for the full Feynman propagator has in the light-front variables only one pole in k^- for given momentum k^+ (see Fig. 3.4).

The propagator in the second order (g^2) is:

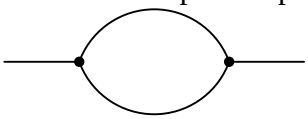
$$\tau^{(g^2)}(x_1,x_2) = \langle 0|\, T_{(+)}\left[\phi_{0q}(x_1)\phi_{0q}(x_2)\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}x^+}{2}(-i)\left(H^{(g)}_{>}(x^+)+H^{(g)}_{<}(x^+)\right)\times\right.$$
$$\left.\times\int_{-\infty}^{\infty}\frac{\mathrm{d}x'^+}{2}(-i)\left(H^{(g)}_{>}(x'^+)+H^{(g)}_{<}(x'^+)\right)\right]|0\rangle\ , \quad \text{(I.6)}$$

I review here the calculation of this expression in more detail. The scattering amplitude calculations in Sections 5.4 and 5.7 follow a similar path.

Since H_I annihilates the vacuum, it has to be separated from it by at least one ϕ_0. Therefore only orderings leading to $\langle 0|\,\phi_0 H_I H_I \phi_0\,|0\rangle$ may lead to non-vanishing results. Also, different orderings of x^+ and x'^+ reduce to re-labeling these variables. This cancels the factor one-half coming from the exponent. The expression for $\tau^{(g^2)}$ can thus be written as:

$$\begin{aligned}\tau^{(g^2)}(x_1,x_2) \;=\; & \frac{(-i)^2}{2^2}\int_{-\infty}^{\infty}\mathrm{d}x^+\mathrm{d}x'^+\,\langle 0|\\ & \phi^{(+)}_{0q}(x_1)H^{(g)}_{<}(x^+)H^{(g)}_{>}(x'^+)\phi^{(-)}_{0q}(x_2)\theta\left(x_1^+>x^+>x'^+>x_2^+\right)+ \quad \text{(I.7)}\\ & \phi^{(+)}_{0q}(x_2)H^{(g)}_{<}(x^+)H^{(g)}_{>}(x'^+)\phi^{(-)}_{0q}(x_1)\theta\left(x_2^+>x^+>x'^+>x_1^+\right)|0\rangle\ , \quad \text{(I.8)}\end{aligned}$$

where $\phi_0^{(+)}$ denotes the annihilating part of the field, and $\phi_0^{(-)}$ the creating part. For the matrix element not to vanish, each creation operator has to disappear by commutation with some annihilation operator. Whenever a particle is created at some interaction vertex H_I or external field ϕ_0, it has to be annihilated at some other part of this expression, and vice versa. Therefore, the structure of this expression can be shown as a number of lines, each representing a creation-annihilation operator pair:



The θ functions can be written as

$$\theta\left(x_2^+>x^+>x'^+>x_1^+\right)=\theta\left(x_2^+-x^+\right)\theta\left(x^+-x'^+\right)^2\theta\left(x'^+-x_1^+\right)\ , \quad \text{(I.9)}$$

associating one θ function with each diagram line.

At this point it is simpler to calculate the Fourier transform of the tau function:

$$\widetilde{\tau}^{(g^2)}(p_1,p_2)=\int\frac{\mathrm{d}x_1^+\mathrm{d}x_1^-}{2}e^{ip_1x_1}\int\frac{\mathrm{d}x_2^+\mathrm{d}x_2^-}{2}e^{ip_2x_2}\tau^{(g^2)}(x_1,x_2)\ . \quad \text{(I.10)}$$

The initial expression can then be extracted by an inverse Fourier transform:

$$\tau^{(g^2)}(x_1,x_2)=\int\frac{\mathrm{d}p_1^+\mathrm{d}p_1^-}{2\cdot(2\pi)^2}e^{-ip_1x_1}\int\frac{\mathrm{d}p_2^+\mathrm{d}p_2^-}{2\cdot(2\pi)^2}e^{-ip_2x_2}\tau^{(g^2)}(p_1,p_2)\ . \quad \text{(I.11)}$$

The Fourier transform (I.10) has the following form. For each *internal line* there is a factor:

$$\int[k]e^{-ik_m^-x^+/2}e^{+ik_m^-x'^+/2}\theta(x^+-x'^+)\ , \quad \text{(I.12)}$$

where the exponents are related to the free evolution of the creation and annihilation operators in terms of which the interaction Hamiltonian H_I is expressed. Once the θ function is expressed as an integral (I.4), this leads to:

$$\int \frac{\mathrm{d}^2k\theta(k^+)}{(2\pi)^2}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik^-(x^+-x'^+)/2} \tag{I.13}$$

($\mathrm{d}^2k=\frac{1}{2}\mathrm{d}k^+\mathrm{d}k^-$). For each *external line* there is a similar factor: the difference is that the Fourier expansion of the fields introduces for the external x^μ an $\exp(ik_m^\mu x_\mu)$, instead of $\exp(ik_m^- x^+/2)$. Integration over x_i^μ in (I.10) substitutes physical p^μ of the external particle for momentum parameter corresponding to this line. The formal requirement $k^+>0$ is thus automatically fulfilled, and $\theta(k^+)$ may be dropped for external lines. For each *interaction vertex* there are factors:

$$(-i)g\int\frac{dx^+}{2}4\pi\delta(k^+_{cre}-k^+_{ann})\,, \tag{I.14}$$

where k^+_{cre} is the sum of the momenta of all particles created in the vertex, and k^+_{ann} is the sum of the momenta of all annihilated particles. Integration over the vertex time x^+ can be performed, leading to $4\pi\delta(k^-_{cre}-k^-_{ann})$. Altogether, each vertex gives:

$$(-ig)2(2\pi)^2\delta^2(k^\mu_{cre}-k^\mu_{ann})\,, \tag{I.15}$$

and exponent factors for each line should be dropped, except for the exponents for the external lines, corresponding to unintegrated x^μ.

These rules may be summarized as follows:

- **For each internal line** there is an integral:

$$\int\frac{\mathrm{d}^2k\theta(k^+)}{(2\pi)^2}\frac{i}{k^2-m^2+i\varepsilon} \tag{I.16}$$

  ($\mathrm{d}^2k=\frac{1}{2}\mathrm{d}k^+\mathrm{d}k^-$);

- **For each external line** ending at a point x^μ:

$$\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik_\mu x^\mu}\,; \tag{I.17}$$

- **For each interaction vertex**:

$$(-ig)2(2\pi)^2\delta^2(k^\mu_{cre}-k^\mu_{ann})\,. \tag{I.18}$$

Accordingly, $\tau^{(g^2)}(x_1,x_2)$ is:

$$\begin{aligned}\tau^{(g^2)}(x_1,x_2) \;=\; &\int\frac{\mathrm{d}^2p}{\cdot 2\pi}e^{-ip(x_1-x_2)}(-ig)^2\frac{i}{p^2-m^2+i\varepsilon}\frac{i}{p^2-m^2+i\varepsilon}\times\\ &\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{i}{k^2-m^2+i\varepsilon}\frac{i}{(p-k)^2-m^2+i\varepsilon}\,.\end{aligned} \tag{I.19}$$

($\mathrm{d}^2p = \mathrm{d}p^+\mathrm{d}p^-/2$). Together, the zeroth and the first order lead to:

$$\tau^{(0+g^2)}(x_1,x_2) = \int \frac{\mathrm{d}^2p}{(2\pi)^2} e^{-ip(x_1-x_2)} \frac{i}{p^2-m^2+i\varepsilon} \times \\ \times \left[1 + \frac{i}{p^2-m^2+i\varepsilon} g^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2-m^2+i\varepsilon)\left((p-k)^2-m^2+i\varepsilon\right)}\right] \quad \text{(I.20)}$$

We are interested in this expression at its pole. Perturbation expansion can only be used if the pole is not shifted too much. The bare expression has the pole at $p^2 = m^2$, so we are interested in this expression for $p^2 \approx m^2 \in (0, 4m^2)$. The d^2k integral can be calculated explicitly (see Appendix I.2), leading to:

$$\tau^{(0+g^2)}(x_1,x_2) = \int \frac{\mathrm{d}^2p}{(2\pi)^2} e^{-ip(x_1-x_2)} \frac{i}{p^2-m^2+i\varepsilon + \frac{g^2}{4\pi m^2} \frac{\delta}{\sqrt{\delta-1}} \arctan \frac{1}{\sqrt{\delta-1}}}\,, \quad \text{(I.21)}$$

where $\delta := 4m^2/p^2$ is assumed to be in the region $\delta \in (1,\infty)$.

This expression is analyzed further in Section 5.5.1.

I.2 Propagator in 1+1 dimensions in order g^2: results of integration

The integral

$$I_{1+1}(p^\mu) := \int \mathrm{d}^2k \frac{1}{k^2-m^2+i\varepsilon} \frac{1}{(p-k)^2-m^2+i\varepsilon} \quad \text{(I.22)}$$

($\mathrm{d}^2k = \mathrm{d}k^+\mathrm{d}k^-/2$) can be performed either by replacing integration over k^- with a sum of residues, or by introducing Feynman parameters. Both ways lead to the same result given below.

$$\delta := \frac{4m^2}{P^2} \quad \text{(I.23)}$$

For $P^2 > 4m^2$ (i.e., $\delta \in (0,1)$):

$$I\left(4m^2 < P^2\right) = \frac{i\pi}{2m^2} \frac{\delta}{\sqrt{1-\delta}} \left[\log\left(\frac{1-\sqrt{1-\delta}}{1+\sqrt{1-\delta}}\right) + i\pi\right] \quad \text{(I.24)}$$

For $P^2 \in \left(0, 4m^2\right)$ (i.e., $\delta > 1$):

$$I\left(0 < P^2 < 4m^2\right) = \frac{i\pi}{m^2} \frac{\delta}{\sqrt{\delta-1}} \arctan \frac{1}{\sqrt{\delta-1}} \quad \text{(I.25)}$$

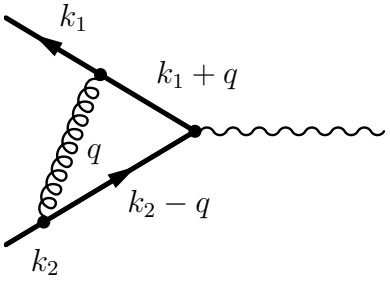
For $P^2 < 0$ (i.e. $\delta < 0$):

$$I\left(P^2 < 0\right) = \frac{i\pi}{2m^2} \frac{\delta}{\sqrt{1-\delta}} \log\left(\frac{\sqrt{1-\delta}-1}{\sqrt{1-\delta}+1}\right) \quad \text{(I.26)}$$

For example, for $P^2 = m^2$:

$$I\left(P^2 = m^2\right) = \frac{i\pi}{m^2} \frac{4}{\sqrt{4-1}} \arctan \frac{1}{\sqrt{4-1}} = \frac{i\pi}{m^2} \frac{4}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} = \frac{i\pi^2}{m^2} \frac{2}{3\sqrt{3}} \quad \text{(I.27)}$$

I.3 1+1 triangle term



The integral:

$$f = i\int \frac{dq^+dq^-}{2(2\pi)^2}\,\frac{1}{q^2-m^2+i\varepsilon}\,\frac{1}{(k_1+q)^2-m^2+i\varepsilon}\,\frac{1}{(k_2-q)^2-m^2+i\varepsilon} \tag{I.28}$$

can be performed either by replacing the integral over k^- by a sum of residues, or by introducing Feynman parameters (for details of how to integrate the Feynman parameters explicitly in this case, see e.g. [119]). Either way, it is easier to first calculate the imaginary part of this function:

$$Im(f) = -\frac{1}{4m^4}\,\frac{\delta}{\sqrt{1-4\delta}}\,\frac{1-2\delta}{1-3\delta} \tag{I.29}$$

($\delta := m^2/s \in \left(0,\frac{1}{4}\right)$), and then the real part, using the dispersion relation:

$$f(s) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds'\frac{Im(f(s'))}{s'-s-i\varepsilon}\,. \tag{I.30}$$

The result is:

$$Re(f(s)) = \frac{1}{4\pi m^4}\,\frac{\delta}{1-3\delta}\left[\frac{2\pi}{3\sqrt{3}}+\frac{1-2\delta}{\sqrt{1-4\delta}}\log\left(\frac{\left(1+\sqrt{1-4\delta}\right)^2}{4\delta}\right)\right]. \tag{I.31}$$

Appendix J

Eigenvalue equations

J.1 Schrödinger's solution in Coulomb potential

The equation:

$$\frac{\vec{k}^2}{2\mu}\phi(\vec{k}) - \int \frac{d^3k'}{(2\pi)^3}\frac{4\pi\alpha}{(\vec{k}-\vec{k}')^2}\phi(\vec{k}') = -B\phi(\vec{k}) \tag{J.1}$$

(i.e., the Schrödinger equation for positronium without spin and with reduced mass $\mu = m/2$) has the ground state eigenvalue:

$$B_0 = \frac{1}{2}\mu\alpha^2 \,, \tag{J.2}$$

and the normalized ground-state wave function:

$$\phi_0(k) = N\frac{1}{(a^2+k^2)^2} \,, \tag{J.3}$$

with $N = \sqrt{8\alpha^5\mu^5}/\pi$ and $a = \alpha\mu$.

J.2 Reduction procedure

The eigenvalue equation for a Hamiltonian H:

$$H\,|\psi\rangle = E\,|\psi\rangle \tag{J.4}$$

for certain low eigenvalues, is replaced using the operation R (see [49, 44]) by an eigenvalue equation for eigenstates $|\varphi\rangle = \sqrt{\hat{P}+R^\dagger R}\,|\psi\rangle$ of the reduced Hamiltonian H_R. This is given by the following formula:

$$H_R = \frac{1}{\sqrt{1+R^\dagger R}}(\hat{P}+R^\dagger)H(\hat{P}+R)\frac{1}{\sqrt{1+R^\dagger R}} \,. \tag{J.5}$$

If one splits the initial Hamiltonian into the free and interaction parts:

$$H = H_0 + H_I \,, \tag{J.6}$$

then one can look for R and H_R in perturbation theory in H_I. This leads to the lowest (second order) expression for H_R:

$$H_R = \hat{P}H\hat{P} + \frac{1}{2}\hat{P}H_I\hat{Q}\Big\{H_I\Big\}_0\hat{P} - \frac{1}{2}\hat{P}\Big\{H_I\Big\}_0\hat{Q}H_I\hat{P} + \ldots \tag{J.7}$$

Appendix K

Mathematical formulae

K.1 UV divergent integrals

Let us consider an integral:

$$\int_a^\infty \frac{\mathrm{d}z}{z}\exp\left(-c_\eta\frac{z}{\Delta^2}\right) = \left|u := c_\eta\frac{z}{\Delta^2}\right| = \int_{c_\eta a/\Delta^2}^\infty \frac{\mathrm{d}u}{u}e^{-u} = \tag{K.1}$$

$$= \int_1^\infty \frac{\mathrm{d}u}{u}e^{-u} + \int_{c_\eta a/\Delta^2}^1 \frac{\mathrm{d}u}{u}\left(e^{-u}-1\right) + \int_{c_\eta a/\Delta^2}^1 \frac{\mathrm{d}u}{u} = \tag{K.2}$$

$$= \ln\frac{\Delta^2}{m^2} - \ln c_\eta - \ln\frac{a}{m^2} + I_\gamma \tag{K.3}$$

Only the first two terms depend on the regularization. I_γ is a constant numerical term:

$$I_\gamma := \int_1^\infty \frac{\mathrm{d}u}{u}e^{-u} + \int_0^1 \frac{\mathrm{d}u}{u}\left(e^{-u}-1\right) = -0.577216\ldots \tag{K.4}$$

Using this, one can calculate frequent divergent integrals: the logarithmically divergent one:

$$\int \frac{\mathrm{d}z}{z+d}\exp\left(-c_\eta\frac{z}{\Delta^2}\right) = \int_0^a \frac{\mathrm{d}z}{z+d} + \int_a^\infty \mathrm{d}z\left(\frac{1}{z+d}-\frac{1}{z}\right) + \int_a^\infty \frac{\mathrm{d}z}{z}\exp\left(-c_\eta\frac{z}{\Delta^2}\right) =$$

$$= \ln\frac{\Delta^2}{m^2} - \ln c_\eta - \ln\frac{d}{m^2} + I_\gamma\,, \tag{K.5}$$

and the quadratically divergent one:

$$\int_0^\infty \mathrm{d}z\frac{z}{z+d}\exp\left(-c_\eta\frac{z}{\Delta^2}\right) = \int_0^\infty \mathrm{d}z\left(1-\frac{d}{z+d}\right)\exp\left(-c_\eta\frac{z}{\Delta^2}\right) =$$

$$= \frac{\Delta^2}{c_\eta} - d\left(\ln\frac{\Delta^2}{m^2} - \ln c_\eta - \ln\frac{d}{m^2} + I_\gamma\right)\,. \tag{K.6}$$

K.2 Area of an n-dimensional sphere Ω_n

The area of n-dimensional sphere is:

$$\Omega_n = \left(\int_0^\pi d\theta_n \sin^{n-2}\theta_n\right)\ldots\left(\int_0^\pi d\theta_3 \sin\theta_3\right)\int_0^{2\pi} d\theta_2 = 2\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \tag{K.7}$$

For example:

$$\Omega_4 = 2\pi^2 \tag{K.8}$$
$$\Omega_3 = 4\pi \tag{K.9}$$
$$\Omega_2 = 2\pi \tag{K.10}$$